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Engineering Preview

AN INTRODUCTION TO ENGINEERING INCLUDING THE
NECESSARY REVIEW OF SCIENCE AND MATHEMATICS

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PREFACE

Engineering Preview is designed to open the field of engineering study to anyone who has the determination to master the subject. The book can be studied most effectively in the senior year of high school or the freshman year of college, at which time the preparation of the student in mathematics is still fresh. However, since the minimum essentials of mathematics are included as one section of the book, those who have been away from formal contact with mathematics for several years may study the book profitably by giving particular attention to the refresher section on mathematics.

The sections of this book treat the background sciences of engineering which are mathematics, chemistry and physics; the two languages of engineering which are mathematics and technical drawing; the three tools for making quantitative calculations which are mathematics, technical drawing, and the use of the slide rule; and some of the basic engineering applications of physics such as illumination, electric power, electronics, mechanics, heat engines, and refrigeration. The applications in these latter sections are from the engineering point of view rather than that of the physicist.

Engineering Preview is not a collection of unrelated chapters but it is an organized presentation of background material needed for the ultimate study of any specialized field of engineering. Even in the chapters on science, the problems and examples are drawn wherever possible from engineering practice. Since each author has written widely in his own field, the book brings the mature experience of seven technical writers together to bear upon the reader's problem of developing a proper background for professional engineering study. It represents the best available technical advice on the subject of what to study and how to study it with the objective of preparation for engineering practice.

The book does not, of course, complete any subject discussed. It takes four years of college study to produce an engineer. No doubt the reader will notice that a part of the material recalls some of his high school work and he will find the rest of it treated

again in somewhat different form in college courses. This is as it should be. We learn by repetition and by seeing a subject presented from more than one point of view. For example, the study of forces may be brought up in a dozen college courses in mechanical engineering. Only by such repetition can we be certain that the mechanical engineer will develop a complete understanding of forces and of their effects upon structures and machines.

Engineering Preview serves to orient the reader in the fields of science and engineering. Almost every person has some question as to his bent in life. An effective way to learn whether you really should become an engineer or scientist is to read and to work simple problems in these fields. If you find this book tiresome and confusing, it is unlikely that you would want to become a technologist; but if it seems clear and stimulating, you are probably destined to use a slide rule. The first section of this book is devoted entirely to orientation, and it even includes a simple self-scoring test giving you a rough inventory of your technological traits. A careful reading of this section will help you decide for or against technology as a career and will lift into focus for you the many subdivisions of science and engineering.

The authors do not recommend this book for a rapid survey unless the reader has covered much of the material once before and wishes to scan it as a review. Probably the equivalent material could not be found elsewhere in less than a half-dozen volumes, so deliberate attention to details and to the solution of problems is recommended. However, it is surprising how fast one progresses if a full hour each day is set aside for study. Such preparation may well change the study of a specialized engineering curriculum from a difficult task to a pleasing experience.

It is a pleasure to acknowledge the assistance of Professor Walter Peterhans who selected many of the photographs and of Professor L. Hilberseimer who made the designs and drawings for the endpapers. The excellence of the drawings is due largely to the careful supervision and personal work of Professor H. C. Spencer. A few illustrations are from the authors' previous books: *Technical Drawing* by Giesecke, Mitchell, and Spencer; *General Chemistry* by Harry Holmes; *Applied Thermodynamics* by Virgil Faires, all published by Macmillan; and *Principles of Electronics* by R. G. Kloeffler, published by John Wiley & Sons.

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CONTENTS

Outlines in Science and Engineering ORIENTATION 3

Technology and culture scientist, engineer, technician the development of engineering the age of machines power, electronics the profession of engineering who should be an engineer parlor test for technological traits reading and using technical books using mathematics physics as engineering background applications of chemistry and chemical engineering a great bridge collapses civil engineering fatigue of metals mechanical and aeronautical engineering experiments on models electrical and communication engineering metallurgy architecture technology and government how to become an engineer

Chemistry THE SCIENCE OF MATTER AND MOLECULES 53

Substances, elements, mixtures, compounds the atomic theory the language of chemistry chemical equations oxygen and oxides gases, liquids, solids, hydrogen, activity series of metals valence and structural formulas, water, carbon, carbides molecular and atomic weights solubility chlorine, bromine, iodine the halogen acids alkalies sulfur periodic table radioactivity the atmosphere nitrogen ammonia nitric acid hydrocarbons rubbers alcohols fats oils, and soaps starch and sugars plastics, silica and glass electrochemistry the metals, alloys, metallic compounds, strategic materials

Technical Drawing THE LANGUAGE OF ENGINEERING 152

Instrumental drawing equipment, pencil lines use of scales and instruments, ink drawing vertical and inclined lettering shape description, projection and the glass box exterior views, surfaces and edges sectioning and interior views violations of projection auxiliary views and revolution dimensioning, size dimensions mating dimensions, isometric projection oblique projection perspective drawing production illustrations, technical sketching drawing threads other fasteners idea sketch, layout drawing, detail drawing, assembly drawing, patent drawing

Mathematics A UNIVERSAL TOOL OF ENGINEERING 253

Zero and the negative numbers, rational numbers, real numbers, Pythagorean theorem, irrationality of $\sqrt{2}$ and π , powers, products, division, square root, rounding off numbers, errors linear equations simultaneous equations, quadratic equations, exponents and radicals logarithms, log tables, trigonometric functions of angles, arcs and radian measure, similar triangles, trigonometric relationships, solving right triangles solving oblique triangles, points and straight lines, equations of straight lines, the circle, parabola, ellipse, hyperbola, curve fitting, families of curves and nomographs

| | | |
|---|----------------------------------|-----|
| <i>The Slide Rule</i> | THE ENGINEER'S COAT OF ARMS | 363 |
| <p>What it is; the Mannheim rule; the scales; how to multiply and divide; square and square root; cube and cube root; trigonometric functions; special scales; the duplex type rule; the log-log type rule; the circular slide rule; special uses.</p> | | |
| <i>Light and Electricity</i> | HALLMARKS OF CIVILIZATION | 391 |
| <p>Electric lighting; incandescent lamp; mercury and sodium lamps; fluorescent lamp; physiology of lighting; psychology of lighting; units of lighting; direct and indirect lighting; design of proper lighting, electron theory of electricity; units of electricity; electrical energy and power; electromagnetism; producing voltage; power generation; electric motors; alternating current; transforming voltage; electronics, thermionic emission; electron-tube operation; gaseous conduction; diode and rectification; grid tube; amplifier, oscillator, modulator, detector; radio transmission and reception; industrial electronics.</p> | | |
| <i>Mechanics</i> | THE EFFECTS OF FORCES | 460 |
| <p>How forces are applied; why materials fail; balance and symmetry; the principle of the lever; equilibrium of forces and moments; force components; vectors and trigonometry; stresses in a bridge truss; velocity and acceleration; friction; Newton's laws; mechanical work; kinetic energy; work and acceleration; impact and momentum; recoil of a gun; rocket propulsion.</p> | | |
| <i>Thermodynamics</i> | CONTROL OF HEAT, COLD, AND POWER | 491 |
| <p>Sources of heat; internal combustion engines; measurement of heat; horsepower, engine efficiency; engine knock; steam power plant; first law of thermodynamics, efficiency of power plant, the electric refrigerator; pumping heat; air conditioning; relative humidity; winter conditioning; dehumidification for summer conditioning.</p> | | |
| <i>Tables for Mathematical Calculations</i> | | 525 |
| <p>Five-place logarithms of numbers; values of trigonometric functions of angles from 0° to 90°; radian measure with trigonometric functions; compound interest; amount of an annuity; important constants or numbers</p> | | |
| <i>Index</i> | | 575 |

Engineering Preview

Engineers and scientists devote their lives to improving the physical well-being of man. From the field engineer, every report shows how some difficulty of construction or operation was overcome. Each product of the designer is somehow better than the old. Through research new methods, new products, new materials are sought and found. Greater production at lower cost is the industrial engineer's creed. A literature filling whole libraries has been required to record our scientific advancement. With clarity, the engineer sees the changes to come. Together from field and factory we proclaim progress for the new world.

Outlines in Science and Engineering

It is a wise precaution for any author to think a bit about the reader before he starts to write. We believe that you will be a hardy reader rather than the “gentle” type. You are probably interested in science and engineering either because you expect to become an engineer or scientist or because you really want to know what makes this technological world revolve around you. In either instance we authors should be able to depend upon your fortitude which will be needed when the digging gets down past the soft topsoil into the hardpan beneath. But, truly, we expect to use simple language and not to strain your resolution unduly. It is said of good engineers that they are not unusually brilliant people but that they are tough-fibered and mentally tenacious. If you had lacked those characteristics completely, you would not likely have become interested in this book, nor would we be interested in inducing you to read it.

*Reading and
writing*

Engineering is applied science. Technology is a more impressive word with about the same meaning. No one needs to be told that he must learn some science before he can apply it. You have probably studied mathematics and perhaps some chemistry or physics in high school, and you will restudy all of these subjects in college or at home by “burning the midnight oil” if you want to become an engineer. But there is an in-between field of refreshing whatever science you may have studied once before and of advancing a little way into the modern concepts of science and engineering that this book intends to cover. Such a book could be made quite imposing by using long technical words and longer sentences. However, like the preacher who admitted that he had probably never saved any souls after the first twenty minutes of his sermon, we doubt our ability to create many engineers by that method. We are, therefore, inclined here to rely upon the dignity of science itself to impress the reader and to let the words fall where they may.

*Science
refresher*

Year by year, a continually increasing percentage of the populace becomes intimately associated with engineering developments. As we look back, it is easy to trace the growth of popular awareness

*Technological
evolution*

of the engineer and his works. During the twenties, our everyday life gradually became associated with the automobile, the radio, and the electric refrigerator. In the thirties, we found that we were face to face with the economic necessity of invention and new technological developments to keep our industries alive. Finally, when the plastics industry loomed over the horizon, it was greeted with great enthusiasm, as much because of its economic significance as because of its contribution to our life and culture. Beginning with the forties, we entered a totally technological war and found our very existence depending upon the speed of our technical progress. With sweethearts, wives, and mothers running machines, making blueprints, and driving rivets, it soon became trite to say "this has become a technological world." We realized, at last, that it had been a technological world for a generation and that the engineer had become important to our way of life.

*Culture and
technology*

A great deal of time has been wasted in discussions of the relative merits of a "cultural" versus a "technological" education. Actually, any education may be cultural, but only a few students in any field obtain a cultural education. The mere study of languages, literature, and history is no assurance that the student has achieved culture. But neither is it true that specialization in one branch of engineering alone will produce a cultured person. Culture is a deep understanding of man and his environment. At the time of Christ, culture was achieved through a casual knowledge of history, local geography, the Mediterranean languages, and the religious and social customs of the people. Of limited usefulness was any knowledge of mathematics, and science was mainly philosophical speculation. Law, philosophy and art, reaching high plateaus, contributed the most enduring influence.

*Depth of
culture*

The situation now has changed so greatly that a cultured man of the Roman Empire would appear pathetically unlearned in many common fields of knowledge today. Without a reasonable understanding of mathematics, science, and technology — an understanding appreciation of their influence now and probable future influence upon man — how can anyone be truly cultured? Then, it seems that one other requirement of culture is often overlooked. No one can develop real understanding in many fields unless he is a thorough student of one. In fact, any deeply cultured person will have been interested in making some contribution to man's ultimate spiritual, social, intellectual, or physical welfare.

SCIENTIST, ENGINEER, OR TECHNICIAN

When you find that your inner interests direct or lead you into technical fields — that is, when you get a thrill out of machinery and blueprints and laboratories — the question still should be raised: Are you to educate yourself as a technician, an engineer, or a scientist? No one can answer this question but yourself although there is some help available through placement examinations that will be mentioned later. However, if a man wants anything strongly enough, he often finds ways of getting around a lack of natural ability. Determination almost always wins. Lack of determination means failure in engineering or in anything else.

But, within reason, it is wise to aim at the field for which your aptitudes best fit you. Several times as many technicians are needed by industry as engineers, and several times as many engineers are needed as scientists. A technician is commonly one who conditions complicated machinery or who acts as troubleshooter or detail draftsman or laboratory assistant. He usually works under an engineer or scientist or production superintendent. He does not need much knowledge of mathematics or of basic science. His knowledge is practical and is composed of “how to do something” rather than “why it is done.” The scientific information in this book might be adequate for the technician while it is just an introduction for the engineer. *Technician*

The scientist is usually looked upon as more theoretically inclined than the engineer, and he is often a better mathematician. If we think of the spectrum as symbolizing the whole field of technology, the red, yellow, and blue primary colors with their secondary colors of orange, green, and violet might be compared to six major fields of engineering (civil, mechanical, electrical, chemical, industrial, and aeronautical). From one end of the visible spectrum extends the well-known infrared or heat band which we will think of as analogous to the broad field of the technician's work, while from the other end of the spectrum we step off into the cold but strangely dazzling field of the ultraviolet or black light radiation in which the scientist primarily operates. *Spectrum analogy*

The analogy may be carried further. The warm belt of technician service is a comfortable, satisfying, steady state of life for which reliability is more important than imagination. It has many compensations for its workers. The color bands of engineering also *Engineer*

are stable and in their fundamental aspects unchanging, but they are brilliant and challenging to the imagination. They can be mixed to produce new shades or tints of color — new fields of technological accomplishment. The possibilities for new creations in engineering technology are as limitless as the paintings that may come from the artist's brush; we both have the entire spectrum at our command.

Scientist

Finally, science is much like the black light of the ultraviolet radiation which is the source of fluorescence. Impartially, yes coldly analytical, the scientist works largely with the unknown. Nonetheless, there are times when the invisible light of his scientifically trained mind falls upon a fluorescent situation and suddenly there lights up before him a scene of startling beauty, his own intellectual creation. The scientist and the research engineer will often experience this thrill of personal creation; meaningless to some, it is the whole of life to others.

It
his career

It is your career that you would build and so you must choose it for yourself. If my analogy will help you, go over it again:

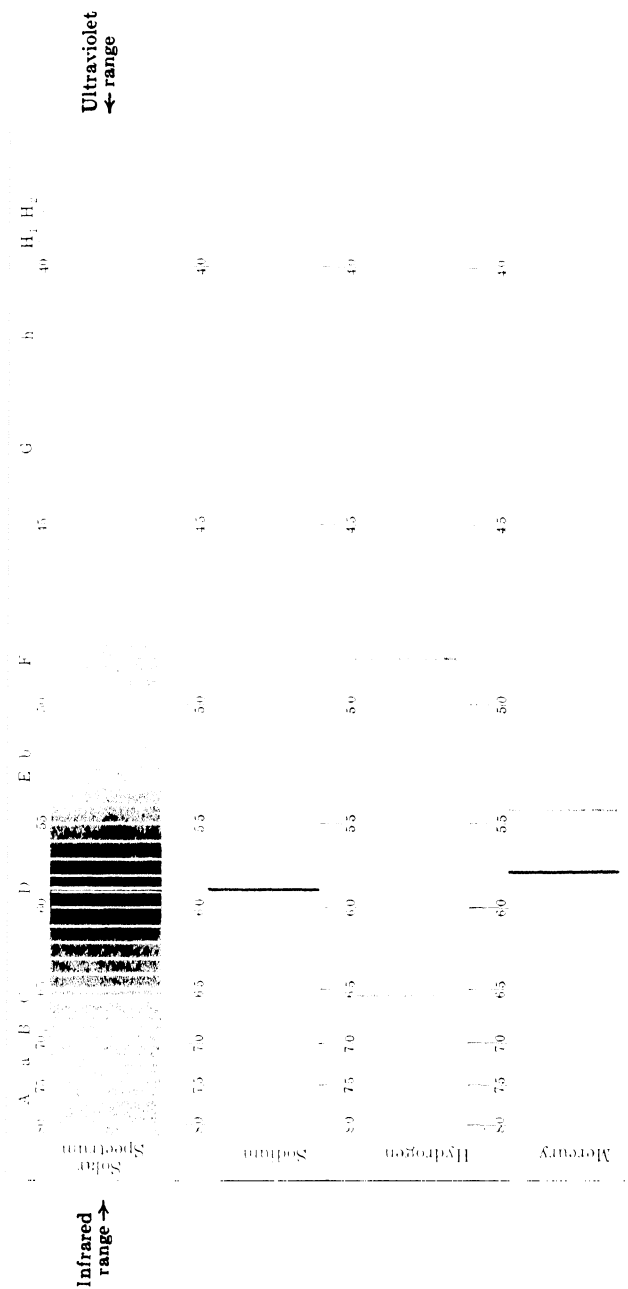
| | | | | |
|-----------------------|---|---------------------------------|---|---------------------------|
| Warmth | → | Light | → | Fluorescence |
| Technician | → | Engineer | → | Scientist |
| Comfort | → | Vision | → | Creation |
| Worker | → | Supervisor | → | Researcher |
| Freedom of mind | | Mental responsibility | | Incessant mental activity |
| Respect of associates | | Recognition for accomplishments | | Possible public applause |

Of course, each classification flows into the adjacent ones. Still, if we pull together all of the corresponding word descriptions, we find for the technician a comfortable worker's job, with freedom from heavy responsibility or mental worry and with the certain respect of his associates. The engineer has a job requiring the light of clear vision, usually carrying the inevitable responsibility of the supervisor either for success or failure and offering the reward of probable recognition for accomplishments. The scientist's job is the brilliant one of creating new things through research which requires a particular kind of mental activity and imagination and offers the uncertain reward of possible public recognition reserved for statesmen, military heroes, and occasionally scientists.

Dep.
cultu

vention

At all levels of technology there is opportunity for invention. The simplest gadget may still be improved. Look for improvements and you may surprise yourself by becoming an inventor.



Adapted from Erdmann's, "Lehrbuch der Anorganischen Chemie"

FIG. 1. SPECTRUM CHART

The solar spectrum with Fraunhofer lines; the bright-line spectrum of sodium; the spectra of hydrogen gas and mercury vapor. The numbers refer to wavelengths expressed in hundredths of a micron. For comparison, 1 100 micron = 0.00001 mm. The dark lines on the solar spectrum are absorption lines representing wavelengths of light that were absorbed by the gases in the atmosphere of the sun. (Reprinted from Black's *Introductory Course in College Physics*, The Macmillan Company, 1941.)

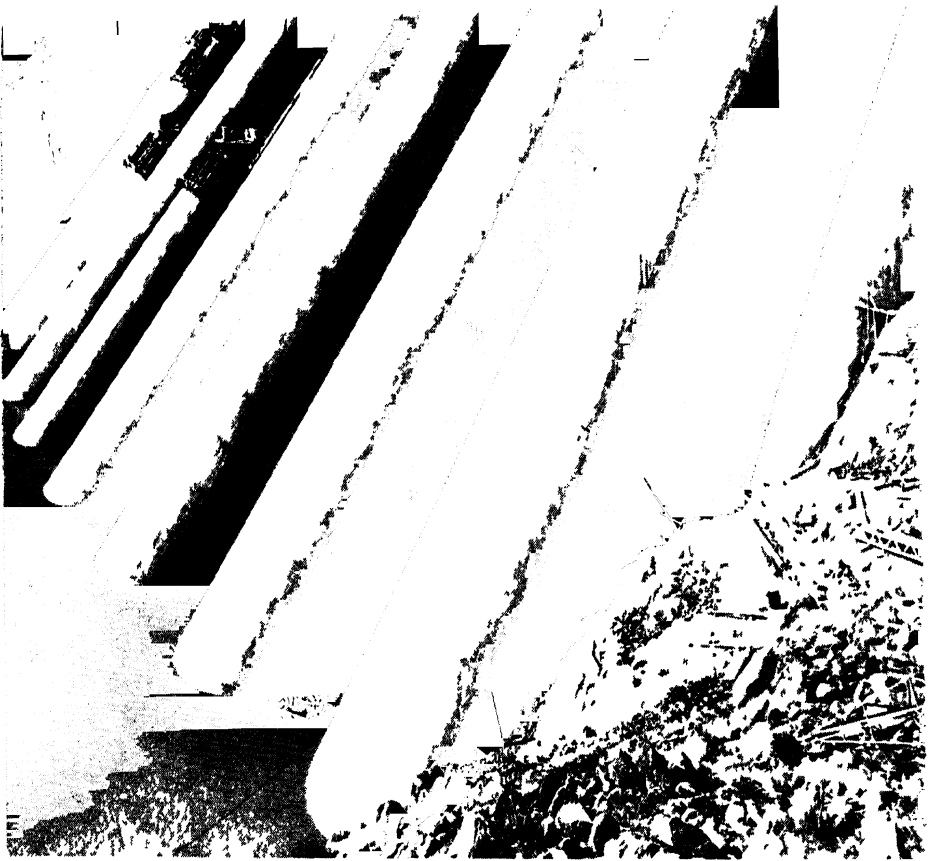


FIG. 2 GREAT ARCH DAM PROVIDE IRRIGATION WATER AND HYDRO
ELECTRIC POWER IN THE WEST

THE STORY OF ENGINEERING

I imagine that the first engineer, to our way of thinking, was a savage — a man without tools and almost without clothes. Fire was the symbol of the beginning of civilization and he used it not only for warmth and light and partially to cook his food but also as a tool. He must have crossed streams on fallen logs and then one day he slowly began to realize how fine it would be to have a log bridge across the stream just outside his cave. A tree grew there, leaning across the stream, and, moreover, it was burned partly through by his own carelessness. Why not renew this fire and keep it tended until the tree fell across the stream?

In his rude mind this simple plan must have been slow to evolve for it has in it all the elements of engineering. We say that engineering is the art and science of bringing the forces of nature to serve the use and convenience of man. Our primitive engineer felt the need of a bridge and his plan for producing one made use of two of the forces of nature — *fire* to burn the tree trunk through and *gravity* to fell it across the stream. When pre-civilized man first walked across a stream on a log bridge of his own creation, engineering had started its course toward the ultimate production of the steel mills, the airplane, the radio, and the mechanized world yet to come. *Primitive technology*

From the log bridge it was but a step to produce the dugout canoe. Fire burned away the inside of the dugout so that only the crudest scraping tools were needed to speed up the work. The dugout provided transportation for a whole family and its few possessions. With it, man began his attack upon distance, and, with the invention of outriggers, he even used it to cross from island to island over the sea. From this stage the development of sailing ships and eventually great steamships followed in natural course. *The dugout*

Perhaps the greatest single engineering achievement of all time was the invention of the wheel and axle. We can picture this invention as mainly accidental. A flat round stone just happened to have a hole in the center; a small boy stuck a stick through the hole and with his hands on either side of the stone leaned over like an animated wheelbarrow and annoyed everybody by running the contraption over their toes! But one man of the tribe could not drop the matter from his mind by cuffing the boy, and at long last the idea of the one-wheeled vehicle had evolved. In due time *The wheel*

the improvement of adding another wheel to produce the cart and then two more to balance the load was inevitable. Man had at last started his long journeys over the land as well as the water. As engineering science progressed, railroads and automobiles came as logical developments using the all-important wheel.

*Irrigation
systems*

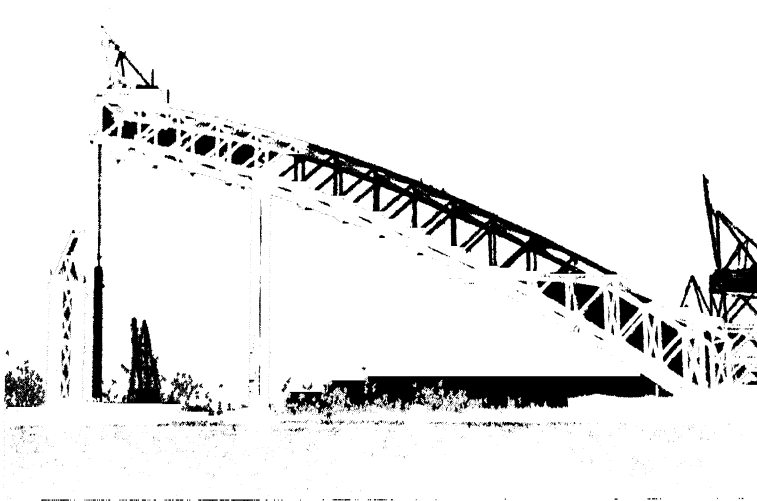
One of the most civilizing influences, and also a great engineering achievement, was the irrigation development of the ancient world. The civilizations of Babylon and Egypt in particular were built upon the agriculture of irrigated lands. As compared to the great concrete arched dams of today (Fig. 2) with their hydroelectric power developments, the irrigation systems of the ancient world would seem simple indeed, but they brought fertility to the land, nevertheless. Today around the site of Babylon there is nothing much but wasteland due to the shifting of rivers and the destruction of dams, canals, aqueducts, and other devices of the sixth century B.C. In Egypt we know that it was necessary, except in floodtime, to raise irrigation water from the level of the Nile to the level of the farm land and that several types of hand lifts were used. By this laborious process, Egypt attained the most powerful civilization of the pre-Alexandrian world. Engineering and agriculture still go hand in hand and still remain great civilizing influences upon mankind.

*The column
and lintel*

Architectural construction followed the growth of organized government and religion. Monumental buildings were not needed until government or religion developed to the point where the protection of the fort or the prestige of the temple became important. For several thousand years through the civilizations of Egypt and Greece, architectural construction was limited to the structural possibilities of the column and lintel. Extraordinary structures in stone were built under these restrictions.

*Temple of
Amen*

An interesting example is the Temple of Amen at Karnak in ancient Egypt, completed about 980 B.C., which was under construction for seven centuries. This building was 1200 ft. long and 350 ft. across its greatest width. It was larger than the great airship dock at Akron, Ohio, which is 1175 ft. long and 325 ft. wide. In sheer mass of stone, the Temple of Karnak exceeds anything in the nature of shelter ever constructed by man. Built without modern tools, the great central stone columns are 66 ft. high and 10 ft., 7 in. in diameter. The lintel slabs of stone spanning from column to column are 36 ft. long and 4 ft. thick. It is believed that the



Courtesy American Bridge Company

FIG. 3 ERECTION OF AN ARCHED TRUSS BRIDGE

This structure combines the action of the arch and the truss. When finished it will be a gracefully curved arch made up of steel bars put together to form triangles which is the principle of the truss. One temporary vertical bent is supporting the arch during construction and another is being erected at the left. Both will be removed when the arch is completed.

building was filled with sand as it was constructed so that these monumental stones could be dragged up a sloping ramp and into place without cranes. The building then was emptied of sand as the final step of construction.

It remained for Roman engineers to make the first major use of the arch. They built long aqueducts on multiple stone arches and also hundred-foot bridge spans of arched construction. Many still exist. For roofing one magnificent building, the Pantheon at Rome, the world's greatest masonry dome was completed in 120 A.D. The diameter of this impressive domed structure is 144 ft. which is also the height of the wall from which the dome springs. Only in the church of Hagia Sophia in Constantinople, 563 A.D., was this work exceeded. This building has a nave 243 ft. long and 107 ft. wide covered by a central dome and two half-domes at the ends, in part supported upon stone arches of about 100-ft. span. Artistically, its interior is one of the supreme masterpieces of all time.

We should not leave the subject of architectural construction without mentioning the late development of the truss. It was not

The arch and dome

The truss

until 1758 that Ulric Grubenmann, a Swiss carpenter, constructed a great wooden truss bridge over the Rhine. Two hundred years earlier, Andrea Palladio (1518–1580), an Italian architect, invented the truss and may have used it in building construction, but his idea was lost with his death. It is strange that the truss, constructed with straight bars joined to form a system of adjacent triangles, which is not a very complex construction, should have developed several thousand years after the lintel and the arch. Soon after its first use, however, it became well understood and evolved rapidly. First built as a wood structure, the truss soon was made with cast-iron and wrought-iron members and then after the Civil War its construction in steel revolutionized bridge building.

THE MACHINE AGE

Iron and steel

Before machines could really change man's daily life, there had to be a cheap strong material for their construction. Simple machines such as the potter's wheel, the spinning wheel, and the hand loom had been built of wood since the dawn of history, but iron was needed for steam engines, power tools, or even farm machines. Iron from meteorites had certainly been known for many thousands of years, but its production from ores was not common until a thousand years before the time of Christ. Bronze was the common metal of early historical times because it could be produced (from copper and tin) at a temperature below 2000° F. compared with 2800° F. at which iron must be cast. Also, it really takes machines to make iron because its great weight presents a handling problem only solved by the use of power. Hence, the machine waited upon the cheap production of iron in large quantities and iron production required the machine.

Bessemer steel

The development of iron foundries had progressed in England to such an extent that a cast-iron arch bridge of 100-ft. span was constructed at Coalbrookdale in 1776. From then until Bessemer developed his converter to make iron into steel (1856), we find a steady increase in the production of cast iron and wrought iron accompanied by an extraordinary development of the machine and the industrialization of England, America, and other parts of the world. The greater strength and reduced cost of steel after Bessemer's invention simply speeded up the process of industrialization, which by that time had become clearly defined.

Age of power

James Watt's development of the steam engine (1765) and

Michael Faraday's explanation of electromagnetic induction, the basic concept of the electric generator (1831), have a special significance because they symbolize the introduction of two forms of power that were to revolutionize civilization. Industrialization through the nineteenth century was accomplished by steam power; the evident symbols were the factory steam engine with its belts, shafts, and pulleys, the steam locomotive, and the steamship.

Toward the end of that period, steam was beginning to be generated for the purpose of producing electric power, and the transformation of industry to the age of power began. By 1943, our over-all power load of the electric utilities of the country had reached 36,900,000 kilowatts or nearly 50 million horsepower. The rapid production of airplanes indicates that the total horsepower of airplane motors will soon exceed this figure while the total potential power of American automobiles is in excess of a billion horsepower. Given assurance of peace, industry could obviously remove the main load of heavy labor from the shoulders of man, at the same time making possible new and higher standards of living for all. *Electricity*

Steam and electric power released man from the need for slave labor, and then it almost made him its slave by pinning him down to monotonous watching, feeding, and tending of machines. As machines have become more and more nearly automatic (but not entirely so), our task of watching machines has become more and more dull. There is a cultural influence in fine craftsmanship, but machine tending is deadening rather than stimulating and at best monotonous. *Electronics*

Now, clearly before us, we see the coming emancipation of man from his self-imposed servitude. Through technology has come a new development — electronic controls for machines. The vacuum tube of radio has been put to new uses such as the electric eye, the electronic ear, the electromagnetic arm. So far, man's senses of taste and smell have defied electronic duplication, but the functions of sight, sound, and touch can be performed more exactly by electronic devices than by human beings, although as yet this field of technology has been only touched by the scientist and the engineer.

In this new age of electronics that we visualize for the last half of the twentieth century, machines will become entirely automatic and vastly more complex than anything we now know. A recent *Automatic machines*

Robot tools

development is a machine nearly 200 ft. long that takes airplane cylinder-head castings into its maw at one end, moves them along automatically past cutting and grinding and polishing tools, to deliver the finished parts at the rate of more than one per minute. The machine automatically shuts itself down if anything goes wrong. Instead of many workmen tending the individual machine tools, one man only is needed and, in fact, he may eventually be freed by a few additional gadgets. There seems little doubt but that such machines will soon be provided with automatic inspection devices which will discard pieces with flaws that would not be seen by the human eye. The shadow on the X-ray screen, the whine of the cutting tool, the reflection of light from a polished surface, the ring of a part when it drops on a table may each be used with electronic devices to discard imperfect parts.

*Technologic
leisure*

Some persons wonder what we can do or will do with the leisure that the development of more automatic machines will provide. Some envision calamitous unemployment. One answer is that no unemployment need result if man's standard of living is permitted to rise — but this is more a problem of economics and the distribution of goods than it is a problem of technology. If more leisure can be provided, there is much use to be made of it. We begin to see the need for a large group of persons in all countries to be educated for world citizenship. No such education would be complete without world travel and the ability to communicate with all peoples through a common language. If technologic leisure should be used to train such world citizens who would then return to become productive citizens of their own countries, it might be that technology would have developed a force able to cope with the problem of "peace in our time."

THE PROFESSION OF ENGINEERING

*The
professional
man*

We will not take time here to go into the fine distinctions between the professions and those callings that are not professions. According to common understanding, engineering is a profession and the engineer is a professional man. So is the scientist. The first recognized professions were the ministry, the law, and medicine. These are sometimes called the *closed professions* because the one way of entering them is by education in a recognized institution followed ordinarily by an examination for a state license to practice in that profession.

Engineering is still an open profession since it is possible for any-
one to prepare himself to serve as an engineer. Of course, there are
state registration law for engineers, but the examinations are open
to anyone who has demonstrated reasonable competence in his
work as an engineer, registration being required only for those
who approve plans. State registration is desirable because engineer-
ing works affect the health and safety of the public which deserves
the protection given by the registration law. An incompetent en-
gineer might build an unsafe building or design a water-supply
system that would permit pollution of our drinking water. Since
we cannot take chances with the public safety, undoubtedly all
states will eventually pass registration laws for engineers. *Registration*

Each branch of engineering is represented by a professional
society to which leading engineers in that field belong. The first of
these (1856) was the American Society of Civil Engineers. Other
major societies are the American Society of Mechanical Engineers,
the American Institute of Electrical Engineers, the American Insti-
tute of Mining and Metallurgical Engineers, and the American
Institute of Chemical Engineers. Engineers consider membership
in one of these societies to be a mark of professional distinction
because only those who have attained a certain level of professional
experience and ability are accepted for membership. *Professional societies*

Each society has more than one grade of membership and, as
the young engineer gains in professional experience, he is advanced
from *junior* member to *associate* member and finally to *full* mem-
ber. For this reason the grade of member in one of these societies
is a badge of professional distinction. The engineer often belongs
to other technical societies devoted to the study of problems in
which he has a special interest, but he should always maintain his
membership in one of the professional engineering societies men-
tioned. Student engineers have a chance to join these societies
before graduation through student chapters that exist at most of
the major schools of engineering. If you are a student engineer,
do not miss this opportunity; you are eligible as soon as you become
a classified student in engineering. *Society membership*

When you become a member of a profession, you thereby accept
certain responsibilities and obligations. Perhaps the greatest respon-
sibility is that you must undertake to look after the interests of the
public; and this assumes, of course, that you will be honest in the
strictest definition of the term. It may help if we cite a few examples
Obligations of the professional man

of professional conduct. There are some callings where the trading of an opinion or a vote is openly condoned. An engineer will not long remain a member of one of the professional engineering societies if he permits his personal gain or loss to influence his professional opinion on technical problems.

*Professional
conduct*

If a bridge is weak, the engineer's report must say so. If he feels that a proposal to build an industrial plant in a certain location is contrary to the public interest, it is his obligation to render an adverse opinion. And, if his study of a proposed manufacturing project convinces him that the product could not be produced and sold profitably, he has no choice but to advise against the project even though he loses thereby a desirable job of designing and building the plant. This, in brief, is what we mean by the professional honesty of the engineer; this is simply the obligation of the professional man.

*Code of ethics
for engineers*

Long codes of ethics or of proper professional conduct have been adopted by the professional engineering societies for the guidance of their members, but these are too complex for study here. Instead, we shall consider only a few points of ethics as outlined by a committee of the Engineers' Council for Professional Development, a joint council founded by the professional engineering societies:

The engineer will avoid conduct and practices likely to discredit the honor and dignity of the engineering profession.

He will interest himself in the public welfare and be ready to apply his special knowledge, skill, and training for the benefit of mankind.

He will not express publicly an opinion on an engineering subject without being informed as to the facts relating thereto.

He will not lend his name to any questionable enterprise or engage in any occupation contrary to law.

He will carry on his work in a spirit of fairness and loyalty to associates, subordinates, and employees, fidelity to public needs, and devotion to high ideals of courtesy and personal honor.

WHO SHOULD BE AN ENGINEER

*Technical
competency*

There is an old saying that you can't make a silk purse from a sow's ear. It is quite as impracticable to make an engineer of a young man who should be a musician or a poet or perhaps a lawyer. Not that you may not become both an engineer and a musician or lawyer. But if you are to become an engineer, you must have most of the well-known qualities of an engineer, and then, if you also have other useful qualities, so much the better.

It is pretty well understood that a good engineer must be reasonably good in mathematics. Now that does not mean that every engineer is a "math whiz" by any means, but it does mean that, on the average, engineering students make about the same grades in their engineering subjects that they make in mathematics. Good in math, good in engineering; poor in math, poor in engineering is the common rule. I cannot say, however, that most engineers like to study mathematics. They are too impatient, too anxious "to study something practical," but they do stick to their mathematics courses and learn far more about the subject than most liberal arts students. They do so because they realize that mathematics is of daily use in engineering.

*Mathematical
ability*

Physics is the first of the basic sciences of engineering, which is largely applied physics. The usual subdivisions of physics — mechanics, heat, light, and electricity — are the engineer's theoretical tools. Along with chemistry and some bacteriology, they make up the background sciences of engineering. It is not likely that a good prospective engineer could really dislike physics, and certainly a chemical engineering student should enjoy chemistry. A knowledge of bacteriology is essential to the sanitary engineer whose job it is to guard the public health.

*Physical
sciences*

There is plenty of chance for self-expression within the profession of engineering. Out in the field, there is the construction man, and in the factory the production man. Construction and production engineers "kick the job along" and see to it that things happen fast. They need less theory and mathematics than the designer who works in the office or the drafting room. The designer's job is making calculations and supervising the production of drawings which transmit design information in the "language of engineering." The designer must be careful, accurate, and sure of himself. The research engineer must be all of these things and be curious and imaginative besides. He must continually find new ways of doing things, new materials, new devices, new ideas. He is responsible for new industries such as plastics, television, air conditioning, aircraft. Research raises our standard of living in peacetime and may be the savior of civilization in time of war.

*Engineering
practice*

"PARLOR TEST" FOR TECHNOLOGICAL TRAITS

It would be wonderful if psychologists and professional testers could pry into our minds, our personalities, and our characters to

*Your bent
in life*

determine for each beyond question the direction of his true bent in life. You should understand before you read this discussion that such claims are made only by astrologers and mystics. Some day psychology should be developed to the point where it will produce tests of great reliability, but the human mind-personality is so unbelievably complex that improvements in placement testing naturally develop slowly. In the meantime, you can make use of tests that have been devised, recognizing, of course, that a test is considered quite successful if it predicts correctly the performance of three persons out of four in a random group.

*Aptitudes and
interests*

Tests have been prepared either to measure your knowledge, your interest, or your aptitude for a subject. It seems likely that there will always be some overlapping here because it is difficult to see how you can show an intelligent interest in any field before you know something about it. However, if a test is devised merely to measure vocational interests, it would seem reasonable to assume that a person who has the same general pattern of interests shown by most successful engineers could be a successful engineer himself. At least, we would feel that his chances of success were improved by his fortunate pattern of interests.

Then again, if you take a series of aptitude tests and score high in the technical aptitudes, you should feel encouraged to study engineering. One such test is the "wiggly block" produced by cutting a wooden cube into many pieces by curved saw cuts. Engineers can usually put the parts back together in short order while successful bankers are much slower. You would do well to take such tests before you spend several years studying engineering or science; any college or university can advise you.

*Simple self-
scoring test*

A thoroughgoing test requires considerable time to take and to score. Vocational-interest tests use several hundred questions and aptitude tests may require several hours of your time. The purpose here in providing a test of twenty-five questions that you can answer and score by yourself in a few minutes is merely to interest you in your own abilities and to provide you with a very rough measure of your interests and aptitudes, especially those that would naturally lead you into the technical field. In fact, it is hoped that your experience in taking this simple test may develop your interest in taking a complete vocational test because you should not take the results of this "parlor test" too seriously. When taken by a group the test may be used as an entertaining game.

All tests apply most effectively to groups while any individual may be an exception. You may have a tenacity or keenness of mind that will make it possible for you to succeed in engineering even though you lack some of the usual interests and special traits of the engineer. You may also fail even though you have the desired technical qualifications if the "will to succeed" is lacking in your nature.

BACKGROUND FOR TECHNOLOGY

Answer each question "yes" or "no." Give the first answer that comes to your mind. These are not questions in the main that require careful analysis. You can answer several questions each minute. Answer all questions before you read the instructions on scoring that follow the questions. Be honest with yourself for otherwise your test score will be misleading.

Group 1

- (a) Do you like mathematics more than three of these subjects: English, history, art, music, civics?
- (b) Were you ordinarily able to work the problems in your mathematics courses without great effort?
- (c) Can you attack a problem in mathematical symbols without confusion or displeasure? For example: Given that $\theta = \zeta$ while $R = \phi$ and $\zeta = R$: Does $\phi = \theta$?
- (d) In high school a mathematics course consumes about one-fourth of one's time. Would you like a lifetime job that consumed a quarter of your time solving mathematical problems?
- (e) Does the difficulty that everyone finds in getting the correct answers to mathematical problems challenge you rather than plague you?

Group 2

- (a) One circular gear turns between two others, the three being in line. If the center one rotates clockwise, both outside gears rotate counterclockwise. Do you find it confusing to visualize this motion?
- (b) A wheel rolling from left to right over the floor turns clockwise, but, if rolled along the ceiling, it turns counterclockwise. Are you uncertain whether this statement is correct?
- (c) An object hanging near a corner of a room would cast a triangular shadow on either adjacent wall or a circular shadow on the floor or ceiling. Do you have difficulty visualizing the object as a cone with axis vertical?
- (d) Read the following sentence over until you know whether the man returns home. *A man walks north 3 blocks, east 1 block, south 1 block, west 2 blocks, south 2 blocks, and east 1 block.* Did you read the directions more than twice before you were sure that he returns home?

- (e) Place a cube on a table. Make two horizontal saw cuts entirely through the block, keeping the pieces together. Then place the saw on the diagonal of the top face and cut vertically downward. Turn the saw 90 degrees and repeat on the other diagonal. Do you fail to see immediately that you have produced 12 similar wedges?

Group 3

- (a) Does it bore you to read about molecules, viruses, atoms, protons, and electrons?
- (b) Is chemical jargon such as $2\text{H}_2 + \text{O}_2 = 2\text{H}_2\text{O}$ confusing to you?
- (c) Could you define the meaning of either *citizenship* or *truth* more readily than a *degree* of longitude or latitude?
- (d) Does it seem meaningless to you to associate color with vibration and wave length?
- (e) Do you pass up newspaper or magazine articles on plastics, electronics, synthetic materials, or other popular but technical subjects?

Group 4

- (a) Did you ever (of your own volition) construct a machine or doorbell or other apparatus that worked?
- (b) Would you like to take a clock or an airplane engine apart just to see and feel its parts?
- (c) Do you enjoy working with tools, shaping material, or fixing household gadgets?
- (d) Would you rather work in a shop or laboratory than listen to good lectures and read interesting books?
- (e) Have you on your own initiative visited a machine shop, a welding job, or carefully inspected any mechanical process because of your special interest?

Group 5

- (a) Do you speak easily and also understand the meaning of most words such as *squalid*, *condone*, and *future*?
- (b) Would you prefer being tied down with more work and greater responsibility if it did not involve larger salary?
- (c) Do you accept yourself and others without worrying about speech, manners, or character?
- (d) Have you been planning your own career instead of waiting for it to develop?
- (e) Have you been an officer of your team, class, or club, editor or manager of your school paper, or been elected or appointed to any official position?

Scoring the
test

Score each group of questions separately beginning with Group 1. If three or more of the five questions in any group are answered "yes," your score for that group is 1; otherwise, the score is 0. After scoring the five groups, make up your code number by placing the group scores in sequence from the first to the fifth. Your code number would be 10100 if the answer "yes" had predomi-

nated in the first and third groups while the answer "no" had predominated in the second, fourth, and fifth groups. There are thirty-two possible code numbers.

Engineers are expected to show strong aptitude for mathematics. *Mathematical aptitude*
If you are in this group, your code number should begin with 1. If it begins with 0, you should take a full mathematical aptitude test as a check.

A trait that seems to appear with great regularity in engineers is the ability and desire to visualize or picture things and motions in space. *Technical visualization*
If you have this ability, your code number will have a 0 in the second place from the left. This capacity will not substitute for mathematical aptitude but does supplement it.

It is difficult to see how an engineer can fail to have a greater *Science interest*
interest in science than the common person. In fact, it has been found that a high score on science tests usually goes along with mathematical aptitude. We would expect the code number of a potential engineer or scientist to have 0 in the third place or the third digit from the left.

Many engineers — particularly mechanical engineers — have the interest and ability to produce things with their hands. This is the special field of the mechanic, but it is shared by laboratory research workers, glass blowers, usually by shop foremen, and often by inventors. *Mechanical bent*
If you are mechanically inclined, your code number should have a 1 in the next to the last place. If you find a 0 there, you need not worry, since this trait is not common to all engineers.

More than one-half of all engineers are in supervisory work *Supervisory quality*
after fifteen or twenty years of experience. Hence, this quality itself points to one type of engineering ability. If you have several qualities of the successful supervisor, your code number should end in 1. If you lack such abilities, you can still be successful as a designing or development engineer and you may later develop the characteristics of the supervisor. These qualities may change gradually through life.

USING TEST RESULTS

This "parlor test" is not complete enough to make each individual score significant. However, if your code number begins with 100 or even 101, you have shown the main characteristics of the engineer, 10011 being the most rounded engineering type. *High and low scores*
The

person with all negative engineering characteristics would be expected to have a code number 01100. However, any code number beginning with 011 or 010 is a negative score in regard to the most essential engineering traits.

Scoring stars

As an added indication, you may give yourself one star for each group of questions in which you wrote down four "no" or four "yes" answers and two stars for each group for which all five answers were alike. Then, the number of stars is a measure of the definiteness of your interests. If you score at least five stars, you may feel somewhat more confident that the test is indicative of your technological traits.

*Why people
are successful*

Even if professional testing can be developed to the point where it accurately measures our interests and aptitudes, it is doubtful that we shall be able to predict by such tests who will and who will not be successful. While writing that your code number should begin with 100 if you have the proper characteristics of an engineer, the picture of one of my former students came to my mind. He certainly would have scored low in mathematics and science aptitudes; by all the rules he should have "flunked out" of engineering school in his freshman year. When he entered my class as a third-year student, he said, "I know I'm dumb in theoretical subjects: I've had to take several theoretical courses twice, but I'm going to be an engineer. If you don't drop me out of your class in the first few weeks, you will find that before the end I will have all the problems worked correctly even if it takes me half the night every night." He passed the course and those that followed it and graduated as a civil engineer. By directing his energies into the practical field of construction, he has become a successful practicing engineer.

*Determination
wins*

If you are not unchangeably set on being an engineer or scientist and if you score low on the test given here and also on more thorough engineering aptitude tests, you would be well advised to try another field. But if you cannot get technology out of your mind and if you are a determined sort of person, for you the tests may be unimportant. All teachers have known students of only average ability who were able to graduate from an engineering college in four years and hold down another job at the same time. City college deans often find that the "honor man" is supporting not only himself but his mother or his own family by night work. Determination wins in every phase of life.

Another point worth making is that the test you have taken

here is probably more indicative of the ease or difficulty you would have in graduating as an engineer than it is of your ultimate success. If you have only average aptitude for mathematics but speak and write easily, you might have difficulty graduating as an engineer and then find yourself successful as a patent lawyer, technical magazine editor, personnel manager, sales engineer, or a promoter of engineering projects. The practical person who is weak on theory may become successful as a production or maintenance engineer and later advance to chief operations engineer or superintendent. The field of engineering is so broad that it provides outlets for men of varied talents. Nonetheless, it is a good idea to test yourself because your greatest success should come when you get into a job that fits your aptitudes as well as your interests.

Promoters and producers

HOW TO READ BOOKS ON TECHNOLOGY

The thrill of the mystery novel is likely to be lacking in scientific or engineering books. Luckily, there is another pleasure to take its place — the sense of power over physical reactions grasped through the understanding of scientific thought. It does not come to you without effort, or all at once, but the satisfaction rising from this ability is enduring.

Since technical reading is not exactly exciting, you ought to prepare for it. There is little profit in picking up a book on electric circuits or airplane design in a room where others are engaged in discussing the cost of living or where a radio is blaring. If your job is to concentrate on electronic circuits, give yourself a chance to succeed. Two may be “company” under certain circumstances, but you study alone or usually not at all — at least not efficiently — and the sense of frustration after an unsuccessful attempt to concentrate may keep you from trying again.

Avoid distractions

Adjust your surroundings to your own temperament. It is best not to get too comfortable. Most people who slump into a deeply cushioned easy chair want to overrelax and go to sleep. Some persons can study in bed — the habit is often developed during an extended illness — but it is a poor method for most people. A straight chair that is reasonably comfortable, a desk where you need not slump over while making notes, and a steady light without glare are the minimum physical aids to successful study.

Now, with surrounding distractions reduced as far as reasonable, you must learn to concentrate on the book before you. Persons

Ideas vs. calories

who flit back and forth between the kitchen and the study room pick up more calories than ideas. We observe that the great scientists stick very closely to a single field of study through an entire lifetime. You must at least give science an undivided hour of mental concentration if you expect to be rewarded with the clear understanding of a new scientific idea.

Scanning

Then there is the question of whether scanning has any usefulness in technical study. Of course, before you quit, you must read not only for ideas but for details as well. To read rapidly over a half-dozen pages for the main ideas and then to cover the same material at a slower rate, reading for details, is a good plan if it is followed consciously. The danger is that you may get into the habit of scanning and fail to develop the patience necessary for digging through a tough chapter paragraph by paragraph, sentence by sentence, or word by word if necessary. You must accept the fact that a single page of a textbook may require hours of study for a complete understanding. Unfortunately, the author may not have written that page clearly or you may have forgotten some basic principle that he thought you would know.

Technological language

Keep in mind that most books on technology are written logically. This means that ideas and concepts are developed step by step. If you miss the derivation of one formula or fail to grasp one logical development, you probably will not understand the next one. It is impracticable to pick out only interesting topics and skip dull ones. Only children pass up the meat course of a fine dinner and hope to get two desserts.

Probably by now I have made the whole subject seem too formal, too rigorous, and too prosaic for you to look forward with enthusiasm to technical reading. Actually, there is a bit of inspiration in every paragraph and a thrill on every page. Mathematical symbols, chemical formulas, tables, drafts, and charts are simply the language of technology. They are like the words of a novel, individually unimportant but, taken together, they weave the plot. You must recognize the words to read the book. Naturally, you must understand this language of science and engineering if you are really to learn how rubber can be made from alcohol, and plastics from wood fiber, why diesel locomotives are replacing steam engines but television did not immediately start to replace radio, how great bridges sometimes collapse, or why the helicopter after a half-century of neglect now becomes a practical

means of flight. The thrill is there if you prepare yourself to experience it.

TECHNIQUES — GOOD AND BAD

Depending upon whether you get started right or wrong, you may become either a real engineer or a "handbook engineer." Now a handbook is a wonderful asset if it is used rightly, but it becomes a snare when it is given too important a place in your library. There are handbooks for every basic field of engineering and then special ones for welding, radio, plastics, and a dozen other limited branches of technology. They give all of the formulas, equations, charts, and tables generally available in the field; in fact, they seem to solve most of the standard problems for you. *Handbook engineers*

Of course, you cannot expect to remember every formula and every accepted rule of engineering. There are just too many. Neither can you take time to derive each equation every time you want to use it. So you keep a handbook handy to pick out the formula, rule, equation, or curve when and as you need it. Now here is the catch question: If you are going to use the handbook anyway, why bother to derive the formulas or plot the curves in the first place? Can you just become familiar with the authoritative sources of material and then assemble and use these data when the need arises? The answer is *No!* That is handbook engineering and it has been proved to be too dangerous to encourage. That is why most states require engineers to prove their competency to design engineering works without the use of handbooks before they can become registered.

Engineering is not a series of convenient pigeonholes from which you pluck facts and figures that apply directly to the problem before you. The same problem seldom occurs twice. We have been busy for years building plants to produce high-octane gasoline; but no two of these plants are identical. If they had all been cut to a single pattern, there could have been no progress, no improvement, no use of successive inventions. Besides this, the plants must handle different kinds of crude oil, with the result that the by-product oils will be different for plants producing the same high-octane gasoline. Handbook engineering fails when confronted with such complex situations. Judgment based upon a thorough knowledge of theory and familiarity with good practice is the only background that is adequate for the designing engineer. Besides, we *High-octane technology*

must not forget that the public safety is involved in many engineering projects.

*Scientific
skepticism*

When you first start to study technology, you are likely to feel that anything in print is bound to be right and that every author is an authority. A halo often seems to fall upon the brow of the writer; we are likely to assume that he never makes mistakes. But the fact is that there are very few books without errors, and an error or confused statement every few pages is common in technical books. Most of these errors are not very important, but some of them are completely misleading.

*Textbook
errors*

One textbook, in fact a good one, was found to contain the following mistakes: a misspelled word that changed the meaning; a decimal point misplaced; a table with two column headings reversed; a misprinted formula; a symbol used twice with different meanings, an unexplained numeral that could not be checked; a reference to the wrong specification; an error in a problem answer; a use of the wrong formula; a factor omitted; an error in a table; a square root sign omitted; a slide-rule error; an error carried over from a previous publication; the unit inch-pound given incorrectly as pound; an illustration with vital dimensions omitted; a plus sign changed to minus; the use of an obsolete specification; a garbled table; a wrong heading to a graph; an error in a reference; a few minor typographical misprints; several statements that might be misinterpreted.

*Hypotheses
and proofs*

Undoubtedly, in reading the book the critic did not find all the errors. If you should read the same book, you might find others. The point is that you cannot afford to accept anything you read until you have checked it. Naturally, you must pass over some statements with the mental reservation that you have not time to check them during a first reading and that they will be brought back to mind for restudy later. And then, there are certain hypotheses that are not subject to proof. For example, we usually assume that a liquid is noncompressible. Actually, at room temperature and a pressure c' about one hundred thousand pounds per square inch, water will be reduced one-third in volume after having passed through a special form of ice. But for low pressures, liquids remain almost constant in volume, and so this assumption is a convenient one for simplifying our calculations whenever it is applicable.

Hypotheses are not subject to logical or mathematical proof but

are justified by observation and experiment as the starting points for theoretical derivations. Keep in mind, nevertheless, that mathematical transformations applied to a starting hypothesis produce formulas that are no more applicable, exact, or rigorous than the original assumption or hypothesis. *Mathematics adds no authority to the final results* — it merely transforms our stated hypothesis into equations, formulas, or graphs for convenient use.

USING MATHEMATICS

Here is a mathematics problem that teaches an important lesson without being difficult to understand. Two streamliner trains are 100 miles apart and are approaching each other at 100 miles an hour. A pursuit plane traveling 400 miles an hour flies from one to the other and back again and again until the trains meet.

Morals and mathematics

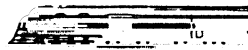
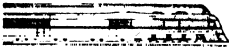


FIG. 4. PLANE FLIES FROM ONE TRAIN TO THE OTHER CONTINUOUSLY

The question to be answered is: What is the length of path that the airplane travels? We will try the complicated solution first. As the trains get closer together, the plane makes the round trip more and more quickly and it passes over the meeting point an unlimited or so-called “infinite” number of times. Its total distance of travel, then, is the sum of an infinite number of round trips that become continuously shorter until they approach zero as a limit. This could be a rather difficult problem in infinite series, but it happens that there is another and much simpler solution. The trains approaching each other and each moving at 100 miles per hour will cover the 100-mile distance between them in 30 minutes. Evidently in 30 minutes the airplane flying at 400 miles per hour would cover 200 miles, which is the answer. If we had been able to sum the infinite number of round trips made by the airplane, the answer would also have been 200 miles.

The moral of this story is that it is a good thing to look around for a simple solution to any problem. You will remember how

Simple solutions

Checking

much easier some arithmetic problems became when you learned to call the unknown quantity x and to solve for it by use of an algebraic equation. All the way through your study of mathematics you should keep looking for the simple way to approach a problem. As a matter of special interest, you will want to know that the engineer has several devices that help him find a simple mathematical solution. First, he usually can nearly guess the answer so he can often work backwards to help simplify the mathematics, and his knowledge of the answer is always one way of checking the result. Then, too, his understanding of the engineering problem often leads him to expect to need a certain kind of mathematics for the solution. For example, the load coming on the central pier of a two-span continuous bridge can be calculated from a simple algebraic equation. For three spans, the two central pier loads must be computed from the solution of two simultaneous algebraic equations.

PHYSICS AS ENGINEERING BACKGROUND

Phenomenon of physics

Closely related to mathematics, the second great tool of the engineer is his knowledge of physics. Mathematics is a theoretical subject; physics introduces theory to laboratory experiment. The physicist tries to explain the things he sees in nature, using mathematics and his ability to experiment in his laboratory. Everyone knows the story of Newton's interest in falling objects which led after years of study to his theory of gravitation. Einstein found that Newton's theory did not explain some movements of the planets, which led him to devise his theory of relativity. Galileo experimented with pendulums and then wrote mathematical equations for their motion. In each of these cases and in most of the physicist's work, we find tied together (a) an unexplained phenomenon of nature, (b) controlled experiments for study of the observed phenomenon, (c) the devising of a mathematical theory that explains the phenomenon. An example is described below.

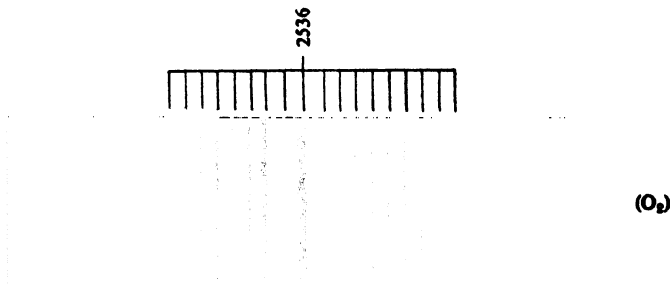
The blue of the sky

Why are the sky, the sea, and the great glacial ice caps blue? We all now realize that color is not a property of the substance itself but the wave length of the light that enters our eyes after being reflected from the thing we see. Blue light is of shorter wave length than red light. When we look at a colorful painting, our eye is really registering the fact that light of wave lengths 2, 3, 5, 6, and 7 is crowding into it from that picture, and through a

miraculous mechanism these numbers or lengths are transferred into our mind picture of the masterpiece. Viewed in blue light, this many-colored picture appears to be only blue and black.

As far back as the start of the sixteenth century, Leonardo da Vinci felt that dust particles in the air caused the blue color of the sky. Some reason had to be found to satisfy the mind of this inquisitive artist-scientist, to whom color produced in colorless air by white sunlight could not be left unexplained. The same theory was again proposed by Tyndall in 1869. He was even able to show in artificial cloud chambers that blue light can be scattered or dispersed when sunlight passes into masses of fine dust particles suspended in air. *Early theories*

It was inevitable, however, that skeptics would point out the increased blueness of the sky as seen from mountain tops above the dust level. It seems simple now to say that it should have been evident that the molecules of the air itself formed a more effective dispersing agent than the most finely divided dust. So it proved to be when Lord Raleigh showed that the blue color and luminosity of the sky were fully explained by molecular scattering of sunlight.



Courtesy Chemical Publishing Company

FIG. 5. RAMAN SPECTRUM OF OXYGEN

Reproduced from *Scattering of Light and the Raman Effect* by S. Bhagavantam, who explains in greater scientific detail the reasons for the color shown by the ocean and the sky.

The simplest explanation of the ocean's color is that it might be a reflection of the sky. Skeptics again arose to show that the ocean was sometimes seen to be bluer when the sky was almost gray. Dissolved coloring matter lost caste as an explanation because the observer was always above the water and could see only by light reflected from near the surface; besides, when sampled, sea water no longer looked blue. There might still be floating particles to *The blue of the ocean*

disperse the light, but tests by C. V. Raman, the famous Indian physicist, showed suspended material to be almost nonexistent particularly in the bluest ocean waters. And so Raman concluded that molecular scattering of sunlight explained the blue of the ocean as well as the azure sky. Evidently, the cold blue brilliance of glaciers was another manifestation of the same effect.

*A scientific
discovery*

Thus, the age-long interest in the blue of the sky and the ocean resulted in the discovery of the Raman effect, an important building block in the house of science. It was not long before Raman had shown that every transparent gas, liquid, or solid would disperse light of a different wave length or a color different from the light falling upon it. The discovery of these radiations resulted in full confirmation of the important quantum theory of dispersion. Further research showed that the wave length of this secondary light radiation was a characteristic of the molecular structure of the gas, liquid, or solid passing the light. Now science has developed the Raman spectrum which is a virtual fingerprint of any substance that will pass light. Raman spectra and X-rays are being used regularly by industry as tools for determining the internal structure of liquids and solids of all kinds.

APPLICATIONS OF CHEMISTRY

*Organic
plastics*

In continually greater measure, chemistry is laying the foundations for new industries. Organic chemistry, in particular, seems to be developing rapidly. Synthetic materials stimulated by the war will continue to be produced in greater and greater quantities. Of these, the organic plastics have attracted the greatest popular attention. The thrill we get out of the development of plastics is difficult for the chemist to understand. He had known about the basic kinds of plastics materials for a generation — some were developed nearly a century ago. Plastics materials naturally catch our fancy because of the extraordinarily colorful plastics gadgets that industry is turning out. But, in a larger sense, our interest is an economic one. Though a long depression from 1930 to 1937, we waited hopefully for the new industry that failed to come to our economic rescue. Then about 1938 we became aware that the new plastics industry was already here and *plastics* became a word to conjure with.

*Phenol
plastics*

Everyone should know a few of the scientific facts about these revolutionary materials. The most common plastic (phenolformal-

dehyde or bakelite) comes from these simple chemical changes: Methyl or wood alcohol passes through a heated metal screen, which changes (catalyzes) the alcohol to formaldehyde. Phenol, obtained from coal tar, is then cooked with the formaldehyde, producing the synthetic resin in a half-finished state. The cooking is completed in the hot metal mold into which this resin, mixed with fine sawdust, cotton fiber, or asbestos fiber, is compressed under pressure. The resulting plastic is a cheap, hard, amber or dark-colored material useful for many purposes, but brittle rather than shock-resistant. Anyone who has had a phenol plastics radio and a small boy in the same house will understand this reference.

For a number of years plastics were always dark brown, black, or other dark shades. The first milk-white plastics which appeared in the early 1930's were most attractive as compared with other industrial products. These were the urea-formaldehyde plastics, which were soon available in all the tints of the rainbow. Ammonia and carbon dioxide (soda-fountain gas) combine to produce urea, which reacts with formaldehyde to form a clear resin solution. The filler must be milk-white to take fine colors. Chemically pure wood or cotton cellulose is used. The resin solution and the filler are mixed with color pigment and then dried to form the molding powder. Molded under low heat and pressure, the plastics product is translucent and colorful. *The new technicolor*

Popular realization of the importance of plastics developed when the crystal-clear acrylic plastics (lucite or plexiglass) came on the market in 1938. So susceptible to color work that the resulting products rivaled jewels in brilliance, the acrylic resins met obvious needs and developed other uses on every hand. They are most valued for light effects, since they conduct light around corners in astounding ways. This is due to their excellent characteristics of internal reflection and transmission which keep the light internally reflected within the plastic itself until the light arrives at a roughened surface or area which lets it escape. The chemical name for this plastics material is methyl methacrylate. *Acrylic plastics*

Another important material is vinyl plastic. Vinyl acetate stems from acetylene gas (formerly used to light automobile headlamps) and acetic acid (vinegar). A complex chemical change known as "polymerization" is needed to convert vinyl acetate to a plastic. Ultraviolet light, heat, and pressure speed up this process of linking molecules together to form *polymers*. When completed, these *Vinyl plastics*

reactions of several chemicals produce a new material, the elastic plastic. Used for safety glass, the vinyl plastic is an improvement because it stretches and does not break through.

*Chemistry and
chemical
engineering*

It is often difficult to distinguish between chemistry and chemical engineering. The subject of plastics production will serve as an example. We know that a plastic can be made from many different raw materials, such as oil, tar, coal, wood, cotton, peanuts, coffee, and cornstalks. The job of the research chemist is to experiment with the raw materials available to find out what kinds of plastics material can be produced and to iron out all difficulties inherent in the chemical reactions involved. It is then the function of the chemical engineer to take over and commercialize the process.



FIG. 6. SANITARY CHEMISTS STUDY WATER POLLUTION

The engineer must not only design a plant to produce the plastics material on a large scale but he must also be certain that the resulting plastic molding powder can be sold profitably on a highly competitive market. The practical engineer brings the dollar sign into the complex chemical reactions involved in the research chemist's process. Of course, some chemists are also able to commercialize their research developments and some chemical engineers direct research, so that the fields are interwoven, but the common division of duties is as indicated.

THE SCOPE OF CHEMICAL ENGINEERING

The main fields of chemical engineering are the design and operation of equipment of chemical plants for sugar refining, paper-making, oil refining, alcohol production, food preservation, rubber synthesis, and other processes in which a chemical change is produced. The operations involved are broken down into the so-called *unit operations* of drying, vaporizing, crushing, grinding, filtering, humidifying, and dehumidifying. The plant process can usually be broken down into certain chemical processes such as alkylation, sulfonation, nitration, hydrogenation, and so forth, known as the "unit processes." When a new product is to be made, the chemical engineer is able to transfer his knowledge of the successful use of each of the unit operations in other plants to the new design. He also has a fund of information about the kinds of machinery that have proved successful in operation under various conditions, although he usually leaves the design of moving machine parts to the mechanical engineer.

The main theoretical tools of the chemical engineer are mathematics, chemistry, and a knowledge of the transfer of heat and the flow of fluids. Plant processes involve chemical reactions which are usually speeded up by increasing the temperature. Heat must also be added at one step and removed at another stage whenever the process involves vaporization and condensation or drying. Materials are commonly handled either as liquids and vapors, or as solids in suspension since the transportation of dry solids is often troublesome and costly. Hence, every step in the design of the process requires a knowledge of the flow of fluids. More and more, the study of heat flow, fluid flow, and chemical engineering thermodynamics requires a good background of mathematics. It goes without saying that a knowledge of physics is also essential.

*Development,
design, and
operation*

Like all other fields of engineering, chemical engineering is broad enough to give people of many interests an outlet and a chance to become successful. The successful chemical engineering student who is also an excellent chemist should consider research and process-development work. After the chemist has produced important new reactions in the laboratory, the development engineer has the job of trying out on a small scale a process which makes use of these new reactions. He must not only understand what the chemist did within his test tubes, but he must be clever at finding and curing the innumerable difficulties arising from this change to "pilot-plant operation." The person with more practical interests will become an operating engineer. In between these two are the plant designer and the engineer who supervises construction or changes in the plant and then directs the adjustments that bring the plant into successful production.

A GREAT BRIDGE COLLAPSES

*Tacoma
Narrows
Bridge*

On November 7, 1940, newspaper extras all over the country headlined the collapse of the world's third longest bridge. A few days later, the newsreels showed millions of people the first bridge failure they had ever seen. To the public this was probably the most spectacular newsreel of a decade; to the engineer it has become a scientific record of the greatest significance. Great popular interest was stimulated and the structural engineer for a short time became a public figure criticized and held in awe. Certainly, after seeing the moving pictures of that writhing, tossing monster, equal in over-all length to a half-dozen battleships, people began to realize the tremendous responsibilities involved in the engineer's job of "controlling the forces of nature."

Slender span

The Tacoma Narrows Bridge was opened to traffic on July 1, 1940. It was a slender, graceful, suspension bridge spanning 2800 ft. between the main towers from which 1100-ft. side spans extended to the shores. This entire bridge was constructed for about 6,000,000 dollars, a low figure only a fraction of the cost of several of the great suspension bridges. A low-cost structure was possible because the traffic needs called only for a two-lane bridge with sidewalks so that the two supporting cables suspended over the 425-ft. towers were but 29 ft. apart. The light weight of the bridge could be supported by comparatively small cables, $17\frac{1}{8}$ in. in diameter, and the designers concluded that the roadway stiffening trusses, which

by ordinary standards would have been 25 ft. or more in depth, would be out of place on this light structure. They used two 8-ft. girders along the outer edges of the sidewalks as seen in Fig. 7.

Even during construction, the Tacoma bridge showed some tendency to sway and to oscillate up and down. Other bridges had acted that way too, but this one did not stop when the floor had been completed. Rumors began to pass around the country about the undulations of "Galloping Gertie," as the bridge had been nicknamed. One man related that he drove across the bridge during a 30-mile wind and saw a car, well ahead of him, disappear in the trough of a wave passing along the floor. At that time this account seemed like the exaggeration of a non-engineer, but the official report of the failure states that oscillating waves as great as 5 ft. had previously been observed, which is about the height of a modern motorcar. The oscillations were seen to change back and forth from six or eight up to thirty-six or thirty-eight per minute.

"Galloping Gertie"

Five-foot oscillations



Courtesy Professor B. W. Farquharson

FIG. 7. TACOMA NARROWS BRIDGE IN MOTION BEFORE ITS COLLAPSE

Note the angle between the light posts and the vertical suspenders which is even visible in this diagonal camera shot

Restraints

Engineers were engaged in studying the bridge. They were planning additions to increase its stiffness and reduce the oscillations that by then were recognized to be dangerous. Some center diagonals and hydraulic buffers had been added before the bridge was opened to traffic in July. These restraints proved inadequate and, on October 7, tie-down cables were attached to the girders of the side spans anchoring them to the ground. The side spans then quieted down, but the undulations of the center span were not greatly influenced. Promptly the engineers began to study the effects of stay cables to go from the tower tops down to the center span girders. These ties were never installed because the bridge collapsed before they could be designed.

The day of the failure

We get the picture best by reading the words of eyewitnesses to the twisting, writhing, and final collapse of this ill-fated structure. The following accounts are from several observers, since no one person saw every detail of the collapse.

7:30 A.M., Nov. 7. "I went to mid-span and took the velocity of the wind, reading 38 miles an hour on the anemometer. The bridge bounce was noticeable but of no great magnitude."

9:30 A.M. "The motion kept building up and at about 9:30 A.M., I drove out and parked in the center of the main span and counted the number of cycles the bridge was making per minute and found it to be thirty-five."

10:00 A.M. "A violent change in motion was noted. This change (from vertical oscillation or bounce to a writhing motion as shown in Fig. 7) appeared to take place without any intermediate stages and with such extreme violence that the span appeared to be about to roll completely over." From then on the main oscillation was such that the center of the span stayed reasonably quiet, one end rising and the other end falling.

Approaching collapse

10:00 A.M. "The center span was swaying wildly, it being possible (from the side) to see the entire bottom side as it swung into a semi-vertical position and then the entire roadway. It was at once apparent that instead of the cables in the main span rising and falling together, they were moving (now) in opposite directions, thereby tilting the deck from side to side."

10:00-11:00 A.M. "It was plain at this stage that the lower limit of undulation was almost exactly flattening the girder to a straight (horizontal) line from tower to tower while the upper limit must have been something over 25 ft. above this level." This twisting made the deck slant nearly 45 degrees back and forth. At times when the near section of the deck twisted clockwise, the far section beyond the center of the bridge was seen to be twisting counterclockwise.

11:00 A.M. "I went to the observation house and noticed small chunks of concrete falling. I supposed they were the cracked sections around the light posts. Hearing someone yell, I rushed down to the parking area and observed that the mid-span was broken completely in two, the ends working against each other with a sidewise motion "

11:10 A.M. "Shortly thereafter two or three sections of roadway fell. Then approximately 600 ft. of girders and roadway fell in. I heard a deafening crash and looking up saw clouds of dust and a snarled mass of suspenders and deck whipping. The east approach dropped below my vision and then everything became quiet."

The Tacoma bridge was by far the most flexible long-span bridge ever constructed. Its collapse was the third major bridge disaster of this century. This bridge was amply strong to resist all loads but its tendency to oscillate under the influence of wind gusts caused its failure. Its shallow stiffening girder and its narrow roadway combined to produce little resistance to twisting action. Consequently, as soon as the bridge fell into this kind of oscillation under the action of the wind, the motion became enormously magnified until the floor system was torn to pieces. Studies of models and mathematical investigations will eventually show us how to build slender bridges that are safe from cumulative vibration, but, in the meantime, we can follow the old rules that have proved safe in the past. This "full-scale experiment" may prove to be the stimulus for an important advance in bridge engineering.

*Lessons
learned*

CIVIL ENGINEERING

In the failure of the Tacoma bridge you have seen some of the problems faced by the structural engineer. He also designs and constructs buildings in steel and concrete, dams, docks, hangars, and every type of structure needed by industry. The highway engineer lays out and builds roads, streets, and airports. The hydraulic engineer designs irrigation systems, levees, and other flood-control projects. The sanitary engineer is responsible for water supply and sewage disposal and also for the treatment of industrial wastes. The municipal engineer often acts as a city manager. The foundation expert is much needed for the design of deep piers, footings on troublesome soil, and underwater construction. These are the divisions of civil engineering and, in the main, they are involved with public works. More than any other, the civil engineer works directly in the interest of the public.

*Structures,
highways,
hydraulics,
sanitation*

"FATIGUE" OF METALS

Engineers sometimes use a word in a technical sense that is descriptive because of its meaning in the everyday world. *Fatigue* when applied to the fracture of engineering materials is such a

*Axles even
break on trains*

word. You may find something of interest in the following experience with a fatigue fracture. A small group of engineers was riding in a railroad dining car when, without warning, the car began to bounce so severely that the dishes upset and seemed about to leave the table. From the car window, we could see a stream of fire flaring backwards along the train. When the train stopped, we climbed out and found the thick steel axle underneath the rear of the diner broken cleanly through. The apparent fire had been a stream of sparks caused by the wheels and axle pounding against the rails.

Fatigue crack

Under the conductor's flashlight, the surface of fracture of the broken axle showed a characteristic pattern. Around its circumference was a smooth band while the center area appeared very rough and coarsely crystalline by comparison, as illustrated by Fig. 8. The trainmen said that the steel had "crystallized." Actually, this was an example of fatigue failure. After long service, a microscopic surface crack had developed, probably starting at some invisible flaw. At each revolution of the axle, this tiny crack opened and closed, its faces being pounded to perfect smoothness as it extended around the axle and gradually worked its way inward. Perhaps months or years after the crack started, it had

Progression

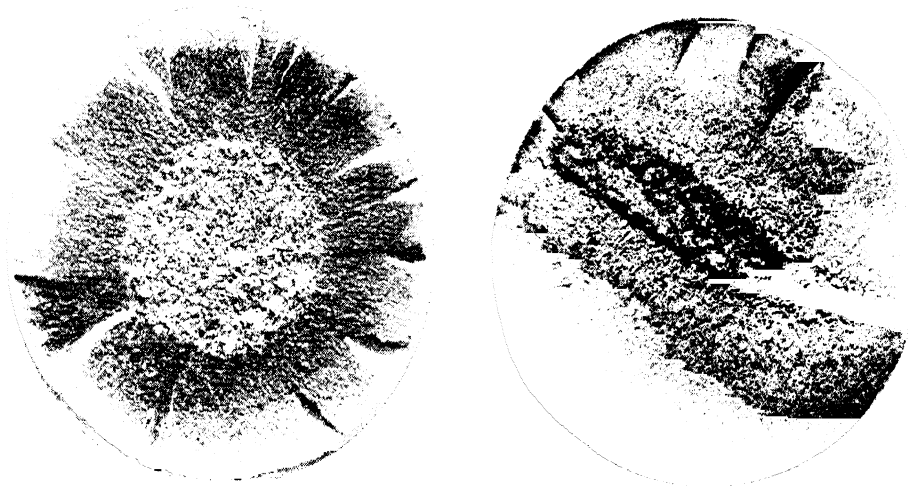


FIG. 8. FATIGUE FRACTURES AS FOUND IN AXLES AND SHAFTS

Courtesy Professor H. F. Moore

The fatigue crack in the shaft illustrated at the left gradually extended all around the bar and reduced the area to the small inside circle; then a sharp fracture occurred producing the rough crystalline appearance at the center. In the rotating shaft illustrated at the right the fatigue cracks worked in from two sides.

penetrated deeply enough to cause the axle to fracture. The final rupture occurred suddenly, producing the usual grainlike surface over the center section of the axle characteristic of shock fractures.

Without thinking about it, you have undoubtedly produced many fatigue fractures yourself. Everyone knows that he cannot break a wire or a strip of metal by bending it once, but that it can be broken readily by bending it back and forth a few times. If you make the angle of bend rather small, you will have to repeat the bending operation several hundred times to break the metal. In fact, several million repetitions may be necessary if you bend the metal part so lightly that it acts as a spring and automatically becomes straight when you release it. Actually, there seems to be a limit of bend below which you can repeat the operation forever without rupturing the metal. This is known as the "endurance limit," which is expressed by engineers as a stress in pounds of load per square inch of material. This definition will be clearer if you think of the load not as a bending force but as an axial load alternately stretching and compressing a length of metal rod. Then, the ultimate stress is the static load that will break the rod divided by the cross-sectional area of the rod; and the endurance limit is about 50 per cent of this ultimate stress. Stresses produced by bending can also be computed, and endurance limits can be set for this case and for other cases.

Life span of metals

The phenomenon of fatigue has not been explained completely, but a working hypothesis that accounts for most observations is as follows. Steel and other metals are crystalline, which means that they are made up of minutely interlocked crystals. Under heavy load or stress there is some slipping between crystals which may be too severe to disappear entirely when the load is removed. Such tiny internal slippage may be of no importance if it occurs just a few times, but if it is repeated indefinitely, and particularly if the slip is reversed from left to right, a crack will begin to form, resulting eventually in a fatigue fracture. In other cases minute or microscopic cracks or flaws may already be present in the material; such flaws tend to form fatigue cracks after repeated loading.

Progressive cracking

For many years engine designers were able to build successful steam engines without giving any consideration to fatigue. These were the old reciprocating engines that you still see as standbys in power plants. They turned at no more than 300 r.p.m. (revolutions per minute) which is about the engine speed of a freight

When fatigue becomes critical

*Automotive and
aviation engines*

locomotive. If any part broke, the designer simply made it heavier since weight was not important. Then the automobile came and speeds of rotation went up to 2000 and 3000 r.p.m.; also, excessive weight became objectionable. In the airplane engine not only is the speed high but every ounce of excess weight must be trimmed away. For power plants, the steam turbine turning at 1800 or 3600 r.p.m. took the place of the slow-speed reciprocating engine, while supercharger turbines may rotate above 10,000 r.p.m. Tests show that fatigue failure occurs within 10 to 15 million reversals in direction of the loading or usually not at all. At 10,000 r.p.m. a turbine will make 15 million revolutions in a single day. Evidently, if any reversing stresses (caused by alternation in the direction of the loading) occur in the turbine, they must be below the endurance limit or its life will be very short indeed. However, since the turbine has no reciprocating parts, only vibration from lack of balance can produce such stresses. Some parts in automobile and airplane engines can be kept small and therefore light in weight by designing them for replacement after some fixed number of hours of service. Necessity often dictates such practical procedures.

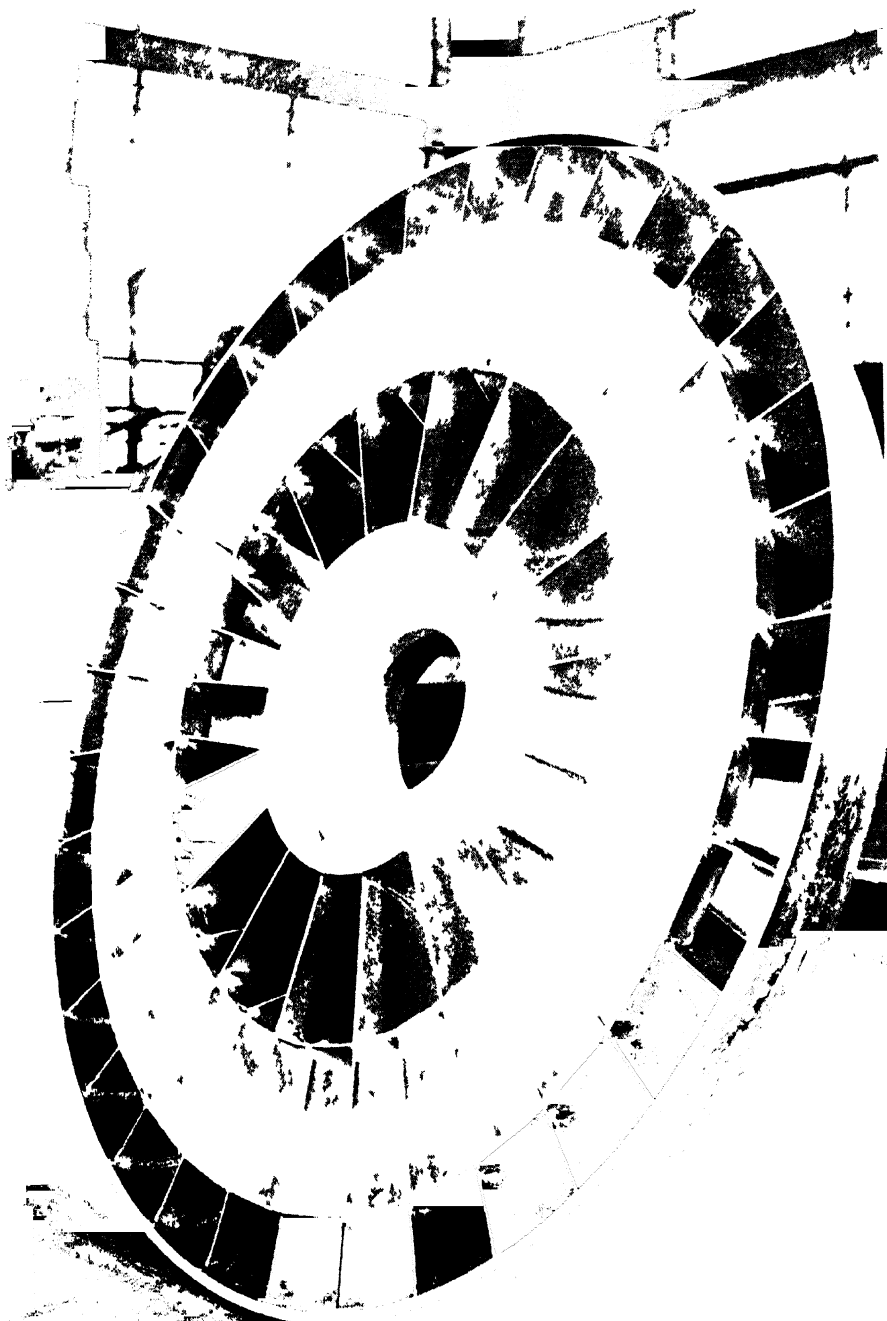
This brief qualitative explanation of fatigue must be supplemented with quantitative data before it can be applied to design. However, it is useful to know something about this phenomenon early in your study of engineering so that you may be looking for its effects.

MECHANICAL AND AERONAUTICAL ENGINEERING

*Machines and
airplanes*

The engineers who have to worry most about fatigue are mechanical and aeronautical engineers because they deal with fast-moving reciprocating machinery. The hydraulic engineer may have to plan the design of a hydraulic turbine and the electrical engineer does design electric motors, but both depend upon the mechanical engineer to design the bearings, the system of lubrication, and the general structure of the machine, based upon the strength of the materials to be used.

Of course, the airplane engine is a natural field of study for the mechanical engineer. The plane itself is a structure rather than a machine since it does not have rapidly moving parts but moves as a whole. Its design is similar to the design of a bridge truss; like the bridge, it carries its own weight and the weight of its cargo. At the same time, the frame of the airplane must resist air pressure



Courtesy Lukens Steel Company

FIG. 9 ROTOR OF A HYDRAULIC COUPLING

This huge all welded hydraulic coupling for smoothly transmitting power is

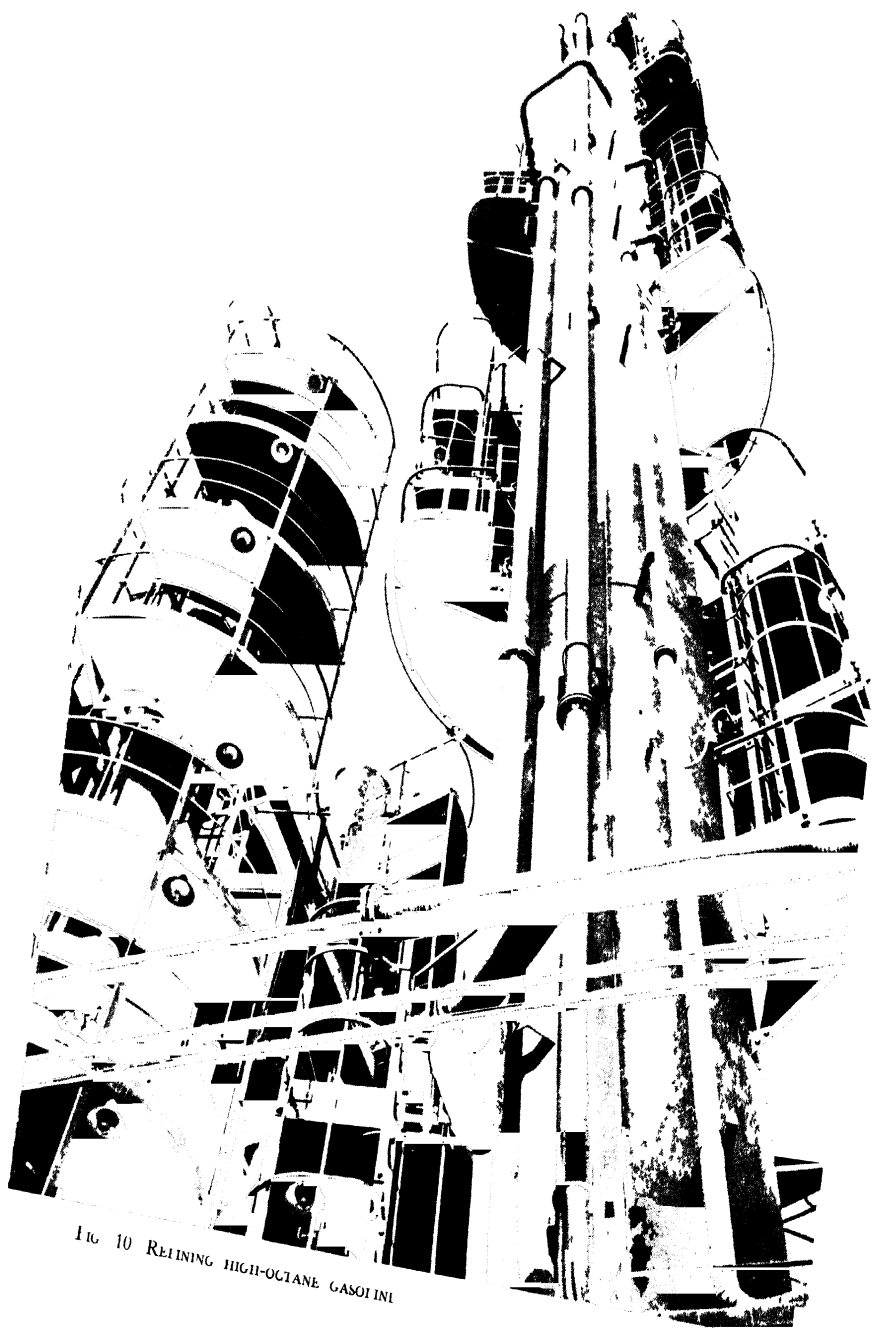


FIG. 10 REFINING HIGH-OCTANE GASOLINE

and aerodynamic forces that produce unusual loads and cause vibration. In this respect the airplane is like the oscillating Tacoma bridge. Unlike the bridge, which oscillated no more than thirty-eight times per minute, the tail or wing of an airplane may vibrate many times per second if there is a slight unbalance and thus become subject to early fatigue fracture. So again we adopt the compromise of designing the plane for a limited life, with the intention of retiring it from service before failure becomes probable.

There are many other fields of mechanical engineering. The steam engineer deals mainly with large power plants. The internal-combustion expert or diesel engineer faces many problems with his engines for automobiles, airplanes, ships, streamlined trains, and small power plants. Air conditioning, which today includes heating, ventilating, humidifying, cleaning, cooling, and purifying air for any building, is an active field for study. Refrigeration engineering is important in the storing, preserving, and transporting of foods. And, of course, we must not forget the production engineer, usually of mechanical-industrial training, whose job it is to keep the machines of industry continually at work turning out goods by mass production. He must be expert in dealing with men, in routing materials through the plant, in developing safety measures, in cost control, and in promoting generally efficient procedures. Years ago he was the unwanted "efficiency expert," but today he is accepted and respected in every progressive industry.

*The scope of
mechanical
engineering*

*Industrial
engineering*

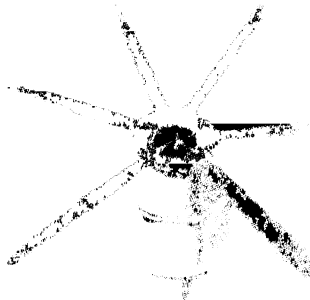


FIG. 11. OIL SPRAY INJECTED INTO A DIESEL ENGINE CYLINDER

From Diesel Engineering Handbook

EXPERIMENTS ON MODELS

*Usefulness of
model studies*

When a new structure, machine, or electrical system is proposed, we face the problem of making a design largely from theoretical or mathematical studies. Practical engineering works are often too complex to analyze exactly by mathematical methods; hence, we must depend upon experience with similar engineering works or upon experiment. Since it is impractical to build a 30-story building or a 1000-ft bridge or a 100-mile electrical network for testing, we naturally turn to model tests. When we were discussing chemical engineering, pilot-plant operation was mentioned as one step in the development of a new process. The pilot plant is a model and the information gained in its operation is used to aid the designer of the full-scale plant. When theoretical calculations prove correct in pilot-plant operation, we feel more confident in applying them to the full-scale plant.

*Mississippi
River model*

Dam and waterway construction is a field in which model experiment has been highly successful. The design of a model to represent its full-sized counterpart is no simple matter, however. The reduction of all dimensions to one-tenth or one-hundredth of natural size might seem a reasonable device, and while this geometric similarity is one essential of perfect similarity, it is not the most important factor to be taken into consideration. Similarity of motion must be studied, which means that the corresponding flow paths must be similar and that the velocities at corresponding points bear a definite constant ratio determined by the relative size of the model to the prototype.

At Vicksburg, Mississippi, United States Engineers have produced a model of the Mississippi River which is about a mile long. In this model, which is illustrated in Fig. 12, it was necessary to dispense with geometric similarity to obtain closer dynamic similarity. The influence of straightening reaches of the river, placing a dam on a tributary stream, or providing floodways to relieve the levees in time of flood have been studied effectively. There is reason to believe that these studies have led to construction work which will reduce the damaging effects of the disastrous floods that occur every decade or so along the lower Mississippi River.

*Models of
structures*

Structural models for buildings and bridges are proving increasingly useful. One type is the bakelite plastics model viewed with plane polarized light or light restricted to vibration in a single

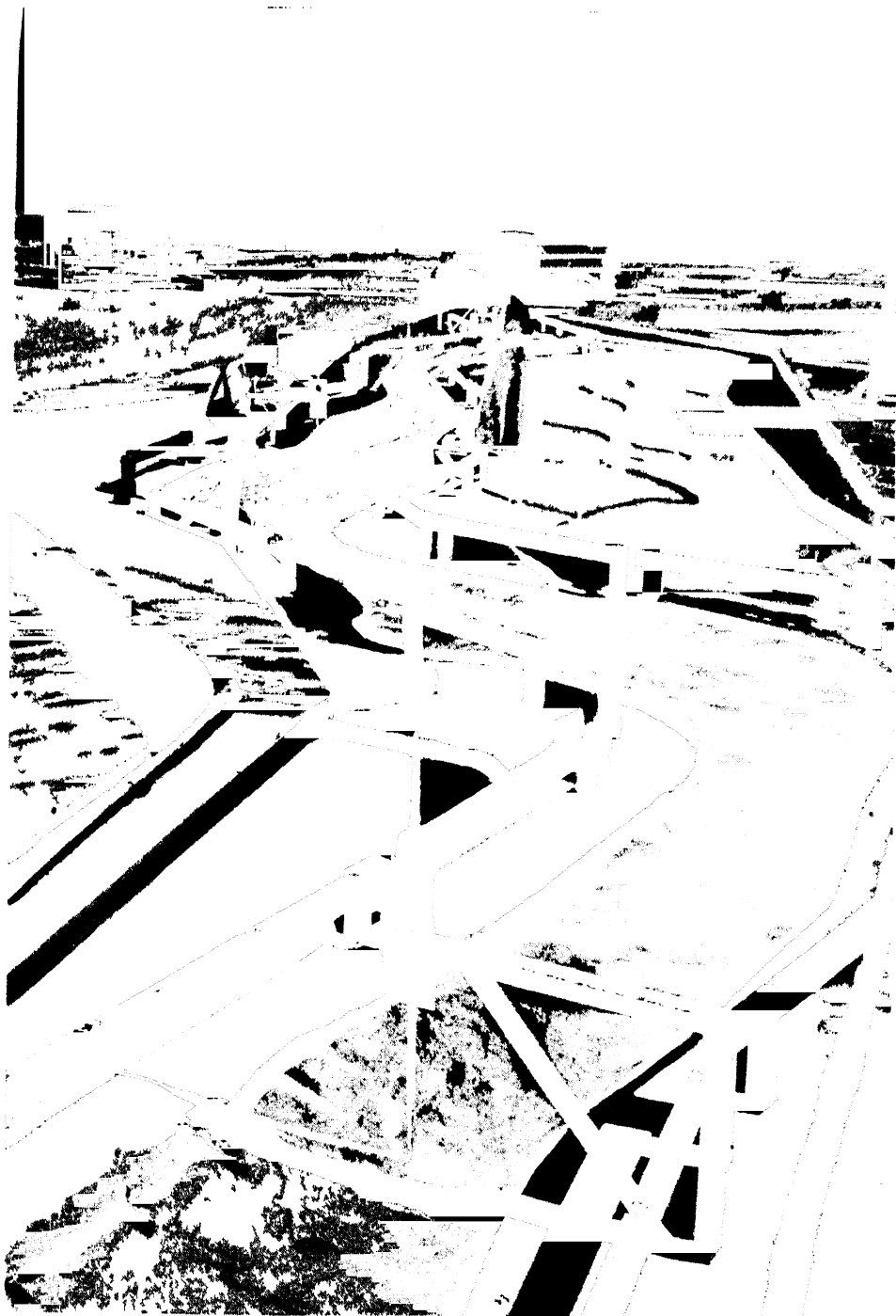
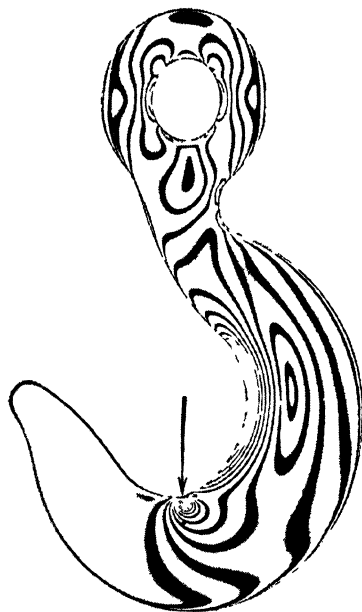


FIG 12 MILE-LONG MODEL OF THE MISSISSIPPI RIVER AT VICKSBURG

Photoelastic models

direction such as the vertical plane. It is actually possible to see the distribution of the stresses in such models because of colored patterns produced by the effect of the polarized light passing through the stressed bakelite model. By using a light of one wave length, we can change the colors into light and dark bands, as



Courtesy M. Hetenyi

FIG. 13 TRANSPARENT MODEL OF A LOADED HOOK VIEWED IN POLARIZED LIGHT

shown by Fig. 13, so clearly defined that stresses may actually be calculated from a photograph of the loaded model. However, the engineer then faces the problem of translating these stresses of the small bakelite model into significant stresses for a steel or concrete structure a hundred times as large. This transference is not really practical so another procedure is used; we develop theoretical equations to compute the stresses in the model and check the accuracy of these equations by photoelastic determination of the stresses in the model. This check gives us greater assurance that our equations may be applied without probability of serious

error to the structure itself. Photoelastic models have also proved useful in the study of stresses in gears and in other machine and airplane parts.

Models of Tacoma bridge

The failure of the Tacoma bridge brought forth numerous models of suspension bridges for study. Many were direct models of Tacoma itself. Most of these models showed a tendency to gallop or oscillate when exposed to an air current crossing the roadway at an angle. Of course, it is no simple matter to design a proper model of such a complex structure as a suspension bridge and particularly a vibrating model. Here, too, a true-scale model would have little meaning. For example, the stiffness of a bent bar varies as the length and breadth but it changes as the cube

of the depth. Nevertheless, these variables can be studied and a simplified model obtained that will simulate the action of the real structure. It seems unlikely that any major suspension bridge will again be built without a model study of its vibrational characteristics.

One of the simplest engineering works to reproduce by a model is an electrical network. Each interconnection between two power systems changes the conditions under which the current flows. Yet the system may be so complex that a mathematical study would be a long, tedious process. Fortunately, a model of almost any electrical network can be arranged on what is commonly called a *calculating board*.

Models of electrical networks

The fundamental relation (Ohm's Law) controlling the flow of current in any system is $E = IZ$, or voltage, E , equals the product of the current flowing, I , and the electrical impedance, Z . Impedance, of course, is any restriction to the flow of current. There are no units here that necessarily depend upon size, and we conclude, therefore, that the circuit may be a hundred miles or a few feet in length without influence upon its electrical characteristics. For convenience, we prefer to make the currents and voltages of the model much smaller than those in the field network but still large enough to measure accurately with available instruments. Evidently, by changing impedances of all parts of the circuit proportionately, any convenient magnitude of voltage or current can be obtained. As a matter of interest, such calculating boards exist at a number of educational institutions and central stations, and they are much used. It is possible in a few hours to plug in a circuit of given resistances (or impedances) on a calculating board to serve as a model of a complex network and to complete the study of the network by measuring the voltages and currents at as many points as desired.

The calculating board

ELECTRICAL AND COMMUNICATION ENGINEERING

The most important engineering advancement of the past half-century has been the development of electric power. The electrical power engineer may be concerned with electric power generation; he may also be interested in electric power transmission or in the design and application of the necessary equipment for transmission, such as transformers, power lines, or lightning protection devices. High-voltage, long-distance transmission is a specialized

Power engineering

field in which there have been important recent developments. Hydropower generation represents another specialized power field. Power-consumption devices, from industrial motors to home vacuum sweepers, provide jobs for many electrical engineers. Perhaps the most important of these special fields is illumination. First the Edison incandescent lamp, then the gas-filled lamp, and finally the fluorescent tube have been successive stages in the development of efficient lighting. The lighting expert finds more need for his services as people become appreciative of the importance of good illumination. For example, the standard of light intensity for proper illumination of a drafting room doubled with the introduction of fluorescent lighting.

*Communication
engineering*

When De Forest began to experiment with radio transmission by use of the vacuum tube (1906), he opened up another field for the electrical engineer that may well become equal in importance to the electric power field. The magic radio of 1920 became the public's darling by 1930 and the widest form of popular entertainment by 1940. Television was nearly ready for commercialization by 1937. The academic interest in extremely high frequencies led to the highly secret Radar which the Army took over for the development of detection devices. Its success in these three fields leads us to conclude that the electronic tube has an important industrial future.

*Industrial
electronics*

Communication or industrial electronic engineers will be needed to devise, install, and operate electronic controls for automatic industrial machinery, for medical heating devices, and for dozens of uses not yet imagined. Evidently, this is a most promising field of study for the man who has mathematical talent and theoretical interests. The development of vacuum-tube devices has passed out of the stage of gadgeteering and it is not likely that major contributions to the field of industrial electronics will be made from now on by the inventor without technical education. In fact, the common belief that "wizards" such as Edison, the Wright brothers, and De Forest were geniuses without scientific education is false. Such men were able and willing to educate themselves in any field in which they happened to be at work.

THE DEVELOPMENT OF RADIO

De Forest

Lee De Forest invented the electron tube in 1906. He was so far ahead of his competitors and of scientists in general that no one

even knew that his *audion* was really an electronic device until five years after he had put it into use for his first love, wireless telegraphy or radio. He broadcast Caruso's voice in 1910 and thus earned his title "the father of broadcasting."

De Forest's first Chicago experiments were in developing a wireless detector. After considerable conflict with his employers because he neglected his job to work on his own ideas, he finally took a night-school teaching position and experimented on his own. From then on, his work progressed rapidly and in 1901 he picked up wireless signals from a boat in Lake Michigan five miles away. *Wireless telegraphy*

From that time there was a race on between Marconi and De Forest to see who could most rapidly develop and publicize his own inventions in wireless telegraphy. Marconi won but De Forest produced some extraordinary promotional stunts and actually headed a company that mushroomed into a multimillion dollar corporation before it collapsed. The fact that the public's hopes were dashed and that investments were lost left the overoptimistic inventor open to severe criticism. He had reaped the expected reward of rash commercialization not backed by long painstaking experimental development.

Broke, De Forest began to hunt for another invention to salvage his lost prestige. It is said that he had noticed a gas light flickering in rhythm with a spark coil and this led him by analogy to study the influence of separate currents upon electric-light filaments. If this story is true, it is a case in which an incorrect analogy led to a great invention. Of course, this effect of one current upon another is the fundamental accomplishment of the electron tube which De Forest developed after hundreds of unsuccessful attempts had merely spurred this erratic genius on. When he succeeded, he knew that he had at hand the one device that could take the weak wireless signal coming through the ether and magnify it to original strength. *Developing the audion*

De Forest in 1906 had radio at his finger tips, but he had to wait until 1920 for its commercialization because the world was not yet ready for such a revolutionary invention. He spent many of the years from 1920 to 1934 fighting in the courts for his rights as inventor of the electron tube. His patents were finally upheld and he received needed royalties. De Forest then returned to his laboratory, hoping to add another milestone to his list of contributions to electrical technology. *Lag in commercialization*

SPECIAL FIELDS OF ENGINEERING

Metallurgy

In twenty years, from 1920 to 1940, the alloys of iron and steel increased a hundredfold in number. Aluminum, dropping greatly in cost, became a competitor of steel for those structures and machines in which weight was a handicap. Magnesium, finally produced from sea water, began to compete with both steel and aluminum for light-weight uses. Following enormously increased war production, aluminum and magnesium may be expected again to drop sharply in cost and to compete on a peacetime basis with steel possibly even for bridge and building construction. The alloys of the light metals may then increase in number for special uses as did the alloys of iron a generation earlier. Rarer metals such as tungsten, vanadium, and molybdenum have become common or at least less rare and have contributed greatly to technology for particular uses or as alloying metals.

Production metallurgy

With all of these developments in the air, it is small wonder that metallurgy has attained the status of a special field of engineering. Sometimes called a metallurgical engineer, the metallurgist has two major opportunities for service. As a production metallurgist, his job is to produce pure or alloyed metals from the raw materials and ores available. The open-hearth furnaces in our steel mills, for example, are under the eye of the production metallurgist. He must control the heat, the contents of the charge, and the time in the furnace to assure a uniform quality of steel ingot.

Physical metallurgy

The other major field of metallurgy is under the control of the physical metallurgist. His work is more scientific in that it is always changing. He is responsible for investigating the usefulness of new alloys. He studies the crystal structure of metals often by X-ray photographs and develops methods of treatment to improve their strength or ductility or resistance to fatigue. He also investigates the causes of corrosion and devises means for reducing the deterioration of metals. In manufacture, the physical metallurgist has the responsibility of selecting special metals for particular products and of devising, along with the mechanical engineer, methods by which these products will be worked during production. Weldability of metals is an important part of his investigations.

Metallurgical engineering was long considered a branch of chemical or mechanical engineering. Certainly the metallurgist

must understand chemistry since physical chemistry along with physics forms the background of most of his work. The scope of metallurgy is such that we must look upon this field as so important that it requires the full-time attention of specialists and not the casual attention of the mechanical or chemical engineer.

The architect is not really an engineer although he is a builder. *Architecture*
Naturally, the fields of architecture and structural engineering flow into each other without a sharp line of demarcation. Some architects become structural engineers and some structural engineers become architects. The main function of the architect, however, is to design the arrangement of the building and its details to serve the needs of the owner and to harmonize with its surroundings. The structural engineer designs the supporting structure and then one or the other follows the building through the construction period to see that it fulfills all plans and specifications. *Structural design*



FIG. 14. A NEW ALLOY OF EXCEPTIONAL HARDNESS AND ELECTRICAL CONDUCTIVITY

Notice the imprint left on the steel bar at the left when the harder alloy at the right is forced down upon it. The alloy is not only harder than mild steel but it has about the electrical conductivity of copper. This is a combination of great importance to industry.

The architect and engineer do not always see eye to eye because the engineer is interested primarily in designing a safe, economical, and simple structure while the architect often complicates the structure to achieve some element of convenience or to maintain style. The modern architect follows closely the ideal of "arrangement to serve the intended function" and is more likely to be in agreement with the engineer than were his classical predecessors.

City planning

The architect must have most of the qualifications of the engineer plus an artistic ability. He has less need for mathematics and is not usually expected to study calculus. He must be imaginative and he should exhibit a special interest in drawing. Some architects have extended their planning function beyond the single building to include residential or business areas, towns, or whole regions; their work is called City Planning or Regional Planning. On the inside of the front cover of this book is a fine example of replanning to eliminate the smoke nuisance of the city. The unsatisfactory appearance, congested areas, and inadequate traffic ways of our cities call for replanning and rebuilding on a large scale. The reconstruction of shattered European cities after the Second World War should bring before us the need for replanning America's metropolitan areas.

Naval architecture

The naval architect is really a marine engineer. He designs and supervises the building of ships of war and ships of commerce or pleasure. His training corresponds to that of the structural engineer since a ship is really a structure similar in many respects to a bridge or a building. The marine engineer is also a specialist in propulsion and in diesel engine operation — the usual motive power for modern vessels. In fact, he needs to be a specialist in heating, ventilation, refrigeration, electrical engineering, and a half-dozen other fields but, since this is manifestly out of the question, he usually acts as supervisor to a group of engineers — each being a specialist in some problem of ship design. Naval architects are trained at only a few colleges since the demand for their services is not great in peacetime.

Engineering specialties

There is a large number of engineering specialties for which courses of study are available in certain institutions. We have already mentioned sanitary engineering which is one phase of public health engineering. Biological engineering extends the scope of these studies to the food industries, where the biological engineer is interested in the handling and packaging of foods and,

especially, in the processing of foods that require fermentation as one step in their preparation.

There are a number of engineering specialties that are directed toward a single industry. The major one has already been mentioned — aeronautical engineering. Others that are considered less critically important today are railroad engineering, automotive engineering, mining engineering, gas engineering, and petroleum-production engineering. Such engineering specialists are much in demand when the industry considered is expanding or rebuilding.

*Mining and
petroleum
engineering*

"A roof prism is used in a gun sight's elbow telescope, which enables an anti-aircraft gunner, for example, to look horizontally into the eyepiece and see his target overhead. The elbow telescope inverts the image; the roof prism's function is to turn the image right side up. Roof prisms are thumb-sized, polished crystals whose two top facets are shaped like a peaked roof. In manufacture, a piece of glass is first sawed roughly to shape, then ground to exact proportions by a delicate hand. In the final product, every facet must be absolutely flat, the right angle between the roof facets must be accurate within one 1800th of a degree."



FIG. 15. ROOF PRISM FOR GUN SIGHT

Courtesy Perkin-Elmer Corp.

The roof prism is the most precise article ever manufactured in quantity. The above quotation is from *Time* magazine, October 4, 1943.

When it becomes stabilized, as in the cases of mining and railroad transportation, the opportunities for young engineers become limited. Other specialties of possible long-term interest are safety engineering, fire-protection engineering, and public-service engineering. You should not pick any specialized field for study until you have familiarized yourself with its purpose and its opportunities. For most persons, the best opportunities are likely to exist in the broad fields of civil, electrical, mechanical, and chemical engineering.

TECHNOLOGY AND GOVERNMENT

When this was written there were 280,000 engineers, 70,000 chemists, 7000 physicists, and perhaps 5000 trained mathematicians in our country. Obviously the key positions in technology are

*Engineers in
politics*

held at any given time by a small part of this group, perhaps by no more than 10,000 or at most 25,000 persons. There have been some misgivings expressed by writers who did not understand engineers and scientists that this small group of professional people might attempt at some time to exert an undue influence upon government. Evidently, if such a small group of technologists could be organized and controlled by unscrupulous leaders, the public safety might be jeopardized because the power supply of the country would be under the control of this small minority. Evidently, too, political pressure could be exercised through such a group.

Public office

Worry on this score seems wholly unnecessary because the entire conception of pressure groups conflicts with the ideal of public service deeply ingrained within the engineering profession. Also, engineers and scientists form the least politically minded group imaginable. The only engineer who became a president of the United States, Herbert Hoover, 1929-1933, had never before run for political office. Few engineers or scientists have been elected to Congress and it is a fact that they are not sought for public office. The reason, it appears, is that the technologist is not trained as a public speaker nor does his work offer him many speaking engagements; he is seldom a spellbinder.

Public speaking

Public speaking is the ability of the lawyer and, of course, the law has always been heavily represented in our elective offices. It would be healthy for the country if more engineers and also scientists could be induced to run for elective office. But, from the practical point of view, until the engineer trains himself as a public speaker he is not likely to be chosen as a candidate because he would stand little chance of being elected. A young engineer is well advised to take advantage of each opportunity to improve his ability to speak well. It will influence every contact that he has with others.

HOW TO BECOME AN ENGINEER

College study

Of course, the most direct procedure for becoming an engineer is to spend four academic years of nine months each in an accredited engineering school, although it has always been possible for an industrious person to study during the summers and to complete the regular engineering curriculum in three years. Shorter courses train technicians who may become engineering assistants, but such short courses are not intended for the education of the professional engineer. The cost of college study varies greatly. A maximum

reasonable range appears to be from four hundred to twelve hundred dollars per academic year. By using the summers as work periods and by serving the university during free hours, many young men practically pay for their own college education. Scholarships and loan funds are also available.

Many engineering schools operate cooperative courses of engineering study. The usual plan is for the student engineer to spend from two to four months in class alternating with similar work periods in industry. An engineering degree can usually be earned in five years of such a cooperative program. Incidentally, the income earned in industry may be nearly adequate to cover all educational expenses. The other common part-time program is that of evening study. Evening classes in engineering and science are offered by most of the universities located in the larger cities. Young people holding daytime jobs come regularly to evening school. By using ten or eleven months of the year for study, it is possible for an employed person to complete the requirements for an engineering degree exclusively in evening school in seven or eight years. *Part-time study*

It is also possible to develop your own background of science and engineering by home study and practical experience. This is the hard way. Every textbook author has received letters from persons who have waded through his text without the benefit of an instructor. Ofttimes such persons have a better grasp of the subject than students who have just passed the course. I should like to encourage home study from standard engineering textbooks. They are harder to use than simple home-study texts, but when you have dug your way through to the end, you will have the confidence that comes from knowing that you have covered the same ground as the engineering college student. If you also make regular use of an engineering library, you may even obtain a better knowledge of the subject than some engineering graduates. *Home study and experience*

American procedures in engineering education follow the belief that at the end of the usual four-year period of college study the engineer should be able to step into industry and perform a useful service; in other words, he should have a practical education. This means that his background of theory has of necessity been neglected in spots and certainly has not been developed as fully as we might wish. Particularly, for those engineers who are to become designers of structures, processes, or machines, and for *Theory, research, and graduate study*

those who wish to conduct research, testing, or development work, a further study of engineering theory is essential. Our graduate schools exist for this purpose. They usually grant the master's degree after one year of postgraduate study and the doctor's degree after three years. Although the jobs for which graduate study is a desirable preparation may represent only about a quarter of all positions in engineering, for the person with a scientific turn of mind these are the most attractive opportunities.

*Your job in
engineering*

The engineer deals with men, money, methods, and materials. He plans, constructs, operates, and produces. Wherever capital is invested or government works are constructed, you will find the engineer. Thus he moves into foreign service and returns home when the job is finished. The engineer is behind the man behind the flight because he designed the airplane. But he is also in front of the man on the front line of operation for he is out ahead constructing a new airport. Wherever he is, you may be sure that responsibility follows him. And because of his willingness to accept responsibility and his ability to produce under difficult conditions, the engineer soon finds himself in an administrative position. After twenty years of practice, more than one-half of all engineers have become administrators. It is literally true that in order to remain an engineer, the engineer must often refuse advancement in salary and position. Strange as it may seem, some do.

Chemistry

THE SCIENCE OF MATTER AND MOLECULES

Every engineer and every student of the physical sciences is expected to understand the essentials of chemistry. Perhaps only the chemist, the chemical engineer, the sanitary engineer and the metallurgist come into daily contact with chemistry, but all scientists and engineers should be able to speak the language of chemistry and all must understand its fundamental concepts.

Chemistry is closely related to the other sciences, to the various fields of engineering, and to the routine of your daily life. The chemical process of digestion makes life pleasant, when it is efficient; the chemistry of the movie film is necessary to your entertainment; and chemistry makes your motorcar possible by contributing alloy steels, storage batteries, lacquer finishes, high-octane gasoline, good rubber tires, and near-perfect lubricants. *Chemistry in daily life*

Think of surgery without anesthetics and antiseptics; of airships without light, strong alloys, and without gas or gasoline; of clothing without dyes; painting without colored pigments. We might almost as well try to build bridges and office buildings without steel and concrete; to dig tunnels and a Panama Canal without explosives.

THE FIELD OF CHEMISTRY

Chemistry treats of the composition and changes in composition of substances. It is also concerned with their properties and their energy relations. We may go further and state that chemistry deals with the detection of substances, their separation from mixtures, and their preparation. *What is chemistry?*

The *detection* of substances is managed so well by chemists that, for example, one part of poisonous mercury vapor in twenty million parts of air may be recognized by the blackening of paper coated with sensitive selenious sulfide. Safe operation of the mercury vapor turbine in power plants is made possible by such a test.

The *separation* of a desired substance from incidental impurities is illustrated well by the winning of gold from the sand or rock

in which it is found and by the removal of the rare gas krypton from air in which it occurs to the extent of one part in one million.

The *preparation* of desired substances is so familiar to the chemist that he makes thousands of useful compounds not found in nature.

*Energy trans-
formations*

Physics is largely a study of the various forms of energy such as light, heat, electricity, sound, mechanical energy, and so on, while chemistry deals more with matter. However, each science finds it necessary to include both matter and energy in its field. Chemistry, like physics, is fundamental.

Matter and energy are closely related. Matter is difficult to define with exactness. We usually consider matter as anything that has mass or weight and that occupies space.

Chemical energy

The term *energy* is applied to work or to anything that can do work. Therefore light, heat, motion, and electricity are forms of energy. Chemical energy locked up in coal or in beefsteak may be released for our benefit.

The free transformation of one form of energy to another is



FIG. 16 APPARATUS SET-UP AND REAGENT BOTTLES IN A CHEMICAL LABORATORY

This is a laboratory demonstration of the production of high-octane or aviation gasoline.

vital to our existence and welfare. Chemical energy stored up in coal is released as heat by combustion. Chemical energy contained in our food is converted into the motion and heat of our bodies. When the composition of a given portion of matter is changed, energy may be released yet, conversely, light, heat, and electrical energy may be applied to a portion of matter so as to force a change in composition. Heat applied to limestone in the kiln changes it into lime and carbon dioxide.

The *Law of the Conservation of Energy*, which holds that in all ordinary transformations energy is neither created nor destroyed, is merely a generalized statement of a great many observations.

A *substance* is a particular kind of material with specific properties, such as gold, sugar, common salt, or sulfur; a *body* is merely a definite portion of material such as a bottle, a kettle, a stove, or a statue. *Substances*

Substances that we can decompose into two or more simpler substances are called *compounds*. We recognize limestone as a compound substance because on being heated in the limekiln it breaks down into quicklime and carbon dioxide. There are nearly 500,000 known compounds and millions of new ones can be made when desired. *Elements and compounds*

An *element* is a substance that cannot be decomposed into two or more substances by our present ordinary chemical methods. Until 1808 quicklime, now recognized as calcium oxide, was called an element, but in that year Sir Humphry Davy prepared calcium and showed that it unites with oxygen of pure dry air to form a white solid identical with quicklime. There was then no denying that quicklime could be prepared by the union of two simpler substances, and is, in fact, a compound of calcium and oxygen.

Although 96 elements are known (all above 92 coming from atomic research), many are rare. Oxygen in its compounds makes up one-half the known crust of the earth and silicon one-fourth. In contrast, it takes seventy-four elements to make up only 1 per cent of the earth's crust. The table on page 62 lists all elements and gives their common abbreviations or symbols along with *atomic numbers* and *atomic weights* which will be discussed later.

To identify different substances we must know, not size and shape, but such characteristics or properties as color, odor, taste, hardness, crystalline form, melting point, boiling point, density, solubility in water or in other solvents, ability to conduct electric- *Properties of substances*

ity, and index of refraction. We may even ask if the given substance burns in air. Hydrogen sulfide, once smelled, is never forgotten and never separated from the thought of rotten eggs.

In fact, we pay money for specific properties of substances. Copper would not sell normally for 10 cents a pound or more to the amount of over a million tons in an ordinary year if it were not such an excellent conductor of electricity. Obviously, if glass were opaque or if it dissolved readily in water, it would have no market for use as windowpanes.

Changes

Changes in properties seem to be necessary to life itself. Earth, air, and water are transformed by nature's chemical processes into green grass, trees, flowers, and yet each year these beautiful new products decay and return to earth, air, and water.

A *physical change* is a more or less temporary change in certain properties such as color, density, conduction of electricity, and ductility. When the original conditions are restored, the original properties return. Ice may be melted and the resulting liquid water changed into steam without any change in the composition of the water.

A *chemical change* is a very definite permanent change of certain properties with formation of new substances and is always accompanied by a gain or loss of energy. Wherever a chemical change takes place we have a *chemical reaction*.

Lavoisier as early as 1785 believed that in chemical reactions the mass of the system is not changed (*Law of the Conservation of Mass*). Naturally, this belief came from careful weighing of the reacting materials and all the resulting products. In the instance of burning wood, the gaseous products as well as the ashes must be caught and weighed.

Pure substances

Every pure compound has a definite composition by weight. Water from any portion of the earth, if pure, is composed of hydrogen and oxygen in the proportion of 1 part of hydrogen to 7.94 parts of oxygen by weight. If other elements are present, they represent impurities and can readily be separated from the water. Any mixture of alcohol and water might look like water, but it would not be a pure substance. Incidentally it would not boil at 100° C. under 1 atmosphere pressure as does water.

Mixtures and compounds

The difference between pure compounds and mechanical mixtures is evident from the *Law of Definite Composition*. The proportions in any mechanical mixture may be varied considerably with-

out any abrupt change in properties, as in a sand-sugar mixture, but if the proportions of carbon, hydrogen, and oxygen in pure sugar were changed, the substance would no longer be sugar.

In a mixture the properties are the sum of the properties of the constituents, as demonstrated so tastefully in a well-flavored soup. The properties of a compound, however, are nearly independent of the properties of the substances used in making it.

A classic illustration of the difference between a mechanical mixture and a chemical compound is found in an experiment with iron and sulfur. If very fine iron filings are mixed with flowers of sulfur, it may be shown that the sulfur still retains its own characteristic properties, as does the iron, and that each may be removed by simple mechanical means. A magnet attracts the iron as if the sulfur were not present. Carbon disulfide dissolves the sulfur out of the mixture as if the iron were not present and, on filtration, a clear solution (the filtrate) is obtained. This filtrate yields the sulfur if it is allowed to evaporate.

The mixture of iron and sulfur may be heated until a hard black solid is obtained which is not attracted by a magnet and from which sulfur is not dissolved by carbon disulfide. Apparently neither free iron nor free sulfur is present — *they have reacted to form a compound.* *Iron sulfide*

The experiment goes as described if the two elements are taken in a very definite proportion. If more iron than is called for by that proportion is used, some free iron is left over and may be removed, leaving the pure product of the reaction. If more sulfur than is called for is used, some sulfur is left over and may be removed. The hard black solid obtained in the above experiment is a *compound* called *iron sulfide*. It has several properties quite different from those of iron or sulfur.

This and similar experiments show us that elements react in very definite proportions to form compounds.

1. How did the capture and control of fire serve man as a tool?
2. How do you recognize copper, lead, paraffin, salt, vinegar, ammonia, water, alcohol, gold, silver, soap, gasoline?
3. What properties make rubber useful? Gold? Diamonds? Water? Asphalt?
4. Give a few illustrations of physical and chemical changes.
5. What type of change is observed when a rubber band is stretched? When a rubber band is burned?
6. Define a body, a substance, a compound, an element, a chemical reaction.

Problems

7. When sand, lime, and soda are fused together to form glass, is the change physical or chemical?

8. How could you separate a mixture of oxygen and ammonia? Ammonia is very soluble in water.

THE ATOMIC THEORY

Dalton's atomic theory

Dalton, in 1803, was convinced that "the ultimate particles of a pure substance, simple or compound, are alike in size and weight."

The smallest particles of these compound substances formed by the union of two or more atoms of different elements we call *molecules*.

In other words, a molecule is the smallest particle of any pure substance with the same composition as any larger portion. If a molecule of sugar were taken apart, these smaller particles would no longer taste like sugar, they would in fact be atoms of carbon, hydrogen, and oxygen. The smallest particles of elements taking part in chemical reactions we call *atoms*. A reaction, then, is merely a regrouping of the atoms in new combinations. Dalton pictured his "ultimate particles" or atoms as definite, concrete grains of matter, indivisible, unaffected by the most violent chemical change.

Conservation of mass

This theory explains why, during a reaction between substances in a closed vessel, there is neither gain nor loss of weight (*Law of Conservation of Mass*). All the atoms of the reacting substances are accounted for in the new substances.

It explains also the *Law of Definite Composition*, for when one element unites with another the combination always takes place between definite numbers of atoms of each kind. For example, if two atoms of hydrogen unite with one atom of oxygen to form a molecule of water, it surely follows that the percentage of oxygen and hydrogen by weight in pure water is always the same.

Molecules are so small that if we empty one liter of water into the ocean, with thorough mixing, and then dip out one liter, we will recapture 2220 of the original molecules of water added to the ocean. A row of 40,000,000 molecules in contact would measure one inch.

Modern atomic theory

Based on study of the X-ray tube and of radium we have come, since 1897, to a more modern conception of the atom, yet Dalton's theory served us well for nearly a century.

The first step was the recognition of cathode rays in the X-ray tube (Fig. 17) as a stream of negatively charged particles (*electrons*)

only $\frac{1}{1837}$ times the mass of the hydrogen atom, the lightest of all atoms. Next came the discovery of the *proton* (a positive charge of electricity with a mass practically that of the hydrogen atom). More recently we discovered another building brick of the atom called the *neutron* (an electrically neutral unit with the mass of the proton).

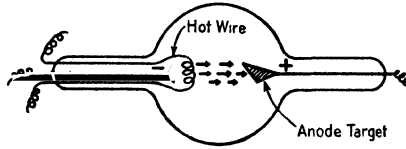


FIG. 17. THE COOLIDGE X-RAY TUBE

Electrons called "cathode rays" are given off by the heated cathode. Their impact upon the anode target produces X-ray radiation.

We now feel certain that the ordinary neutral atom is a miniature solar system with a compact central nucleus made up of neutrons, a number of free protons, and enough planetary electrons to balance electrically the protons in the nucleus (Fig. 18). So small are these three fundamental units, neutrons, electrons, and protons, that the atom is mostly space. So, too, is the solar system.

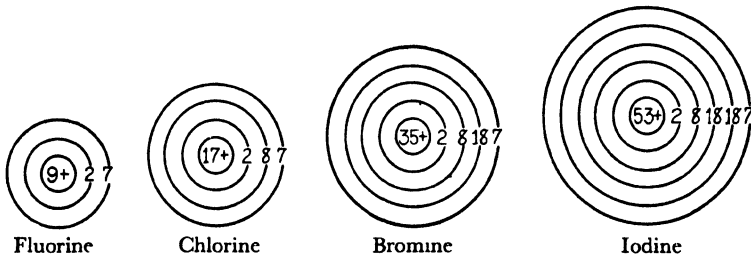


FIG. 18. PLANETARY SHELLS OF ELECTRONS AROUND ATOMIC NUCLEI

Positive protons in any atomic nucleus are electrically balanced by an equal number of negative planetary electrons in the electron shells or "energy levels." Neutrons in the nucleus are omitted here for simplicity.

If the nucleus of an atom were magnified to a 2-in. sphere, the nearest electron would be 2000 ft. away. The gold atom, according to Rutherford, has 100,000 times the diameter of its own nucleus.

In all probability the planetary electrons revolve at great velocities, 150,000 miles per second, around the nucleus, some in circles and some in elliptical orbits (Fig. 19).

The simplest atom is hydrogen, made up of a single proton and a single electron. (The most complex atom is uranium, with 92 protons, 92 electrons, and 146 neutrons.) Since the hydrogen atom weighs 1.008 and the electron mass is only $\frac{1}{1837}$ of the hydrogen

Mass of
proton

mass, it is evident that *the proton has a mass of approximately one*. In other words, the mass of a few electrons is negligible, and practically all the weight, or mass, of an element is concentrated in the nucleus, a relatively minute fraction of the volume of the entire atom.

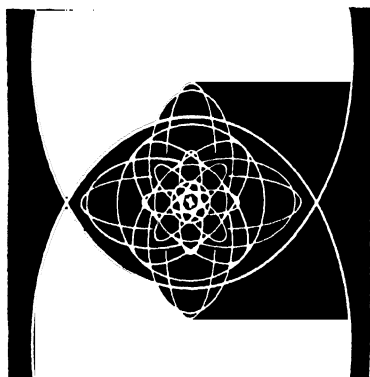


FIG. 19. COMPLEX ORBITS OF ELECTRONS IN THE TANTALUM ATOM

It is helpful at this stage to note that with many common atoms the outer shell of electrons may contain from one to eight electrons. In chemical reactions this number is often raised to eight either by a transfer of electrons from another atom (electrovalence) or by a sharing process (covalence).

Exchanging
electrons

Although the electrons (negative) are attracted by the nuclear protons (positive), some of the outer planetary electrons are known to escape to other atoms. Some of the most active atoms, like those of the metals potassium and sodium, have a very great tendency to give away outer electrons, while some of the most active non-metallic atoms like those of chlorine and fluorine exhibit a reluctance to give away electrons, but show an eagerness to take on electrons. Sodium reacts with chlorine, because the chlorine atom

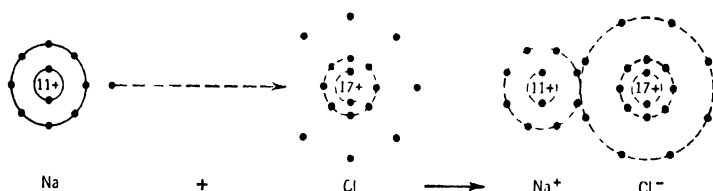


FIG. 20. REACTIONS BETWEEN ATOMS OF SODIUM (Na) AND CHLORINE (Cl) SHOWING ELECTRON TRANSFER

eagerly takes on the electron that the sodium atom is so willing to discard. (See Fig. 20.) Chemical reaction takes place and a molecule of sodium chloride, common salt, is formed.

Problems

1. If a neutral atom contained 40 protons as well as many neutrons in the nucleus, how many electrons must be present? Where are those electrons?
2. If the single electron in the hydrogen atom were knocked out, what would be left? What would be its weight?

THE LANGUAGE OF CHEMISTRY

According to our present conception, symbols stand not only *Symbols and formulas* for the elements but for definite parts by weight of the elements. For example, H represents a single atom of hydrogen and it may also mean the atomic weight in grams, that is, 1.008 g. The symbol O means either a single atom of oxygen or 16 g. of oxygen. See the table on the following page. On page 86 we explain the derivation of atomic weights.

The formula O_2 for the oxygen gas tells us that there are two oxygen atoms in each molecule with a molecular weight of $2 \times 16 = 32$. To represent the formula of a compound we simply join the symbols of the constituent elements together and attach small subfigures to represent the actual numbers of atoms in each molecule. H_2O as a formula for water tells us that two atoms of hydrogen and one atom of oxygen make up a single molecule of water. The weight of the whole molecule, 18.016, is of course the sum of the weights of all the atoms in it. ($2 \times 1.008 + 16.0 = 18.016$.) H_2SO_4 , the formula for sulfuric acid, tells us that a molecule of the acid contains two atoms of hydrogen, one of sulfur (atomic weight, 32.06), and four of oxygen. The formula weight is 98.076 g.

With the use of symbols and formulas we can now represent *Chemical equations* chemical changes (reactions) very simply and clearly. For example, the equation



tells us that one molecule of copper oxide reacts with one molecule of hydrogen to form one molecule of water and one atom of copper. The weight of one molecule of CuO is obviously the sum of the weights of all the atoms in it. $Cu = 63.57$ and $O = 16.0$. Therefore $CuO = 79.57$. The weight of $H_2 = 2.016$; of $H_2O = 2.016 + 16$, or 18.016; and of $Cu = 63.57$.

Before introducing chemical problems it may be helpful to *Metric units* define a few metric units.

Metric Units

1 meter = 1000 millimeters (mm.) = 39.37 inches

1 meter = 100 centimeters (cm.)

1 liter (1 l.) = 1000 cubic centimeters (cc.) = 1.06 quarts

1 cubic foot = 28.32 liters

1 kilogram (kg.) = 1000 grams (g.) = 2.2046 pounds

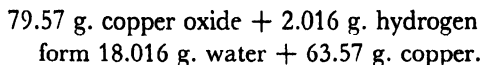
1 cc. of water weighs 1 g. (at $4^\circ C.$)

ELEMENTS AND ATOMIC WEIGHTS

| | <i>Symbol</i> | <i>Atomic Number</i> | <i>Atomic Weight</i> | | <i>Symbol</i> | <i>Atomic Number</i> | <i>Atomic Weight</i> |
|------------|---------------|----------------------|----------------------|--------------|---------------|----------------------|----------------------|
| Aluminum | Al | 13 | 26.97 | Molybdenum | Mo | 42 | 95.95 |
| Antimony | Sb | 51 | 121.76 | Neodymium | Nd | 60 | 144.27 |
| Argon | A | 18 | 39.944 | Neon | Ne | 10 | 20.183 |
| Arsenic | As | 33 | 74.91 | Nickel | Ni | 28 | 58.69 |
| Barium | Ba | 56 | 137.36 | Nitrogen | N | 7 | 14.008 |
| Beryllium | Be | 4 | 9.02 | Osmium | Os | 76 | 190.2 |
| Bismuth | Bi | 83 | 209.00 | Oxygen | O | 8 | 16.0000 |
| Boron | B | 5 | 10.82 | Palladium | Pd | 46 | 106.7 |
| Bromine | Br | 35 | 79.916 | Phosphorus | P | 15 | 30.98 |
| Cadmium | Cd | 48 | 112.41 | Platinum | Pt | 78 | 195.23 |
| Calcium | Ca | 20 | 40.08 | Potassium | K | 19 | 39.096 |
| Carbon | C | 6 | 12.010 | Praseodymium | Pr | 59 | 140.92 |
| Cerium | Ce | 58 | 140.13 | Protactinium | Pa | 91 | 231 |
| Cesium | Cs | 55 | 132.91 | Radium | Ra | 88 | 226.05 |
| Chlorine | Cl | 17 | 35.457 | Radon | Rn | 86 | 222 |
| Chromium | Cr | 24 | 52.01 | Rhenium | Rc | 75 | 186.31 |
| Cobalt | Co | 27 | 58.94 | Rhodium | Rh | 45 | 102.91 |
| Columbium | Cb | 41 | 92.91 | Rubidium | Rb | 37 | 85.48 |
| Copper | Cu | 29 | 63.57 | Ruthenium | Ru | 44 | 101.7 |
| Dysprosium | Dy | 66 | 162.46 | Samarium | Sm | 62 | 150.43 |
| Erbium | Er | 68 | 167.2 | Scandium | Sc | 21 | 45.10 |
| Europium | Eu | 63 | 152.0 | Selenium | Se | 34 | 78.96 |
| Fluorine | F | 9 | 19.00 | Silicon | Si | 14 | 28.06 |
| Gadolinium | Gd | 64 | 156.9 | Silver | Ag | 47 | 107.880 |
| Gallium | Ga | 31 | 69.72 | Sodium | Na | 11 | 22.997 |
| Germanium | Ge | 32 | 72.60 | Strontium | Sr | 38 | 87.63 |
| Gold | Au | 79 | 197.2 | Sulfur | S | 16 | 32.06 |
| Hafnium | Hf | 72 | 178.6 | Tantalum | Ta | 73 | 180.88 |
| Helium | He | 2 | 4.003 | Tellurium | Te | 52 | 127.61 |
| Holmium | Ho | 67 | 164.94 | Terbium | Tb | 65 | 159.2 |
| Hydrogen | H | 1 | 1.0080 | Thallium | Tl | 81 | 204.39 |
| Indium | In | 49 | 114.76 | Thorium | Th | 90 | 232.12 |
| Iodine | I | 53 | 126.92 | Thulium | Tm | 69 | 169.4 |
| Iridium | Ir | 77 | 193.1 | Tin | Sn | 50 | 118.70 |
| Iron | Fe | 26 | 55.85 | Titanium | Ti | 22 | 47.90 |
| Krypton | Kr | 36 | 83.7 | Tungsten | W | 74 | 183.92 |
| Lanthanum | La | 57 | 138.92 | Uranium | U | 92 | 238.07 |
| Lead | Pb | 82 | 207.21 | Vanadium | V | 23 | 50.95 |
| Lithium | Li | 3 | 6.940 | Xenon | Xe | 54 | 131.3 |
| Lutecium | Lu | 71 | 174.99 | Ytterbium | Yb | 70 | 173.04 |
| Magnesium | Mg | 12 | 24.32 | Yttrium | Y | 39 | 88.92 |
| Manganese | Mn | 25 | 54.93 | Zinc | Zn | 30 | 65.38 |
| Mercury | Hg | 80 | 200.61 | Zirconium | Zr | 40 | 91.22 |

It is a very useful practice when working problems to consider these weights as expressed in grams. Of course, the relative values are just the same whether expressed in the minute units of *atomic*

weights, or in *grams*, or *tons*. Expressed in grams we should read the equation thus:



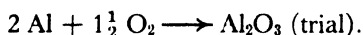
These so-called equations name the reacting substances and their products. "Balancing the equations" consists in prefixing such numbers before the molecular formulas as indicate the relative number of molecules concerned. *Balancing equations*

We admit that unless the student knows from his own (or some other chemist's) observations and tests in the laboratory just what the products are, it will be mere guesswork for him to attempt to balance the equation.

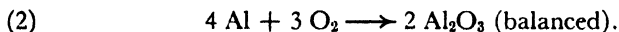
Since facility in balancing equations comes only with considerable practice, it might be well to try another example. Suppose we tell you that aluminum powder can be burned in air or in oxygen gas and that the product is aluminum oxide, Al_2O_3 . The oxygen molecule is always represented as O_2 .



To have any aluminum oxide at all we must write down at least one molecule, that is, Al_2O_3 . This forces us to start with 2 Al on the left and also with three atoms of oxygen. It is hardly consistent to represent 3 O as $1\frac{1}{2} \text{ O}_2$ because *half molecules cannot occur free*. However, as a trial we may write *Balancing by trial*



To obtain whole numbers, each side of the equation should be multiplied by 2, giving us



A chemical equation, then, must represent the reacting substances, the products formed, and the relative weights involved. It does not, however, name the conditions — such as temperature — necessary to cause the reaction to take place, nor does it tell how rapidly it occurs nor to what extent the reaction develops.

If the correct formula for black iron oxide is Fe_3O_4 , it is a simple matter to calculate the percentage of iron, for example. From the formula we learn that the molecular weight is the sum of three times the atomic weight of iron (3×55.85) plus four times the atomic weight of oxygen (4×16), or a total weight of 231.55. *Percentage composition of compounds*

The iron fraction is evidently $\frac{167.55}{231.55} = 0.7236 = 72.36$ per cent of the oxide.

Problems

1. The hard sapphire (except for traces of coloring matter) is aluminum oxide, Al_2O_3 . Synthetic sapphires are now made by the thousands for jewel bearings in the control instruments of airplanes and battleships and therefore are of strategic importance. Explain the meaning of the formula, Al_2O_3 .

2. Red lead, so common in the priming paint coating on metals, is Pb_3O_4 . Explain this formula. What percentage of the weight of Pb_3O_4 is oxygen?

Ans. 9.33%.

3. What per cent of lead is found in pure lead nitrate, $\text{Pb}(\text{NO}_3)_2$?

Ans. 62.56%.

4. What is meant by the formula, $\text{C}_2\text{H}_6\text{O}$ (alcohol)? Determine the weight of the molecule

Ans. 46.07.

5. (a) What weight of red iron oxide, Fe_2O_3 , may be obtained from 250 g. of oxygen and sufficient iron? (b) What weight of black Fe_3O_4 ?

Ans. (a) 831.77 g.; (b) 904.5 g.

Note. Atomic weights of lead, Pb, nitrogen, N, carbon, C, iron, Fe, and other elements are given on page 62.

OXYGEN

Occurrence

Oxygen makes up one-half of all matter that we know. Two-thirds of the human body is combined oxygen. Water contains 89 per cent; ordinary clay, sand, limestone, and granite about 50 per cent. One-fifth of the air is free oxygen, the other four-fifths are nitrogen and other gases. This is the only occurrence of free oxygen except the little that is found dissolved in water. Elsewhere it is found in compounds.

Preparation of oxygen

There are at least four general methods of preparation of oxygen, but its most important commercial source is the atmosphere. Air is chiefly a mixture of oxygen and nitrogen but separation is not easy.

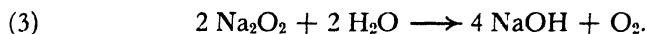
1. Air Liquefaction Process. The separation is accomplished by liquefying air and then allowing it to evaporate. The more volatile nitrogen escapes first, leaving the oxygen, which is usually stored in strong steel cylinders at 100 atmospheres pressure. The gas thus obtained naturally contains a little nitrogen, which does no harm for commercial uses. This modern process is now the only commercial method used on a large scale.

2. Electrolytic Method. The *electrolysis of water* (decomposition by passage of an electric current in the presence of a little added sulfuric acid) releases hydrogen at the cathode and oxygen at the

anode. This process is used commercially to some extent because the product is very pure and because there is considerable demand for the hydrogen.

3. Thermal Decomposition of Oxygen Compounds. The laboratory methods, where convenience is more important than cheapness, in most cases consist in heating oxygen compounds. When potassium chlorate is heated to 400° C. or higher, we obtain $2 \text{KClO}_3 \longrightarrow 2 \text{KCl} + 3 \text{O}_2$. With addition of a mere pinch of black manganese dioxide, MnO_2 , the reaction takes place at 200° C., your introduction to a *catalyst* (chemical reaction helper). *Laboratory methods*

4. Sodium Peroxide Method. Sodium peroxide (made by heating sodium in air) reacts with water to form oxygen and sodium hydroxide.

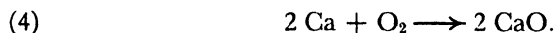


This method is expensive but very convenient since the water may be allowed to drop slowly from a dropping funnel on the peroxide, releasing oxygen as needed.

Oxygen may be liquefied at and below -118°C. , which may then be called its *critical temperature*. At higher temperatures it remains a gas. At the critical temperature, 50 atmospheres pressure are required to liquefy oxygen; hence this is the *critical pressure*. Of course, at temperatures below -118°C. , a lower pressure than 50 atmospheres (735 lb. per sq. in.) would cause liquefaction. *Physical properties*

Oxygen is only moderately active chemically at ordinary temperatures, but at elevated temperatures it unites with nearly every element. *Chemical properties*

If the metal sodium is heated and then placed in a jar of oxygen, it burns brilliantly with a yellow flame. Calcium burns almost as brilliantly, forming ordinary quicklime or calcium oxide.



An oxide is a compound of oxygen with one other element. *Oxides*
Water is an oxide of hydrogen; and rust, Fe_2O_3 , is an oxide of iron. When substances react, heat is usually (but not always) liberated. If the reaction becomes so vigorous that light is also produced, it is called *combustion*. As a rule this term is applied to a union of oxygen with other substances, but many other reactions produce light.

Reactions accompanied by the evolution of heat are *exothermic*, and those that require the application of external heat to sustain

them are *endothermic*. In measuring heat we use as a unit the *calorie (cal.)*, the amount of heat required to raise the temperature of 1 g. of water 1° C. (at 15° C.). The large calorie (Cal.), one thousand times as great, is sometimes called the *kilogram calorie*.

*Heat of
combustion*

The heat of combustion of any pure substance is the number of calories liberated when the formula weight (molecular weight in grams) is burned. For example, a molecule of carbon monoxide weighs 28.01, hence its formula weight is 28.01 g. When 28.01 g. of this gas burns, it liberates 67,623 calories and becomes carbon dioxide.

When enough hydrogen is burned in air or oxygen to form 18.016 g. of water, 68,310 cal. of heat are evolved and exactly this much energy as heat or electricity is required to decompose 18.016 g. of water into hydrogen and oxygen. Obviously the substances which give off the greatest *heat of formation* require the greatest amount of external energy for their *decomposition*.

Uses of oxygen

Oxygen is a most useful element. Essential to animal life in burning out waste tissue and furnishing heat, beneficent in its purification of sewage and other waste matters that would otherwise be a nuisance and a menace to health, vital in developing heat, light, and power from fuel — it is, in short, the central element. Oxygen, compressed in steel cylinders, is used for the oxy-hydrogen (or oxyacetylene) torch. This is a burner by which hydrogen or acetylene is burned in pure oxygen instead of in air. The extremely hot flame produced cuts through iron like a hot knife through butter (by melting and burning the iron).

A college freshman, impressed by man's great dependence upon an atmosphere containing 20 per cent oxygen, asked, "What did we breathe before Priestley discovered oxygen in 1774?"

*Influencing
reaction speed*

The speed of chemical reactions is greatly influenced by temperature, by the concentration of the reacting substances, by the presence of catalysts, and by the amount of contact surface. A rise in temperature of 10° C. doubles or trebles the speed of most chemical reactions. A rise from 20° C. to 200° C. means a million-fold increase in reaction speed. This is understood when we remember that a rise in temperature means that the molecules move faster, consequently hit each other harder and oftener. A reaction requiring 1 sec. for completion at 200° C. will require 11 days at zero. Admiral Byrd found that he could not use flashlights in the

south polar regions because the intense cold slowed down the necessary chemical action of the dry batteries.

With two substances that cannot be mixed thoroughly, such as a solid and a gas, the amount of surface of the solid is of great importance in combustion. Splinters or shavings burn much faster than a solid log — a matter of “opportunities for contact.” Coal dust is now blown into cement kilns with air and burned like a spray of oil.

Fine, dry dust from flour, starch, grain, sulfur, coal, leather, alfalfa, sugar, soap, cocoa, fertilizer, or feeds may also be *inflammable*. A dust explosion is merely a *fast chemical reaction* with formation of expanding gases. *Dust explosions*

A *catalyst* is a substance that aids or retards a reaction without itself being permanently changed or consumed. In some cigar lighters the vapors of methyl alcohol mixed with air pass over a thin roughened platinum wire. There is little oxidation without the platinum. Perhaps the oxygen atoms become “excited” by contact with the platinum catalyst with a resulting displacement of some planetary electrons out of their regular orbits. Manganese dioxide aids the release of oxygen from potassium chlorate at 200° C. by reacting first to form another compound that decomposes readily, releasing oxygen and the original manganese dioxide. Since the manganese dioxide is not used up in the reaction, it is a catalyst. *Catalysis*

Catalysts are of great importance in industry; for example, in the manufacture of sulfuric acid, nitric acid, ammonia, and aviation gasoline.

1. Why do we select oxygen as the first element to study?

Problems

2. How much phosphoric acid, H_3PO_4 , is formed by reaction of 112 g. of phosphorus pentoxide, P_2O_5 , with sufficient water?

Ans. 154.64 g.

3. If oxygen suddenly became very soluble in water, what changes would take place in the world?

GASES, LIQUIDS, AND SOLIDS

Boyle (1660) observed that *the volume occupied by a given sample of any gas kept at a constant temperature varies inversely with the pressure*. Accurate measurements showed that if the pressure was doubled on a given volume of a gas (keeping the temperature constant) the volume was reduced to one-half. With three times the pressure the volume was reduced to one-third. This is illustrated by Fig. 21. *Boyle's gas law*

This generalization can be formulated mathematically thus:

$$\frac{P_1}{P_2} = \frac{V_2}{V_1} \quad \text{or} \quad P_1 V_1 = P_2 V_2$$

from which we obtain

$$(5) \quad V_1 = V_2 \times \frac{P_2}{P_1}$$

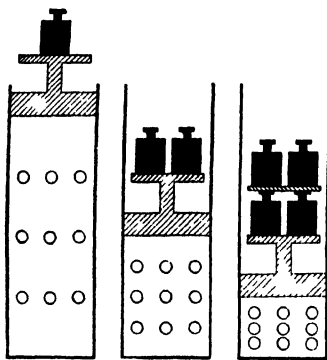


FIG. 21. BOYLE'S LAW ILLUSTRATED
The volume occupied by a given number of molecules of a gas varies inversely as the pressure

Here P_1 and V_1 are the first pressure and volume, and P_2 and V_2 any other pressure with its accompanying volume. Since the product $P_1 V_1$ is equal to any other *pressure-volume product*, such as $P_2 V_2$ for the same weight of the gas, it is sometimes said that the product of the pressure times the volume of a gas is a constant value.

It is our custom to refer to the average atmospheric pressure at sea level (enough to hold up a 760-mm. column of mercury in a barometer) as the standard pressure.

*Volume
calculation*

We do not always measure gases at exactly 760 mm. pressure, but we calculate from Boyle's law what the volume would be if the pressure were changed to the standard 760 mm. For example, 400 cc. of air at 740 mm. would occupy what volume at 760 mm.? Using the formula

$$V_1 = V_2 \times \frac{P_2}{P_1}$$

we may let V_1 represent the volume to be calculated and P_1 the standard pressure, 760 mm. To make it clearer,

$$(6) \quad V_1 \text{ (standard)} = V_2 \text{ (observed)} \times \frac{P_2 \text{ (observed)}}{P_1 \text{ (standard)}}$$

Substituting known values, we have

$$V_1 \text{ (standard)} = 400 \times \frac{740}{760} = 389 \text{ cc.}$$

High vacuums

High vacuums, or even partial vacuums, minister to our comfort and serve well in many ways. They made possible the original incandescent lamp (although the type has since changed), the X-ray tube, the radio tube, the photoelectric cell of television, and the thermos bottle. By exhausting into a vacuum we have increased the efficiency of steam engines and turbines.

Charles in 1787 observed that on cooling any gas from 0°C. to -1°C. it contracted $\frac{1}{273}$ of its volume. It was a natural inference that if cooled to -273°C. its volume would be nothing at all. However, in practice all gases become liquid before they have been cooled to -273°C. , and after they become liquid they no longer obey the gas laws. There is a real convenience, however, in calling this imaginary point of no transitional motion of molecules and no gaseous pressure, *absolute zero*, -273°C. In honor of the eminent physicist, Lord Kelvin, this temperature is usually termed zero degrees Kelvin and written 0°K. *Charles's law*

In recent years temperatures lower than 0.01°K. have been reached, giving reality to the conception of absolute zero.

Charles's law is merely an observation that *the volume of a given sample of gas varies directly as the absolute temperature (if the pressure is kept constant)*. For example, to double the volume that a gas occupies at 20°C. by heating we should double the absolute temperature. This temperature is $20^{\circ} + 273^{\circ} = 293^{\circ}\text{K.}$; and $2 \times 293^{\circ} = 586^{\circ}\text{K.}$ or 313°C. *Example of expansion*

When heated to 313°C. (not 313°K.), the gas would have twice its original volume. Charles's law is, therefore, formulated as follows:

$$(7) \quad \frac{V_1}{V_2} = \frac{T_1}{T_2} \quad \text{or} \quad V_1 = V_2 \times \frac{T_1}{T_2},$$

where T_1 and T_2 refer to absolute temperatures. When we speak of a liter of oxygen weighing 1.429 g., we mean a liter at 760 mm. pressure and 0°C. Since gases expand on heating, there can be no accuracy without reference to a definite standard temperature. Obviously, we do not care to measure gases in rooms at 0°C. , but we can apply Charles's law and calculate what the volume would be if the gas were cooled to 0°C. For example, what is the volume of 400 cc. of air measured at $+18^{\circ}\text{C.}$ when cooled to 0°C. ?

$$(8) \quad V(\text{standard}) = V(\text{observed}) \times \frac{273}{273 + t^{\circ}},$$

where t° is the Centigrade temperature. Substituting numerical values, we obtain

$$V(\text{standard}) = 400 \times \frac{273}{291} = 375.25 \text{ cc.}$$

The observed volume of a gas can be brought to 760 mm. and 0°C. by applying both corrections at once. For example, 400 cc. *Standard conditions*

of dry air at 740 mm. and -18°C . would have what volume at standard temperature and pressure?

$$V(\text{standard}) = V(\text{observed as 400}) \times \frac{740}{760} \times \frac{273}{255} = 412.68 \text{ cc.}$$

Partial pressures

Since the total pressure of a mixture of gases (as the air) is the sum of the pressures that would be exerted by each if the others were not present, it is obvious that the barometric pressure of any gas measured must be corrected for that part of the pressure due to water vapor present. A gas collected over water will be saturated at that temperature.

Moisture correction

Suppose the problem is to correct to standard conditions 400 cc. of air collected over water at 740 mm. and 18°C . Now the *observed pressure* of 740 mm. is not all due to the air, for in this case 15.5 mm. of it is due to water vapor.

$$V_1(\text{standard}) = V_2(\text{observed as 400}) \times \frac{740 - 15.5}{760} \times \frac{273}{291}.$$

ACTION OF GASES

Gases must be composed of flying molecules because such a kinetic theory explains the following observed facts:

| <i>Facts</i> | <i>Explanation</i> |
|---|--|
| 1. Gases diffuse and mix readily. | Relatively wide spaces between flying molecules permit diffusion. |
| 2. Gases are very compressible. | A given volume of gas is mostly space. |
| 3. Gas pressure extends in all directions. | The molecules fly in all directions and bombard the walls of the containing vessel. |
| 4. Boyle's law holds. | Pressure crowds molecules closer together. |
| 5. Charles's law holds. | Molecules fly faster with rise in temperature. |
| 6. Boyle's law and Charles's law both fail when gases are nearly liquefied. | The gas laws refer to change in the amount of free space between molecules rather than to total volume. At high pressures the molecules themselves occupy a very appreciable fraction of the total volume. Also attraction between molecules when close together aids compression. |
| 7. Liquefaction of gases | Forcing molecules into rather close contact brings in the aid of molecular attraction in restricting freedom of movement. |

*Facts**Explanation*

8. Evaporation of liquids

Molecules at the surface of a liquid escape beyond the range of attraction of the liquid.

9. Each gas in a mixture exerts the same pressure it would show in absence of other gases.

Molecules have plenty of room for free movement, even in the presence of other kinds of molecules.

Liquefaction of gases by pressure is comparatively easy to accomplish with ammonia, chlorine, carbon dioxide, and sulfur dioxide, but both high pressure and great cooling are required to liquefy such gases as oxygen, nitrogen, hydrogen, and helium.

*Liquefaction
of gases*

LIQUIDS

The boiling point of a liquid is the temperature at which the upward pressure of its vapor equals the downward pressure or weight of the atmosphere above it. Obviously the boiling point varies with the pressure of the atmosphere (Fig. 22). The bubbling observed in any boiling liquid is due to the expansion of bubbles of vapor forming within the liquid. Water has a relatively high heat of vaporization, requiring 540 cal. to change 1 g. of water at 100° C. into steam at 100° C.

The boiling point of any pure liquid at 760 mm. pressure remains constant (100° C. for water, 78.4° C. for alcohol, and so on) until all the liquid has changed into vapor. Not so with mixtures of two or more. For example, alcohol-water may start boiling somewhat above 78.4° C. but the temperature gradually rises, with escape of *vapor of steadily varying composition*, until the *distillate* (condensed vapor) attains a concentration of 95 per cent alcohol in the commercial column still.

Water in the teakettle absorbs heat from the flame while boiling. Liquefied sulfur dioxide in a beaker resting on the table (not on the stove), will boil, absorbing heat from its surroundings, cooling the room as it does in the coils of your refrigerator. The boiling point of

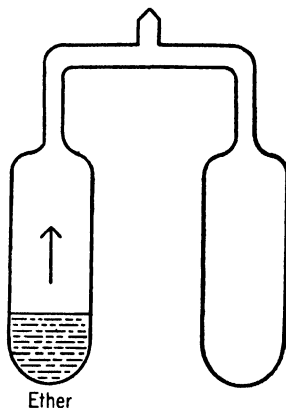


FIG. 22. THE EFFECT OF PRESSURE ON BOILING POINT

When the empty bulb on the right is chilled with dry ice, the ether in the bulb on the left boils (if air had previously been removed from the sealed apparatus). After the ether vapors condense to liquid on the right, the process may be reversed. The movement of gas molecules is decreased by chilling.

Boiling point

liquefied sulfur dioxide is -10°C . and that of liquefied ammonia is -32.8°C .

Distillation

The process of evaporating and recondensing a liquid (by cooling) is termed *distillation*. To expedite matters the liquid is usually boiled, sometimes under less than atmospheric pressure. Vacuum distillation is desirable with liquids or solutions likely to decompose at the normal boiling temperature.

SOLIDS

Freezing point

Liquids, on cooling sufficiently, all lose their fluidity and become solids. They freeze to solids without further lowering of temperature until all the liquid has solidified. When this change of state occurs at a definite temperature, the solid formed is always crystalline.

Water sometimes may be chilled below 0°C . without separation of ice crystals. However, the addition of a minute fragment of ice starts crystallization or freezing. Consequently, we are forced to define the freezing point (same as melting point) as *the temperature at which liquid and solid will remain in contact with each other without change of temperature*.

Heat of fusion

To increase the freedom of movement of the molecules of a solid so that a fluid results, requires increased kinetic energy which involves both mass and velocity. This is introduced by heating the solid. There is a rise in temperature until some liquid is formed and then no further rise until all the solid is melted. To convert 1 g. of ice at 0°C . into water at 0°C ., 79.7 cal. of heat are required. This is termed the *heat of fusion of ice*. The corresponding value for sulfur is 9.37 cal. and for aluminum 76.8 cal. Amorphous solids (such as plastics and glass), unlike crystalline solids, have no definite melting points, no definite heat of fusion.

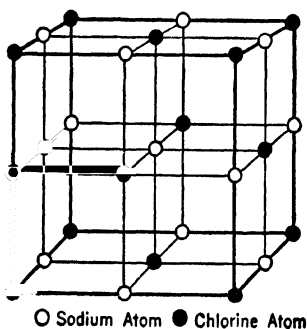


FIG. 23. CUBIC ARRANGEMENT OF ATOMS IN A CRYSTAL OF SODIUM CHLORIDE

Ions in crystals

As Bragg pointed out in 1914, the atoms in crystalline solids are arranged in a geometric pattern or lattice (Fig. 23). The lattice units are not single atoms in all cases. The familiar groups or radicals (such as $-\text{SO}_4$ in Na_2SO_4 and CaSO_4) are found in

some crystalline salts as lattice units and may carry positive or negative charges. Such charged atoms or groups are termed *ions*.

Since the attraction between atoms in a crystal lattice varies inversely as the square of the distance, it is evident that crystals cleave or split most readily along planes where these distances between atoms are greater than in adjoining parts of the solid. This is notably true with mica and graphite. X-ray photographs show that in solid carbon dioxide, CO_2 , each carbon atom is much closer to a pair of oxygen atoms than to the other carbon atoms. In this sense the molecule, CO_2 , is a unit in the space lattice.

The commonest lattice system is the *face-centered cubic* (Fig. 24) with atoms at the corners of a system of equal closely packed cubes and also at the centers of each cube face.

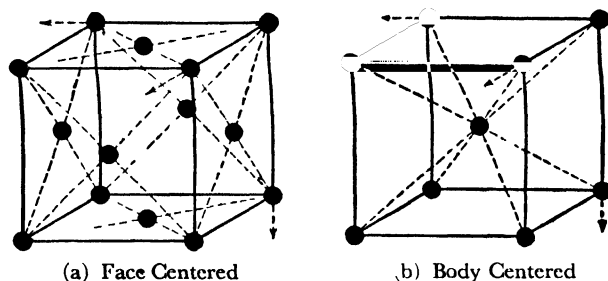


FIG. 24. ARRANGEMENTS OF ATOMS IN CUBIC CRYSTALS

Wyckoff¹ states that X-ray analysis has so far shown that:

1. There is no metal crystallizing in the face-centered system which is not ductile throughout a considerable range of temperature.
2. All "noble" metals (such as gold and silver) are in the face-centered system.
3. All the best conductors of heat and electricity are in the face-centered system.

The ordinary crystal visible to the eye is built up of smaller units or "cells" each having a shape like the larger crystal.

1. In the tropics drinking water is cooled by storing it in large porous jars. Explain how cooling takes place.

2. What is the effect on the air pressure of the tires of a car driven at high speed with a heavy load? Consider the friction of tire on road.

3. Assume that a gallon of gasoline weighs approximately 3200 g. and that C_8H_{18} represents its composition. (Actually gasoline is a mixture of such hydrocarbons.) How many kilograms of water will be formed, as vapor, during the oxidation of this gallon of gasoline in the engine cylinder of your car? The carbon atoms form carbon dioxide and the hydrogen atoms form water. *Ans.* 4.54 kg.

¹Structure of Crystals, Reinhold Publishing Corporation, New York.

4. Calculate the volume at 0° C. and 760 mm. pressure of 450 cc. of a gas measured at 25° C. and 750 mm. *Ans.* 406.8 cc.

5. Ten liters of nitrogen at 20° C. and 1 atmosphere pressure are heated to 500° C. What is the new volume? *Ans.* 26.38 l.

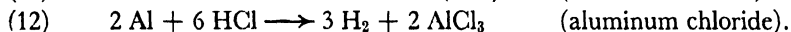
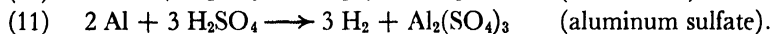
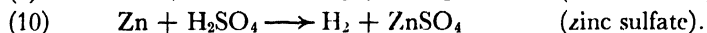
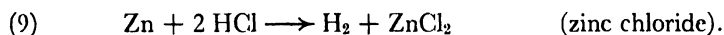
6. Three liters of oxygen at 22° C. and 740 mm. pressure were collected over water. Calculate the volume, dry, at 0° C. and 760 mm. Allow 19.6 mm. for the pressure of water vapor. *Ans.* 2.63 l.

HYDROGEN

In the combined form hydrogen is found in water, plant and animal tissues, petroleum, asphalt, and natural gas. Among laboratory reagents we learn that it is an essential element of all acids and bases. We will discuss six ways in which it may be prepared.

*From metals
and acids*

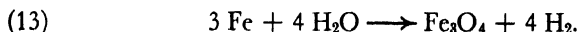
1. Hydrogen is a constituent of such common materials as water, acids, and bases. In the laboratory it is convenient to displace hydrogen from dilute acids by the more active metals. The hydrogen is collected by displacement of water.



When a metal displaces hydrogen from an acid, a *salt*, such as a chloride or a sulfate, is formed. The symbol Zn used above represents zinc; Al represents aluminum, while HCl is the formula for hydrochloric acid.

*From metals
and water*

2. The most active metals, such as potassium and sodium, violently displace hydrogen from cold water. Calcium acts at moderate speed. Less active metals react with water only when heated. Magnesium displaces hydrogen from boiling water if a little of a magnesium salt is present to keep the surface of the metal clean. Steam passed over red-hot iron reacts as follows:



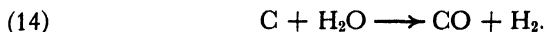
By electrolysis

3. When a current of electricity is passed through water made conducting by the presence of some substance such as sulfuric acid or sodium hydroxide, we are able to collect two volumes of hydrogen at the *cathode*, where the current leaves, to one of oxygen at the *anode*, where the current enters.

From water gas

4. "Water gas," a mixture of CO and H₂, is the principal commercial source of hydrogen in the United States. Water gas is

prepared by the process of passing steam over coke at about 1000°C . yielding a mixture of carbon monoxide and hydrogen.

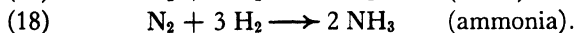
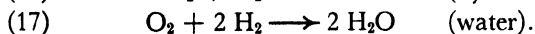
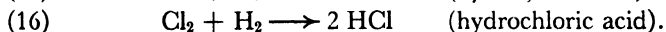
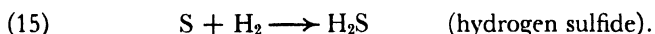


Of course, the coke is soon cooled by the steam below the reaction temperature. Steam is then shut off and a blast of air turned on. Part of the coke burns, heating the remainder up to 1000°C ., and preparing it for another blast of steam

5. Any compound of hydrogen and carbon, such as petroleum, *From hydrocarbons* may be broken down or *cracked* into its elements if sufficiently heated without contact with air.

6. The gas obtained when 1 ton of bituminous coal is turned *From coal gas* into coke contains about 27 lb. of free hydrogen. In Germany the separation of this hydrogen has been an economic success, but water gas is a preferred source in the United States.

Hydrogen, when pure, is a colorless, odorless, tasteless gas. A *Properties of hydrogen* liter weighs 0.08987 g. at standard conditions, so it is the lightest substance known. Consequently, as indicated by Graham's law, it has the greatest speed of *diffusion* of all gases. The loss by diffusion through balloon fabric is a serious problem. Deuterium (heavy hydrogen) is discussed in the section on water. Hydrogen is rather inactive at ordinary temperatures, but at higher temperatures it unites with most of the nonmetals and many of the metals. Some typical reactions are



The hydrogen flame is intensely hot, as 68,390 calories are released when 2.016 g. (formula weight) of hydrogen burn. *Combustion* Hydrogen, when hot, removes oxygen and even some other elements from many compounds. Such a process is called *reduction*, the exact opposite of *oxidation*. For example, $\text{CuO} + \text{H}_2 \longrightarrow \text{Cu} + \text{H}_2\text{O}$.

For many years hydrogen has been used in the oxyhydrogen blowtorch to weld metals or cut up large masses of scrap iron. It also has a large and growing use in hardening liquid fats. The catalytic and high-pressure hydrogenation of petroleum (chemical addition of hydrogen) to yield more gasoline now consumes large quantities of hydrogen. Hydrogen is also used in the reduction of

tungsten from its ores, and in the manufacture of ammonia by direct union of nitrogen and hydrogen (Haber process).

ACTIVITY SERIES OF METALS

(Most active)

Oxidation
of metals

- | | | |
|------------------|------------------|------------------|
| 1. Lithium, Li | 9. Manganese, Mn | 16. Lead, Pb |
| 2. Potassium, K | 10. Zinc, Zn | 17. Hydrogen, H |
| 3. Sodium, Na | 11. Chromium, Cr | 18. Copper, Cu |
| 4. Barium, Ba | 12. Cadmium, Cd | 19. Mercury, Hg |
| 5. Strontium, Sr | 13. Iron, Fe | 20. Silver, Ag |
| 6. Calcium, Ca | 14. Nickel, Ni | 21. Platinum, Pt |
| 7. Magnesium, Mg | 15. Tin, Sn | 22. Gold, Au |
| 8. Aluminum, Al | | (Least active) |

Hydrogen is placed in the *activity series* above to show its relative affinity for oxygen. Silver and the metals below it do not unite directly with oxygen, but the higher in the list above silver a metal is, the more vigorously it unites with oxygen. Since the metals below hydrogen in this activity series do not displace hydrogen from the carbonic acid of natural waters and thus turn into salts, they are the only ones found free as minerals. Only the metals above hydrogen displace it from acids.

Reversible
reactions

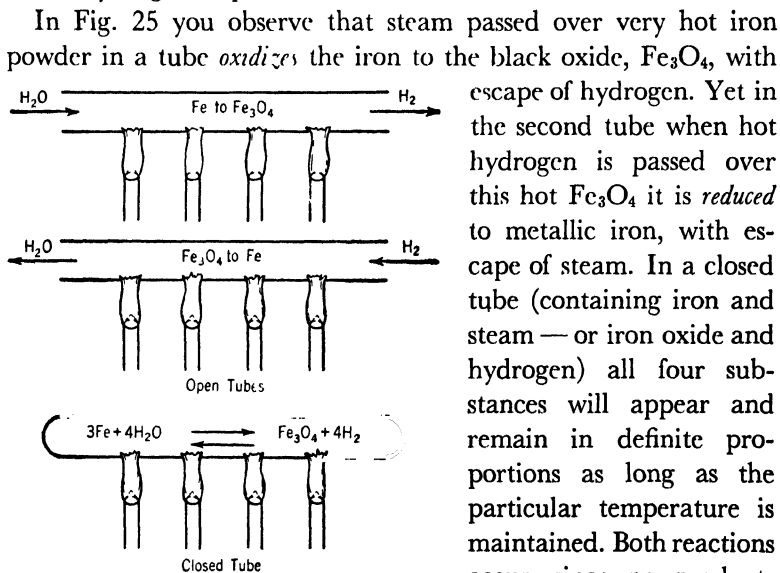


FIG. 25. REACTIONS IN OPEN AND CLOSED TUBES

of the two opposing reactions soon become equal. We arrive at an equilibrium to be disturbed right or left only by change of temperature or by removing at least one of the products. Many

Equilibrium

reactions are reversible if all products are kept in contact, but others are not.

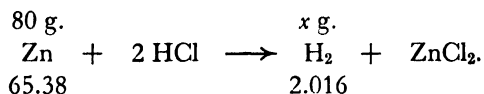
How many grams of hydrogen are formed when 80 g. of zinc react with sufficient hydrochloric acid to dissolve the metal?

*Solving
chemical
reactions*

1. Write the equation: $\text{Zn} + 2 \text{HCl} \longrightarrow \text{H}_2 + \text{ZnCl}_2$.

2. Select the two substances that are being discussed.

3. Select four numerical values relating to this pair of substances, two in grams and two in terms of atomic weight units. Of course x g. is the unknown weight of the product in grams. The 80 g. in the problem and the x g. desired, together with the weight of the molecules (or atoms) compared in the equation, give you these four values. From the balanced equation and the problem we see that one atom of zinc, Zn, and one molecule of hydrogen, H_2 , are to be compared. Write the numerical values appropriately above and below Zn and H_2 . The weight of Zn is 65.38 and of H_2 is 2.016.



4. Write these values in a chemical proportion exactly as indicated by step 3. Solve for x . The proportion could logically be written, $80:65.38 = x:2.016$. Expressed as a fraction the proportion becomes

$$\frac{80}{65.38} = \frac{x}{2.016}, \text{ or } x = 2.47 \text{ g. of hydrogen.}$$

1. How many grams of iron will be required to release 88 g. of hydrogen from sulfuric acid? *Problems*
Ans. 2437.9 g.

2. Will silver react with hydrochloric acid? Aluminum? Gold?

3. Hydrogen was passed over heated copper oxide. The water formed was collected and found to weigh 56 g. (a) What is the weight of the hydrogen that reacted with the copper oxide? (b) What is the loss in weight of the CuO ?

Ans. (a) 6.26 g.; (b) 49.73 g.

VALENCE

Valence is most easily understood by reasoning from the following experiment.

Suppose we weigh out 23 mg. (milligrams) of sodium, 24 mg. of magnesium, and 27 mg. of aluminum and introduce them under inverted tubes of dilute hydrochloric acid. (The sodium should be closed in a gelatin capsule which dissolves slowly.) The hydrogen *Experiment*

gas released pushes the water solution down the tubes as shown in Fig. 26. By experiment (at 0° C. and atmospheric pressure) we observe that

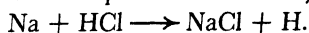
23 mg. sodium release 11 cc. hydrogen which weigh 1 mg.

24 mg. magnesium release 22 cc. hydrogen which weigh 2 mg.

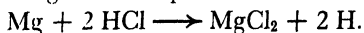
27 mg. aluminum release 33 cc. hydrogen which weigh 3 mg.

Since 23, 24, and 27 are the atomic weights of these metals, we must have weighed out an equal number of atoms of all three. Consequently,

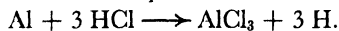
1 atom of sodium displaced 1 atom of hydrogen



1 atom of magnesium displaced 2 atoms of hydrogen



1 atom of aluminum displaced 3 atoms of hydrogen



Definition of valence

Valence is a number which indicates how many atoms of hydrogen or chlorine one atom of a given element will hold or displace.

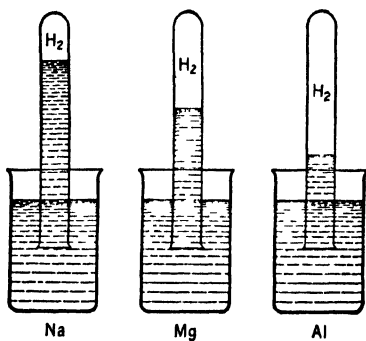


FIG. 26. RELATIVE VOLUMES OF HYDROGEN DISPLACED BY MILLIGRAM-ATOMIC WEIGHTS OF SODIUM, MAGNESIUM, AND ALUMINUM

The valence of aluminum is 3 because one atom of it displaces three atoms of hydrogen, or holds in a compound three monovalent atoms of chlorine. Why is the valence of magnesium 2? Prove your point by reference to magnesium compounds.

Atoms with a valence of 1 are monovalent; with a valence of 2, divalent; and with a valence of 3, trivalent. In water, H_2O , the oxygen is *divalent* because one

atom of it holds two atoms of *monovalent* hydrogen. From analysis and the use of atomic and molecular weights we know that Al_2O_3 is a correct formula. Why is aluminum *trivalent* in this case?

We have already noticed in equations, such as

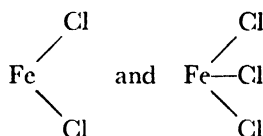


that one sulfur atom and four oxygen atoms seem to hold together as a group or *radical* during reactions, passing from one compound

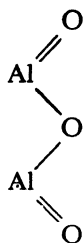
Radicals

to another unchanged, much as a family of Arabs might move from one tribe to another without the loss of any of its members. We call the SO_4 group of atoms the *sulfate radical*. It is incapable of existing free without its negative charge, and its compounds are usually called *sulfates*. For example, H_2SO_4 is known as "sulfuric acid," but it could be called *hydrogen sulfate*. Sodium sulfate is Na_2SO_4 , magnesium sulfate is MgSO_4 , and aluminum sulfate is $\text{Al}_2(\text{SO}_4)_3$. In nitric acid, HNO_3 , and the nitrates derived from it such as NaNO_3 and $\text{Cu}(\text{NO}_3)_2$, we find the NO_3 group of atoms holding together pretty well through all sorts of reactions; so we call it the *nitrate radical*.

It is convenient, although wholly diagrammatic, to represent the valence units between elements by straight lines or "bonds of valence." The two structural formulas

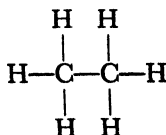


represent ferrous chloride and ferric chloride, in which iron is seen to have valences of 2 and 3. The *-ic* compounds of a metal are those in which the metal has a higher valence than in the *-ous* compounds. The formula



merely represents Al_2O_3 , showing the usual valences of 3 for aluminum and 2 for oxygen.

In ethane, C_2H_6 , the valence of carbon might seem to be 3, yet, if the structural formula is written, it is seen that each carbon atom is holding the other and that the correct valence is 4.



Electrovalence

When sodium unites with chlorine to form molecules of sodium chloride (NaCl), the sodium really loses an electron to the chlorine atom (page 60). This leaves an excess of *one positive charge* on the sodium atom — and, in the compound formed, the sodium atom has a positive valence of 1. When the neutral chlorine atom gains an electron, it possesses an excess of *one negative charge* — and in the compound it has a negative valence of 1. (These charged atoms are called *ions*.) The bond between atoms set up by electron transfer is called an *electrovalent bond*. It is limited to acids, bases, and salts.

In any compound the algebraic sum of the positive and negative valences is zero. Thus in H₂O the total positive valence of the two hydrogen atoms is 2 and the negative valence of the oxygen atom is also 2.

Covalence

In many reactions electrons are not transferred but are shared in pairs by two atoms. Each atom contributes one electron to make a pair by which the two atoms are held together with a valence of 1. If the valence between these atoms is 2, it is necessary that two pairs be shared. (See Fig. 27.)

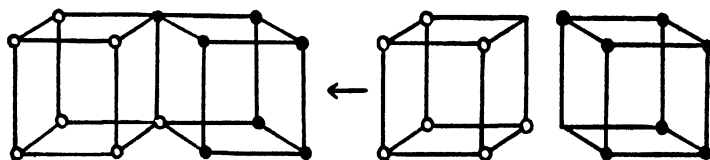
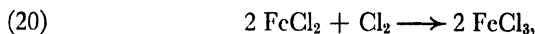


FIG. 27. A MOLECULE OF FLUORINE FORMED FROM TWO ATOMS BY THE SHARING OF ONE PAIR OF ELECTRONS

Oxidation and reduction

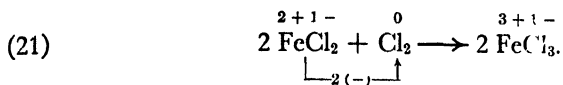
Although the original conception of *oxidation* means the addition of oxygen, and *reduction* means the loss of oxygen, we advance now to a broader view. Increase in valence of the central positive element in a compound has come to be considered as oxidation and decrease in valence as reduction. For example, the process



involving an increase in the valence of iron from 2 to 3, is recognized as oxidation of ferrous chloride to ferric chloride. The free chlorine bringing this about is called the *oxidizing agent*. Note that oxygen is not even involved.

Valence changes

In the reaction above, one iron atom in each molecule of ferrous chloride has lost one electron (to a neutral chlorine atom) and so has been oxidized.



*Oxidation is the loss of electrons by the atom or substance oxidized.
Reduction is the gain of electrons by any substance reduced.*

Assuming that chlorine is monovalent and that oxygen is divalent, state the *Problem*
valence of each of the metals in the following compounds: MgO, AgCl, AlCl₃,
FeO, Fe₂O₃, SnO₂, Cr₂O₃, NiCl₂, AuCl₃.

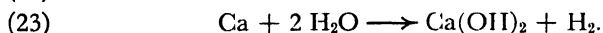
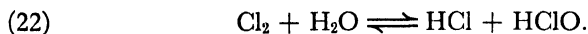
WATER

Water, in a sense, is the central liquid of our life. It makes up two-thirds of the human body and 50 to 75 per cent of all plant tissue. Even in engineering, water is important: for marine transportation, water power, steam power, cooling systems, and many other uses.

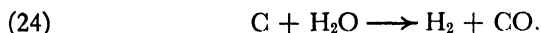
One cubic centimeter of water weighs one gram at 4° C.; so by *Properties of*
definition its *density* is 1 g. per cc., a standard of comparison for all *water*
liquids and solids. Lead, for example, has a density of 11.34 g. per cc.,
mercury, of 13.6 g. per cc. At 0° C. ice is lighter than water and floats.

The boiling point of water is 100° C. when the air pressure is 760 mm. as measured by the mercury barometer. The *specific heat of water* is 1, which means that one caloric of heat is required to raise the temperature of one gram of water one degree C. This relatively high value makes a hot-water heating system effective.

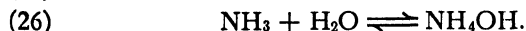
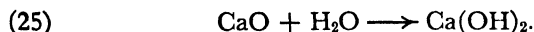
Chemically, water is reactive although sometimes extra heat is *Reactions*
needed.



Even carbon, if red hot, reacts to form the excellent fuel called *water gas*.



Quicklime, CaO, and ammonia, NH₃, react with water to form bases.



Calcium carbide, CaC₂, with water yields acetylene, C₂H₂.

Heavy water (deuterium oxide), D₂O instead of H₂O, occurs *Heavy water*
as one molecule out of every five thousand of ordinary water.

Deuterium is the name for hydrogen of atomic weight 2.016 whereas common hydrogen weighs 1.008. These two forms are explained by Fig. 28.

Hydrates

Certain substances crystallize from their water solutions on slow evaporation of the liquid, carrying definite amounts of the water

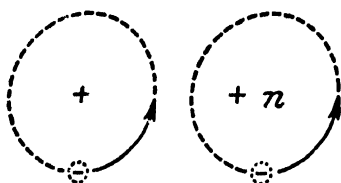


FIG. 28. ATOMS OF HYDROGEN AND DEUTERIUM

The hydrogen atom (left) has one planetary electron revolving around one nuclear proton; the deuterium atom (right) has, in addition, one nuclear neutron designated by the letter n .

with them. This *water of crystallization*, as it is sometimes called, is not held in a wet-sponge fashion, for no amount of pressure will squeeze it out. Moreover a sponge can hold any amount of water within wide limits, but not so with these *hydrates*. Hydrates must be definite chemical compounds because atomic weights are essential in expressing their composition. Some substances, it is true, can crystallize with two or three definite amounts of water,

but the shape of the crystals, solubility, and some other properties are then different in each case.

Loss of water

These hydrates feel dry to the touch but easily lose their combined water on heating. For example, blue vitriol is a hydrate with five molecules of water of crystallization, $\text{CuSO}_4 \cdot 5 \text{H}_2\text{O}$. When heated above 200° , all the water is lost and a white powder is obtained, CuSO_4 . Some hydrates such as washing soda ("soda ash"), $\text{Na}_2\text{CO}_3 \cdot 10 \text{H}_2\text{O}$, lose water in ordinary air and crumble to a powder, but others do not.

Drying agents

Gases are dried by passing them through granules of anhydrous calcium chloride, CaCl_2 , over fragments of sodium hydroxide, NaOH , or quicklime, CaO (or a mixture of both), through ultra-porous "silica gel," SiO_2 , or aluminum oxide, Al_2O_3 , or by freezing out the moisture. Some substances take up water chemically, while others condense it in submicroscopic capillaries (as with silica gel).

CARBON, ITS OXIDES AND CARBIDES

The importance of carbon and its compounds is evident as we learn that they make up our fuels, whether coal, wood, oil, or gas; our lubricants; our foods; our clothing; and that they make possible the change from iron to steel. Drugs, dyes, explosives, auto finish, pencils, picture film, printer's ink — these and a host of

other necessities are based upon carbon and its compounds. *Organic chemistry is the study of these compounds of carbon.*

The element occurs in nature in two crystalline forms, the diamond and graphite, and is readily prepared from its compounds in such amorphous forms as coke, lampblack, charcoal, bone black, and carbon black.

The diamond, a pure form of crystalline carbon, is the hardest substance known and hence scratches all other substances. Faulty specimens are used for the cutting edge of rock drills and in the form of dust to polish good diamonds or to make cutting wheels for the machine shop. *Diamonds*

Graphite, another form of carbon, crystallizes in flat plates which slip over each other with pressure. This explains its use in the common lead pencil and as a lubricant (suspended in oil or water). Carbon and graphite electrodes make possible the modern electric furnace and electrolytic cell. Iron takes up 1 or 2 per cent of carbon from the coke used in the blast furnace, but much of this dissolved carbon crystallizes out on cooling as graphite. *Graphite*

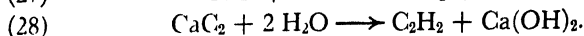
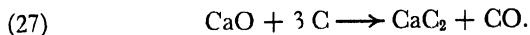
When bituminous coal is heated in retorts without contact with air, many volatile products escape. These include ammonia, illuminating gas, and coal tar, from the last of which benzene, toluene, phenol (carbolic acid), creosote, naphthalene, and many derivatives of great value are secured. Coke is left behind in the retort. It is used in making pig iron from iron ore, for melting metals in foundries, as a domestic fuel, and in the manufacturing of some carbon compounds. Just as coke is the nonvolatile product of coal distillation, so is charcoal the residue remaining when wood is distilled in the absence of air. *Coke, charcoal*

When a smoky gas flame is chilled, the incandescent carbon of the yellow flame deposits as a form of soot called *carbon black*. An inferior carbon called *lampblack* is prepared by chilling the smoky flame from tars and oils. *Carbon black*

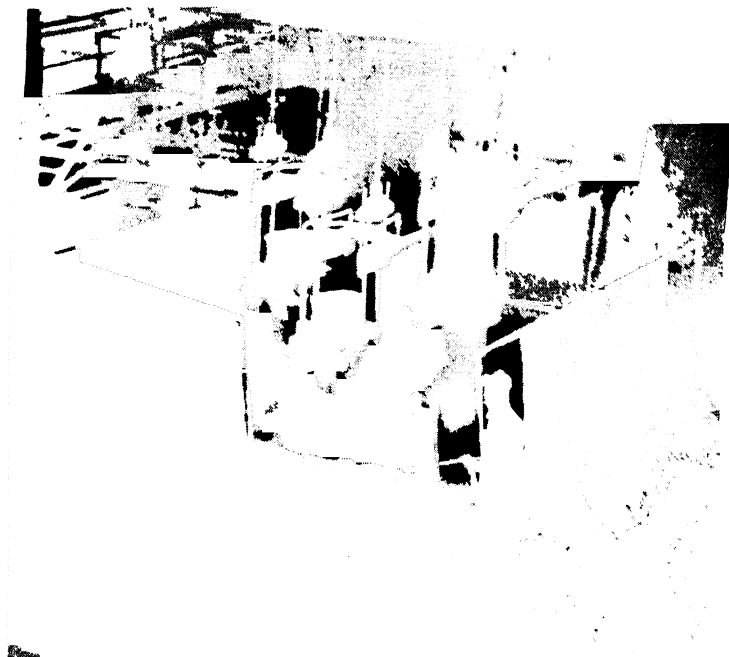
Carbon black doubles the tensile strength of rubber for tires and quadruples its resistance to abrasion. In a set of tires for a small car there are 12.5 lb. of carbon black. Over 330,000,000,000 cu. ft. of natural gas are burned annually in the United States to make about 500,000,000 lb. of carbon black.

A carbide is a simple compound of carbon with one other element. A carbide of iron, Fe_3C , is found in steel and has great influence on the properties of the steel. Calcium carbide, CaC_2 , which *Carbides*

reacts with water to form acetylene, C_2H_2 , is the product of electric-furnace heating of lime and coke (Fig. 29).



*Preparing
calcium
carbide*

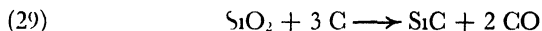


Courtesy Monsanto Chemical Company

FIG. 29 AN ELECTRIC FURNACE

The heat of the arc between large carbon electrodes is necessary for some chemical reactions. Resistance heating is also used.

A carbide of silicon, sold by one firm as carborundum, SiC , one of the most useful *abrasives* known, is likewise prepared in an electric furnace but from a mixture of coke and sand.

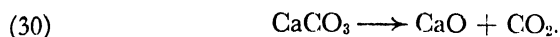


Tungsten carbide and tantalum carbide are also used because of their extreme hardness. Boron carbide is hard enough for use as the nozzle of sandblast apparatus.

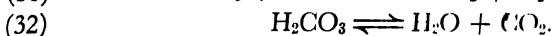
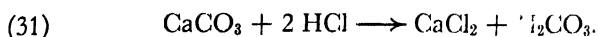
*Carbon
dioxide, CO_2*

The combustion of coke or carbon compounds such as coal, wood, and oils with plenty of oxygen present yields carbon dioxide gas. Any other elements present would probably burn to their oxides.

In the lime kiln carbon dioxide is driven off by heating limestone (calcium carbonate) to a red heat, quicklime being left behind.



In the laboratory a convenient method is the treatment of any carbonate with a dilute acid.

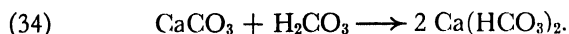


Carbonic acid, H_2CO_3 , a very weak acid, is unstable as shown above. The *reversibility* of the last reaction reminds us that carbon dioxide reacts with water to form soluble carbonic acid. Under pressure, like all gases, its solubility may be greatly increased. When this pressure is released at the soda fountain for your pleasure, your nose is tickled by carbon dioxide escaping from the drink. "Dry ice" is frozen carbon dioxide.

Complete neutralization of carbonic acid, H_2CO_3 , by bases such as *Bicarbonates* sodium hydroxide, NaOH , or calcium hydroxide, Ca(OH)_2 , yields ordinary calcium carbonate, CaCO_3 , or sodium carbonate, Na_2CO_3 .

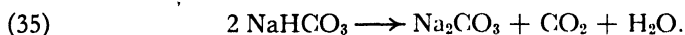


However, an excess of carbonic acid may react with calcium carbonate (as in limestone caves) to yield the soluble calcium bicarbonate.

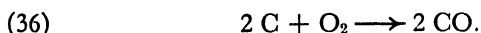


Hardness of water is due, in part, to this soluble bicarbonate.

Sodium bicarbonate, NaHCO_3 , is mixed with solid acidic salts to form baking powder. When heated dry it breaks down into the ordinary carbonate and carbon dioxide.



Carbon monoxide, a colorless, poisonous, almost odorless gas, *Carbon monoxide* is formed by burning carbon, or its compounds, with insufficient air.



It is also formed when carbon dioxide is passed over red-hot carbon.

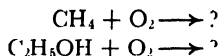


It is a valuable fuel gas as shown by the great use of "water gas," a mixture of $H_2 + CO$.

A concentration of 0.02 per cent of carbon monoxide, CO , may cause headache in a few hours; 0.06 per cent will produce unconsciousness in two hours. Automobile exhaust gases contain about 7 per cent CO ; hence, the danger of warming up the engine in a closed garage.

Problems

1. Balance the equations below following the procedure from p. 63.



The first equation represents the burning of natural gas and the second the burning of alcohol.

2. How many grams of carbon dioxide result upon heating adequately 7 kg. of pure marble (calcium carbonate, $CaCO_3$)? Ans. 3.078 kg.

MOLECULAR AND ATOMIC WEIGHTS

Avogadro's theory (1811), since confirmed, states that *equal volumes of all gases, under the same conditions of temperature and pressure, contain the same number of molecules.*

Exact molecular weights

From the above it is evident that if we weigh equal volumes of gases under like conditions, we get weights that bear the same ratios as the weights of single molecules. In other words, we can thus obtain *the relative molecular weights* of all gases. (Solids must be handled differently.)

For good reasons the weight of the oxygen molecule was finally set at 32 and all others compared to it. For example, if a liter of a new gas weighs ten times as much as a liter of oxygen, its molecule must weigh ten times 32 or 320 in the system of relative weights. In the next section, methods of determining the molecular weights of solids will be given.

Atomic weights

In 1858 Cannizzaro laid the foundations for the determination of atomic weights. He suggested, in line with Avogadro's law, that:

1. The molecular weights of all the volatile compounds of the element studied should be determined.
2. These compounds should all be analyzed and the part of each molecular weight due to the particular element determined.
3. The smallest weight of the element found in any of the molecular weights of its compounds should be selected as the atomic weight.

By this logic the atomic weight of chlorine, for example, would be found from a study of many chlorine compounds to be 35.46. Similarly, $H = 1.008$. Since the molecular weight of oxygen was taken as 32 and the weight of a single atom found to be 16, the formula of oxygen gas must be O_2 .

Let us derive the formula of water. Analysis would show that about eight-ninths of any quantity of water taken, or of one drop, or of the smallest particle with all the properties of water (the molecule), is oxygen. We apply this approximate fraction to the molecular weight, $\frac{8}{9} \times 18.016 = 16$. Since the weight of one atom of oxygen is 16 (as just shown), there must be only one atom of oxygen in the molecule of water. The rest of the molecular weight is $18.016 - 16 = 2.016$, which is, of course, the weight of two atoms of hydrogen, and the formula must be H_2O . Of course, *molecular weights are expressed in atomic weight units.*

The molecules of most, but not all, elementary gases are diatomic. For example, a molecule of chlorine is Cl_2 , and of nitrogen, N_2 . Yet mercury vapor is monatomic and its molecular formula is Hg .

One liter of oxygen at atmospheric pressure and $0^\circ C$. weighs 1.429 g. Hence the number of liters occupied by 32 g. of oxygen gas = $1 \frac{32}{1.429} = 22.4$. Now, by Avogadro's law, this is the volume occupied by the molecular weight in grams of any gas (molar volume). If we weigh a convenient volume of any gas, correct the volume to $0^\circ C$. and 760 mm. pressure, we can calculate the weight of 22.4 l. This weight is also the relative molecular weight. Note that the gram molecular volume, 22.4 l., is approximately 1 cu. ft.

Another useful trick follows. Suppose you want to know the weight of a liter of methane gas. The formula is CH_4 . The combined weights of one atom of carbon and four atoms of hydrogen give the molecular weight, 16.042. Of course 22.4 l. weigh 16.042 g. (the gram-molecular weight). Therefore, 1 l. weighs $1 \frac{16.042}{22.4} = 0.7162$ g.

1. Does the molar volume of water (weighing 18 g.) refer to liquid or gaseous water? The molar volume is nearly 1 cu. ft. or exactly 22.4 l. *Problems*

2. If 1 l. of nitrogen at $0^\circ C$. and 760 mm. weighs 1.251 g., what must be the molecular weight? *Ans.* 28.02

3. Find the molecular weight of a gas, of which 500 cc. at 18° and 752 mm. weigh 2.5 g. *Ans.* 120.65

4. What will be the volume occupied by 12 g. of carbon tetrachloride, CCl_4 (molecular weight 153.84), in the vapor state at 0° and 760 mm.? *Ans.* 1.747 l.

SOLUTIONS

True solutions A solid and a liquid may be mixed so intimately that separate particles of the solid are no longer seen. With certain reservations, to be given later, we may call this mixture a *solution*. Salt water is a good illustration. If the mixture is less intimate, so that minute particles of the solid may be seen with the microscope, we may call it a *suspension*. Two liquids may also mix so completely as to deserve the term *solution*. If less thoroughly mixed, minute droplets of one liquid suspended in the other may be seen, under proper conditions. Such a mixture is called an *emulsion*. Milk is an example.

True solutions are nonsettling and are homogeneous to the eye. Suspensions or emulsions settle, sooner or later, and are not homogeneous under severe optical tests. In a true solution the particles of a dissolved substance are single molecules or, at most, groups of a few molecules. Consequently, a solution is a homogeneous molecular mixture of substances. It should be noted that the proportions of the components of a solution (sugar and water, for example) may be varied within limits. This distinguishes a solution from a compound.

Solid solution Some alloys are mixtures of metals or intermetallic compounds so intimate as to deserve the name *solid solutions*.

Calcium carbonate (limestone) does not actually dissolve in hydrochloric acid but the calcium chloride formed does dissolve.

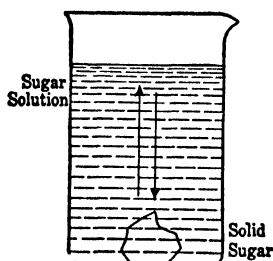


FIG. 30. SOLUTION EQUILIBRIUM

In a saturated solution of sugar a sugar crystal dissolves at the same rate that sugar in solution deposits on the crystal.

The liquid in which a solid or gas has been dispersed to form a solution is called the *solvent* and the substance dissolved is the *solute*. By concentration we mean the amount of solute dissolved by a definite quantity of the solvent, often referred to as so many grams per 100 cc. of solvent.

A *saturated solution* is one which will remain unchanged in concentration if placed in contact with some of the undissolved solute. It is said to be in *equilibrium* with excess of solute. That is, when we stir sugar, for example, into water, it continues to dissolve until the rate at which the sugar molecules in solution deposit

on the lumps of solid sugar just equals the rate at which the solid throws molecules into the solution. (See Fig. 30.)

There are many degrees of solubility. Limestone (CaCO_3), called *calcium carbonate* in its pure form, dissolves to the extent of 0.0013 g. in 100 cc. of water, while 143 g. of potassium hydroxide dissolve in the same quantity of water. By a "10 per cent solution" we mean one containing 10 g. of solute and 90 g. of water (or other solvent).

Solubility measurements

The temperature effect on solubility must be considered in any of these quantitative statements. Nearly all solids are more soluble

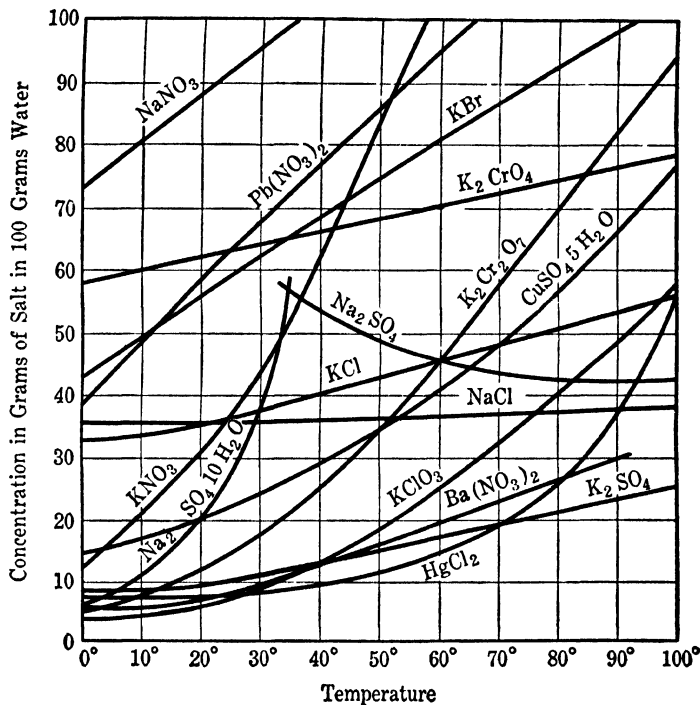


FIG. 31. SOLUBILITY CURVES — TEMPERATURE CENTIGRADE

with rise in temperature, but there are a few exceptions, notably some calcium salts of organic acids. (Gases, however, are less soluble with rise in temperature.) Potassium nitrate is seven times as soluble at 70° C. as at 8° C. There can be no accuracy, then, in a solubility statement unless the exact temperature of the measurement is given. This is usually 20° C.

The solubilities at various temperatures may be plotted in the form of curves, using the ordinates to represent the number of grams of solid dissolved by 100 cc. of water. The abscissas represent temperatures. With the curves in Fig. 31 the solubility at any

Temperature effects

temperature may be read at a glance. What is the solubility at 18° C. and 90° C. of lead nitrate? Of sodium chloride at 77° C.?

*Solubility of
gases in
liquids*

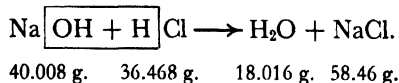
In 1803 Henry observed that at any definite temperature the solubility by weight of a gas in a liquid is proportional to the pressure of the gas, except for the very soluble gases which, like ammonia, react with water. With rise in temperature all gases are less soluble in liquids.

A *molar solution* contains a molecular weight in grams (one mole) of solute in a liter of solution. Since the molecular weight of sulfuric acid is 98, a molar solution of this acid contains 98 g. per liter.

A *normal solution* of an acid, base, or salt contains one gram-equivalent weight of the solute in one liter of solution.

*Equivalent
weights*

The *gram-equivalent weight* of an acid contains 1.008 g. of acidic hydrogen (which requires 36.468 g. of HCl, for example) while a gram-equivalent weight of any base is the weight containing 17.008 g. of basic hydroxyl ($-OH$) as is the case with 40.008 g. of NaOH. The acid and base are exactly equivalent to each other as shown in the equation below. The base exactly neutralizes the acid. By "acidic" hydrogen we mean only those hydrogen atoms replaceable by a metal.



With H_2SO_4 , the formula weight (98.076 g.) would contain 2.016 g. of acidic hydrogen. Hence, 49.038 g. of this acid would be the acid equivalent of 36.468 g. of HCl and either the H_2SO_4 or the HCl would neutralize 40.008 g. of NaOH.

*Normal
solutions*

In the case of a salt such as "blue vitriol" or copper sulfate, CuSO_4 , we note that one copper atom has taken the place of the two hydrogen atoms in H_2SO_4 and so we divide the weight of the molecule by 2, as we did with the acid, to obtain the equivalent weight. Since "blue vitriol" is a hydrate, $\text{CuSO}_4 \cdot 5 \text{H}_2\text{O}$, we include the weight of the 5 H_2O (90.04) in the formula weight which is then divided by two, $\frac{249.71}{2} = 124.85$ g., to arrive at the equivalent weight. Consequently, one liter of solution containing 124.85 grams of $\text{CuSO}_4 \cdot 5 \text{H}_2\text{O}$ is a *normal solution*. The symbol for normal is N . Twice normal is $2N$ and one-tenth normal is $0.1N$ or $N/10$, with other decimal values representing exact concentrations.

A most interesting point is brought out if solutions of gram-molecular weights of such substances as alcohol, glycerine, sugar, and acetone, in 1000 g. of water, are frozen. They all freeze at about -1.86°C . Hence, the *molar depression* of the freezing point of water is 1.86°C . We can usually obtain the molecular weight of a soluble substance by noting how many grams of it must be dissolved in 1000 g. of water to lower the freezing point to -1.86°C . Acids, bases, and salts are excepted.

Freezing-point lowering

In a similar way the boiling point rise caused by one formula weight in grams of solute per 1000 g. of solvent is 0.52° for water (with the exception of acids, bases, and salts as explained later).

1. How could you make 1 l. of a normal solution of hydrobromic acid (HBr)? *Problems*
Of potassium hydroxide? Of barium hydroxide, $\text{Ba}(\text{OH})_2$?
2. (a) How many grams of $\text{Na}_2\text{SO}_4 \cdot 10\text{H}_2\text{O}$ are there in 800 cc. of *N/5* solution? (b) Of Na_2SO_4 in 800 cc. of *N/5* solution? *Ans.* (a) 25.78 g.; (b) 11.36 g.
3. Prove that a solution is a mixture rather than a compound.

CHLORINE, BROMINE, IODINE, AND FLUORINE

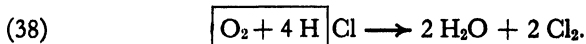
These extremely active elements (with fluorine), called the *halogens*, resemble each other so much that they are grouped as a family. In fact, the first three sometimes occur together in salt deposits.

Occurrence of the halogens

Chlorine occurs in nature chiefly in common salt (sodium chloride), NaCl . The ocean contains 3 per cent of salt, but salt beds, often hundreds of feet below the ground surface, are the chief commercial source. *Bromine* is obtained from the sodium bromide occurring as an impurity in some salt deposits and from sea water. *Iodine* is obtained from an impurity in Chile saltpeter and from a seaweed. *Fluorine* is found in fluospar (calcium fluoride), CaF_2 , and in cryolite, $\text{AlF}_3 \cdot 3\text{NaF}$.

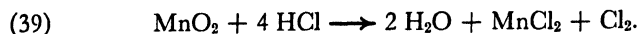
The most convenient laboratory methods for the preparation of chlorine depend upon the oxidation of hydrogen chloride into water and free chlorine:

Preparation of chlorine



Oxygen alone does this only with extreme slowness unless aided by a catalyst at a favorable temperature, but good oxidizing agents such as potassium permanganate, manganese dioxide, sodium dichromate, and lead dioxide will oxidize it rapidly without a catalyst. A concentrated solution of hydrogen chloride in water

(called *hydrochloric acid*), when warmed with the oxidizing agent, gives this reaction.



*Preparation
of bromine*

Bromine may be prepared similarly by oxidation of hydrogen bromide, HBr, and iodine by oxidation of hydrogen iodide, HI. Most of our bromine is displaced from sodium bromide by the more active chlorine.



*Chlorine by
electrolysis*

The leading commercial method of making chlorine, especially in the United States, is the electrolysis of water solutions of sodium chloride. Chlorine is evolved at the anode, which must be made of carbon, usually in the form of graphite, because the gas is so active it would react with metal anodes. At the cathode, usually of iron, sodium is released, but of course it reacts at once with water to form sodium hydroxide, NaOH, and hydrogen. The sodium hydroxide is a valuable by-product. Since chlorine in the presence of water reacts with sodium hydroxide, it is necessary to keep the solutions around the two electrodes from mixing. The current of electricity enters at the anode and leaves at the cathode.

Molten sodium chloride conducts electricity and can thus be decomposed into chlorine and sodium. With no water present the sodium accumulates. This is one of the commercial sources of metallic sodium.

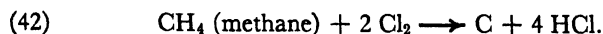
*Properties of
chlorine*

Chlorine, a very poisonous, greenish-yellow gas, can be liquefied at 18° C. by 16.5 atmospheres pressure. It is shipped under pressure in tank cars containing 15 tons of the liquid. Dry chlorine does not attack steel unless it is very hot.

Chlorine unites with most elements to form chlorides, although extra heat may be required.



It will even pull the hydrogen out of some compounds to form hydrogen chloride.



Chlorine not only dissolves in cold water (6.3 g. per liter) but reacts with it:



This is a reversible reaction, but the hypochlorous acid, HClO , slowly decomposes into hydrochloric acid and oxygen.

Two-thirds of all the chlorine produced in America is used in bleaching paper pulp, the remainder in bleaching textiles, in water purification, in laundry bleaching, and so on. Until recent years chlorine was used in combination with lime as "bleaching powder."

Uses of chlorine

Most city water is purified by the aid of chlorine, and the soldiers' drinking water at the front may be made safe by a little *hypochlorite*, NaClO . Chlorine is needed in the preparation of one synthetic rubber (neoprene) and in making glycol (antifreeze), degreasers, and certain solvents. Chlorine is essential to most of the poison gases used in modern warfare. Rolling clouds of chlorine were used in the earlier gas attacks in World War I.

Bromine, although less active, is similar in its properties to chlorine. However, it is a heavy, red-black liquid at room temperatures. Iodine is a violet-black solid at room temperatures, easily volatilized by heating without melting (*sublimation*).

Bromine is used in making some aniline dyes, in making potassium bromide (a valuable nerve sedative), and in the form of silver bromide it is the sensitive material on photographic plates. In 1924 *tetraethyl lead* came into extensive use as an "antiknock" addition to gasoline. About 2 cc. of ethylene bromide, $\text{C}_2\text{H}_4\text{Br}_2$, per gallon of gasoline are required to furnish enough bromine to change the lead set free in the explosion into PbBr_2 (lead bromide).

Bromine

Iodine is rather expensive, finding application as silver iodide in photography, iodides in medicine, and in the manufacture of some dyes.

Iodine

Fluorine as the free element is not yet very important, but some of its compounds have great value. *Hydrofluoric acid* and its acid salts are used in etching glass; calcium fluoride, fluor spar, is used as a slag former in the steel industry; antimony trifluoride and hydrogen fluoride are important catalysts in certain organic chemistry reactions; and "Freon," CF_2Cl_2 , is a safe refrigerating substance (substituting for ammonia and sulfur dioxide).

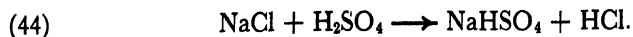
Fluorine

THE HALOGEN ACIDS

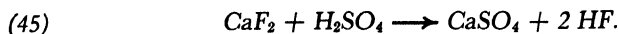
The most used laboratory and commercial method of preparation of hydrogen chloride, HCl , is the heating of any suitable chloride (sodium chloride is the cheapest) with concentrated sul-

Hydrochloric acid

furic acid. The gas is dissolved in water for convenient use as *hydrochloric acid*.



Hydrogen fluoride, HF, is prepared in similar fashion from calcium fluoride (*fluorspar*).

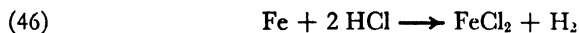


This method gives a badly contaminated product if it is used to release HBr and HI from metallic bromides and iodides. These two acids are so unstable that they are oxidized, to some extent, by the concentrated sulfuric acid which, when hot, can give up some of its oxygen to the hydrogen of the halogen acids, forming water and free bromine or iodine with by-products of SO₂ or H₂S. Phosphoric acid (not an oxidizing agent) is sometimes substituted for the sulfuric acid.

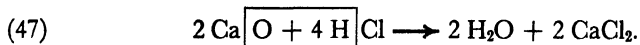
Uses of hydrogen chloride

Hydrogen chloride is a colorless gas of sharp, choking odor, extremely soluble in water. The concentrated "chemically pure" acid of commerce contains 37 per cent hydrogen chloride and 63 per cent water. It is one of the strongest acids.

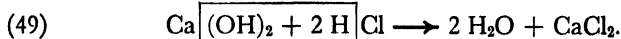
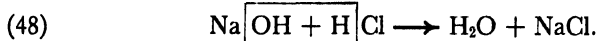
Metals more active than copper displace hydrogen from it. (See p. 76.)



Oxides of the metals react with it to form water and a chloride of the metal:

*Preparation of salts*

Hydroxides of the metals (bases) react with hydrochloric acid to form water and metallic chlorides. This suggests general methods of preparing sulfates, nitrates, and other salts.



The base and acid can be mixed in such proportions that the solution is no longer sour (acidic) or soapy (basic), nor does it change the color of litmus, a dye (p. 96). We say the acid has neutralized the base and vice versa.

Since hydrochloric acid attacks all metallic oxides, hydroxides, and carbonates, it is useful in preparing metallic chlorides. It also is useful in "pickling" steel wire before galvanizing with melted

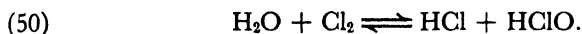
zinc. The black furnace scale, Fe_3O_4 , formed when hot iron or steel remain in the air, must be removed to permit adhesion of zinc. Sheet steel is usually pickled in diluted sulfuric acid (cheaper).

Hydrofluoric acid, HF, is the only acid that rapidly attacks glass or silica; so it is useful in etching glass apparatus (thermometers, and so on), polishing silicate rock, and removing sand from straw (for hats). Recently anhydrous (water-free) hydrogen fluoride has found favor as a catalyst in the petroleum refinery. This acid is so corrosive and toxic that it is very dangerous stuff to handle.

Hydrofluoric acid

Chlorine reacts with water to form, in an equilibrium, hypochlorous acid, HClO , a strong oxidizing agent.

Other chlorine compounds



Sodium hypochlorite, NaClO , is a useful bleaching agent and bactericide. Its calcium salt occurs in "bleaching powder." The chlorates, such as NaClO_3 , readily give up oxygen on heating so are used in making matches, fireworks, and some explosives.

Balance the following:

Problems



BASES OR ALKALIES

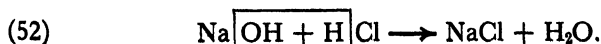
Electrolysis of melted, dry sodium chloride yields the very active metal sodium at the cathode and chlorine at the anode. If "salt" water is electrolyzed, sodium hydroxide, NaOH , is the cathode product. Much sodium hydroxide is also prepared by treating boiling soda solutions with slaked lime in some excess; thus we obtain

Sodium hydroxide



The precipitated calcium carbonate (same as limestone or marble) is filtered off, calcined to form more lime, and the filtrate is concentrated. In commerce, sodium hydroxide is sold in white sticks or lumps. It is very corrosive on the hands or in the lungs.

Bases, compounds of metals with —OH groups such as NaOH , $\text{Ca}(\text{OH})_2$, $\text{Ba}(\text{OH})_2$, are useful for neutralizing acids. Water and a salt are the products.



The hydroxides of sodium and potassium are the strongest, or most reactive, common bases. Ammonium hydroxide, NH_4OH , ranks as a weak base. Another weak one is slaked lime, $\text{Ca}(\text{OH})_2$, but it is so cheap that it has great use. Although only mildly soluble in water, it is effective as "milk of lime," a suspension of excess lime in water. As fast as that in solution reacts, the rest dissolves and reacts. Ferric hydroxide, $\text{Fe}(\text{OH})_3$, is a typical insoluble base.

IONS AND ELECTROLYSIS

*Acids, bases,
and salts*

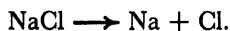
All acids contain hydrogen, replaceable by active metals, although there are thousands of hydrogen compounds (sugar, for example) that do not contain acidic hydrogen and do not taste sour. *Litmus* paper, an "indicator," is turned red by acids.

Bases contain hydroxyl groups, $-\text{OH}$, linked to a metal. They neutralize acids and are recognized by the change of red litmus to blue. A salt may be formed by reaction between an acid and a base or by displacement of hydrogen from an acid by certain metals or by other methods. Copper sulfate, CuSO_4 , is a salt.

*Evidence for
the ionic theory*

Arrhenius in 1887 sought an explanation of the fact that acids, bases, and salts in water solution conduct electricity and are decomposed in the process, while no other substances dissolved in water act in this way. He was also concerned with the abnormally large (and quite unexpected) lowering of the freezing point of water by acids, bases, and salts as a unique class.

For example, a molecular weight in grams of glycerine in 1000 g. of water froze at -1.86° . So it proved with thousands of such substances but not with a salt like sodium chloride. A solution of 58.43 g. (the molecular weight in grams of sodium chloride) in 1000 g. of water froze at nearly twice 1.86° below zero. Since it was known that freezing point lowering depended upon number of dissolved particles, young Arrhenius was forced to conclude that *each molecule of salt split into two in water solution*.



A fallacy?

The reader should at once recognize the necessity of accounting for more particles than indicated by a gram-molecular weight and he may see that this explanation accounts for them, but he does not see how ordinary metallic sodium can exist in contact with water. For that matter he is skeptical about the presence of ordinary chlorine in salt water. Chlorine water is greenish yellow, poisonous,

and a powerful oxidizing agent, and he knows by experience that salt water has none of these properties. He remembers that metallic sodium reacts violently with water, but there is nothing exciting to be seen when salt is dissolved in water.

The reply by Arrhenius to his critics was that this was not common sodium metal but electrically charged sodium atoms (*ions*) and that the chlorine was not ordinary chlorine but charged chlorine atoms (*ions*) with entirely different properties. Solutions of copper salts are blue due to charged copper atoms (*ions*) while common copper is red and insoluble.

Rewritten to meet this explanation the equation above now becomes



because in electrolysis it is the positive sodium particle or ion that migrates to the negative pole, while the negative chloride ion is attracted to the positive pole. Other acids, bases, and salts ionize similarly although some, like Na_2SO_4 , yield three ions per molecule. *Positive and negative ions*

Since bathers in the ocean never seem to be electrically shocked, there must be as many positive charges developed from each molecule as negative charges.

Hence



The number of charges on an ion is the same as the valence.

A modernized ionic theory holds that in the crystal lattice of solid acids, bases, and salts, the charged ions already exist and are simply pried apart by water. Certainly we know that fused salts conduct electricity and are decomposed by it.

Suppose in the cell of Fig. 32 we use a solution of hydrochloric acid and pass a direct current through the circuit. The electric current is merely the passage of a stream of electrons along a conductor. The electrode where the current enters (or electrons leave) is called the *anode* and the one by which it leaves, or electrons enter, is called the *cathode*. Ions attracted to the anode are anions while those attracted to the

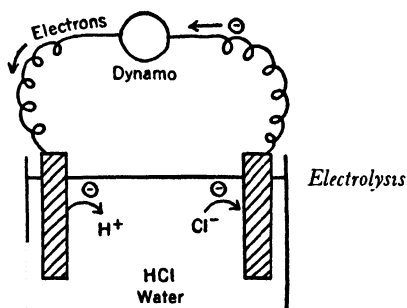


FIG. 32. ELECTROLYSIS OF HYDROCHLORIC ACID

cathode are cations. The passage of the current merely keeps the anode positively charged and the cathode negatively charged. Hydrogen escapes at the cathode and chlorine at the anode. In other words, hydrochloric acid is *electrolyzed*.

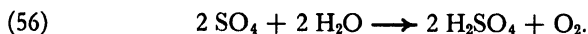
Ion travel

Hydrogen particles from the entire solution travel to the cathode and are there released. Why, unless they are electrically attracted? The cathode is negatively charged and consequently attracts positively charged bodies. The conclusion is that the hydrogen particle is simply an atom carrying a positive charge of electricity which in this form is soluble in water. On contact with the negative electrode it receives an electron from the circuit, its charge is neutralized, and the plain hydrogen atoms then unite in pairs to form ordinary hydrogen gas molecules, which escape.

The chlorine particles move from all parts of the solution to the anode, or positive electrode. This attraction must be due to the presence of a negative charge on the chlorine particle or atom. On contact with the electrode the charge on the chlorine atom is neutralized; that is, it gives its extra electron to the pole while common chlorine escapes and may be detected by the usual tests. Both the hydrogen and chlorine are primary products of electrolysis.

Electrolysis of sulfuric acid

When sulfuric acid is electrolyzed, its hydrogen is liberated at the negative electrode, as with all acids, but the discharged sulfate radical is not found as a molecular substance at the positive electrode; instead, oxygen escapes. Evidently the sulfate radical is attracted by the positive electrode because of the negative charge carried by the radical. After discharge the simple radical, unable to exist in molecular form, reacts with water.

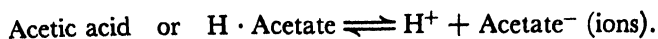


Of course this sulfuric acid of the equation, a secondary product of electrolysis, is itself electrolyzed, and so on indefinitely.

Degrees of ionization

Under similar conditions (surface and temperature) normal acetic acid releases 1 cc. of hydrogen in contact with zinc in the same time that the zinc discharges 65 cc. of hydrogen from normal sulfuric acid and 100 cc. from normal hydrochloric acid. The only possible explanation is that there are 100 times as many hydrogen ions present at any one time in the hydrochloric acid solution as in the acetic acid solution. Yet equal normality would seem to guarantee equal concentrations of hydrogen ions.

Ionization is considered to be an equilibrium process:



Now we might say that only a small fraction of the acetic acid molecules are ionized at any instant. On the other hand, in the reaction



nearly all the molecules are ionized at a given instant. We refer, therefore, to acetic acid as a "weak acid" and to hydrochloric as one of the "strongest acids." Sulfuric acid is intermediate. Similarly, ammonium hydroxide is a weak base, sodium hydroxide a strong base. Carbonic acid, H_2CO_3 , is even weaker than acetic acid. Salts with few exceptions are well ionized.

It is a curious fact that a liter of normal acetic acid will neutralize just as much base as a liter of normal hydrochloric acid despite their relative "strengths." Refer to the equilibrium equations given above. If neutralization of H^+ ions from acetic acid by the OH^- ions of the base (to form H_2O) prevented the reaction from going to the left (because there would no longer be any hydrogen ions) soon all the acetic acid molecules would have ionized and the *equilibrium would have been disturbed to the right*. *Equilibrium disturbed*

In the light of this section, we may now state that acids yield hydrogen ions in aqueous solution and that bases yield hydroxyl ions.

SULFUR AND SOME OF ITS COMPOUNDS

In ordinary years the Louisiana-Texas sulfur field yields nearly 3,000,000 tons of remarkably pure sulfur. At depths of 900 ft. it is melted by hot water and steam, led down through pipes with air to make it lighter. Through one of the three or four concentric pipes the frothy, aerated sulfur rises. *Occurrence of sulfur*

Combined sulfur is very common. Iron pyrites, FeS_2 , once called *fool's gold*, is found in most of our states and in nearly all countries. In fact some of our most important metals occur combined with sulfur, as lead sulfide or galena, PbS ; zinc blende, ZnS ; and chalcopyrite, CuFeS_2 . The sulfates also occur in great quantities. The most common are calcium sulfate or gypsum, $\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$; barite, BaSO_4 ; celestite, SrSO_4 ; and Epsom salt, $\text{MgSO}_4 \cdot 7\text{H}_2\text{O}$.

Sulfur is a pale-yellow solid, tasteless, and nearly odorless, insoluble in water but *soluble in carbon disulfide*, CS_2 . When sulfur is *Properties of sulfur*

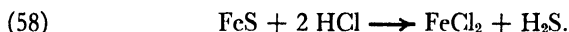
melted it first turns into a thin, pale-yellow liquid, easily poured, then shows a rapid increase in viscosity at 159.5° C. At 230° C. the liquid becomes black and very viscous but at higher temperatures the viscosity again decreases.

Sulfur is very active when heated, uniting with all metals except gold and platinum. It has a high electric insulating power, is water resistant, is inactive toward most acids, and has a fair degree of compressive strength.

Great quantities of sulfur are burned in the manufacture of sulfuric acid and nearly 40,000 tons are used yearly in vulcanizing rubber. Free sulfur is a necessary part of black gunpowder (with carbon and saltpeter). Some insecticides and fungicides require sulfur.

*Hydrogen
sulfide, H₂S*

Hydrogen sulfide is found in some natural waters, such as the so-called sulfur springs, in volcanic gases, and in Mexican oil wells. Hydrogen unites with hot sulfur vapor to form hydrogen sulfide. The most convenient laboratory method of preparation is the addition of some nonoxidizing acid to a suitable sulfide, preferably ferrous sulfide, FeS, because the latter is very cheap.

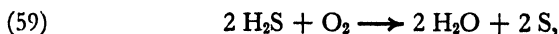


Since hydrogen sulfide in aqueous solution is an acid (although weak), the sulfides of metals are its salts. Many occur in nature and many are formed by the addition of soluble sulfides to salt solutions. *Any sulfate can be reduced to the sulfide by heating with carbon.*

*Properties
of H₂S*

The odor of H₂S is very disagreeable. Air containing one part of this gas in one thousand *is rapidly fatal*, nearly as dangerous as prussic acid but for its warning odor. Since this gas is evolved in some manufacturing processes, it is an industrial poison.

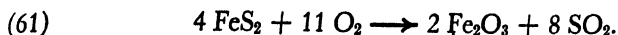
A solution of hydrogen sulfide open to the air soon has a film of sulfur on the surface, showing how easily it is oxidized. A rapid oxidation is secured by burning the gas in air.



*Sulfur
dioxide, SO₂*

Sulfur burns in air with a pale blue flame, forming sulfur dioxide. In the United States this is a great commercial source. It is also formed as a by-product when sulfide ores are roasted in air. As a result, many smelters have attached sulfuric acid plants to utilize the waste sulfur dioxide.

The cheapest sulfide ore is iron pyrites, FeS_2 , which burns like coal.



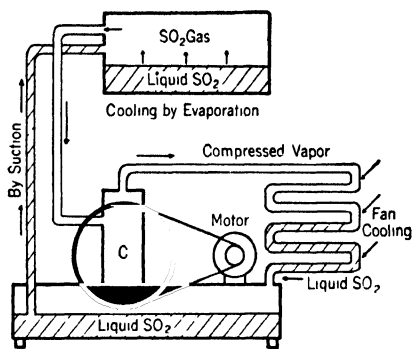
The sulfur dioxide obtained in this way from roasting sulfide ores is, of course, diluted with several volumes of nitrogen from the air.

Sulfur dioxide is a colorless gas with a peculiar choking odor. It is 2.2 times as heavy as air and is easily liquefied — under 3 atmospheres pressure at 20°C . In fact, an ice-salt mixture will liquefy it at atmospheric pressure.

Properties and uses

Sulfur dioxide is used to some extent as a refrigerant (because it is volatilized and liquefied at convenient temperatures). (See Fig. 33.) Huge quantities are necessary for the manufacture of sulfuric acid and for calcium sulfite used in papermaking. A great deal of SO_2 is used in *bleaching* and some in fumigation, although formaldehyde has almost displaced it from that field.

The sulfuric acid industry is now so enormous that the United States makes over 8,000,000 tons annually. There are two rival processes of manufacture, the lead-chamber process for 62–70 per cent acid and the contact process for concentrated acid of high purity. The less concentrated acid is used in making superphosphate fertilizer and in removing from sheet steel the black oxide (Fe_3O_4) formed in hot rolling.

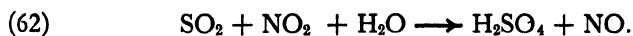


Sulfuric acid, H_2SO_4

FIG. 33. REFRIGERATION WITH SO_2 . (AFTER DEMING)

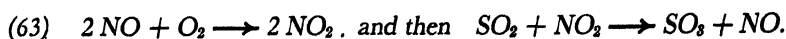
The essential reactions of the lead-chamber process involve the oxidation of sulfur dioxide to the trioxide in the presence of enough water to form sulfuric acid. A mixture of air and the dioxide reacts very slowly, so oxides of nitrogen are introduced to increase the speed of reaction.

Lead-chamber process



If the nitric oxide, NO , were wasted, the lead-chamber process would be too costly. But luckily this nitric oxide has a property — seemingly unimportant to the student but of thrilling interest to the men owning lead-chamber plants — of rapidly uniting with

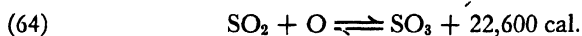
oxygen from the air to form the useful nitrogen dioxide, able to oxidize SO_2 into SO_3 .



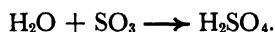
In fact nitric oxide is a *catalyst*, making the oxygen of air unite with sulfur dioxide in a roundabout way and itself being regenerated time after time.

Contact
process

In the contact process a mixture of sulfur dioxide and air is led through a series of iron tubes containing a porous supporting material such as asbestos or magnesium sulfate holding the catalyst, finely divided platinum at $400^\circ C.$, or vanadium pentoxide at $500^\circ C.$



The sulfur trioxide is caught in 98 per cent acid, reacting with the water present. Steady addition of water or dilute acid maintains this concentration.



NORMAL ANNUAL USE OF SULFURIC ACID BY UNITED STATES INDUSTRIES

| | | | |
|-----------------------|----------------|-----------------------|----------------|
| Fertilizers | 2,100,000 tons | Paints, pigments . . | 500,000 tons |
| Petroleum refining . | 1,210,000 tons | Explosives | 190,000 tons |
| Chemicals | 975,000 tons | Rayon and cellulose . | 400,000 tons |
| Dyes, coal products . | 740,000 tons | Textiles | 116,000 tons |
| Steel pickling . . . | 980,000 tons | Miscellaneous . . . | 400,000 tons |
| Metallurgical uses . | 570,000 tons | Total | 8,181,000 tons |

THE PERIODIC TABLE

The Russian, Mendeleeff, in 1869, tabulated the elements in periods of eight in order of increasing atomic weights and showed that the properties of the elements are periodic functions of their atomic weights. See the table on the following page. So fundamental was this great conception that he was able to leave gaps in the table for elements yet to be discovered. Later his prophecy was fulfilled and the new elements fitted into the system. There were a few obvious flaws such as the wrong order of tellurium and iodine but these were corrected by Moseley in 1914 when he arranged the elements in periods in increasing order of their *atomic numbers*.

Atomic
numbers

Strangely enough, these atomic numbers are all whole numbers. The atomic numbers are approximately half the atomic weights,

PERIODIC TABLE SHOWING ATOMIC NUMBERS IN RED

| PERIOD | GROUP 0 | GROUP I | GROUP II | GROUP III | GROUP IV | GROUP V | GROUP VI | GROUP VII | GROUP VIII |
|--------|--------------|---------------|--------------|--------------|--------------|--------------|--------------|--|--------------|
| 0 | | H 1.0080 | | | | | | | |
| I | He 4.003 | Li 6.940 | Be 9.02 | B 10.82 | C 12.01 | N 14.008 | O 16.0000 | F 19.00 | |
| II | Ne 20.183 | Na 22.997 | Mg 24.32 | Al 26.97 | Si 28.06 | P 30.98 | S 32.06 | Cl 35.457 | |
| III | A 39.944 | K 39.096 | Ca 40.08 | Sc 45.10 | Ti 47.90 | V 50.95 | Cr 52.01 | Mn 54.93 | Fe 55.85 |
| | | Cu 63.57 | Zn 65.38 | Ga 69.72 | Ge 72.60 | As 74.91 | Se 78.96 | Br 79.916 | Co 58.94 |
| IV | Kr 83.7 | Rb 85.48 | Sr 87.63 | Y 88.92 | Zr 91.22 | Cb 92.91 | Mo 95.95 | Ma ? | Ni 58.69 |
| | | Ag 107.880 | Cd 112.41 | In 114.76 | Sn 118.70 | Sb 121.76 | Te 127.61 | I 126.92 | Ru 101.7 |
| V | Xe 131.3 | Cs 132.91 | Ba 137.36 | Rare Earths | Hf 178.6 | Ta 180.88 | W 183.92 | Re 186.31 | Rh 102.91 |
| | | Au 197.2 | Hg 200.61 | Tl 204.39 | Pb 207.21 | Bi 209.00 | Po 210.0 | | Pd 106.7 |
| VI | Rn 222 | | Ra 226.05 | Ac ? | Th 232.12 | Pa 231 | U 238.07 | Elements are discussed on next page | Os 190.2 |
| | | | | | | | | | Ir 193.1 |
| | | | | | | | | | Pt 195.23 |

The *atomic number* appears before the symbol of each element in red and the *atomic weight* below. With atomic numbers 81-84 and 88-92 are to be assigned a number of other elements, for the most part radioactive, which have the same chemical properties as the element in the table but somewhat different atomic weights.

and represent, in reality, *the number of positive charges of electricity* (protons) *in the nucleus* of the atom of each element. In addition to the free protons there are (except in the case of hydrogen) some neutrons. Electrons are present in the outer shells of the atom. These protons in the nucleus are electrically balanced in the neutral atom by an equal number of planetary electrons; hence the atomic number is measured either by the number of free protons in the nucleus or by the number of planetary electrons.

In Group I at least one of the valences of each element is unity (Na^1 , K^1 , Ag^1). In Group II the valence is 2, and so on as we proceed to the right. To the right of Group IV, although valence toward oxygen increases, it decreases in hydrogen compounds. For example, N_2O_5 and NH_3 . As you see, correct formulas may be deduced from this useful table. Furthermore, the elements on the left are decided base formers (NaOH), those in the middle are weakly basic or weakly acidic — $\text{Al}(\text{OH})_3$ acts both ways — while those on the right are strongly acid-forming (HNO_3 and HCl). Proceeding down any column we find that the elements become more metallic (compare Si and Pb). Group 0 elements with *no combining power*, and therefore *no valence*, were discovered after Mendeleeff's time.

Use of the table

RADIOACTIVITY

Because of Becquerel's work with the penetrating radiations from uranium compounds, 1896, the Curies (Madame Marie and her husband, Professor Pierre Curie of the Sorbonne) decided to examine some of the residues from pitchblende, a uranium mineral, after all the uranium had been extracted. They found that these residues gave off similar but more powerful radiations than the uranium salt itself.

The Curies

Finally the Curies isolated polonium, a powerful source of radiations that could go through objects opaque to visible light and record their effects on a photographic plate or discharge an electroscope. As a climax radium was discovered in the residues, in 1898, with an activity 1,000,000 times that of uranium.

Soon after the discovery of radium, scientists found that it gives off three types of radiations. The *alpha rays* are positively charged atoms or nuclei of helium moving 15,000 miles per second (in a vacuum). They do not travel 15,000 miles per second in air which quickly reduces their velocity to zero. In fact, they are stopped

Radium rays

completely before they have gone 8 cm. through the surrounding air. After colliding with several thousand air molecules, their initial velocity is reduced to nothing.

*Electron
streams*

The *beta rays* are identical with cathode rays (see page 58), but move faster and are 100 times as penetrating. These cathode radiations and beta rays are known to be merely *streams of electrons*. When thrown off by radium they move nearly 186,000 miles per second (the velocity of light). Since they have only $\frac{1}{7400}$ the mass of alpha particles, they are deflected much more by a magnetic field. As negative particles they are deflected in the opposite direction from alpha particles. In spite of their tremendous initial velocity, they are completely absorbed after smashing through 1 cm. of water or 1 mm. of lead or 10 mm. of aluminum.

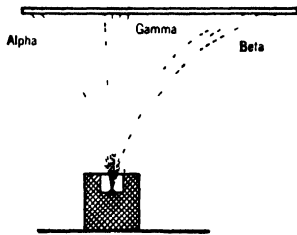


FIG. 34. THREE TYPES OF RADIUM RADIATION IN A MAGNETIC FIELD

The *gamma rays* resemble X-rays but are far more penetrating. They are electromagnetic disturbances, not streams of small particles, and hence are not deflected in a magnetic field (Fig. 34). Although alpha particles are stopped by 0.1 mm. thickness of aluminum foil, gamma rays penetrate 1 ft. of steel.

Since gamma rays affect photographic film, they are used to detect flaws in steel parts.

Radioactivity

The half-life of radium is 1590 years. Half the radium remaining at the end of that period decays in another 1590 years, and so on. This decay is due to the emission of electrons and alpha particles. There are other radioactive elements such as polonium, actinium, mesothorium, and protactinium. The cheapest is mesothorium for half of it is lost in 6.7 years. That is why luminous paint for watch faces is often prepared from a mixture of one to ten parts of mesothorium (worth about 15 cents) with one hundred thousand parts of sensitive, phosphorescent zinc sulfide.

*Importance of
radium*

Radium competes with X-rays in the treatment of cancer and in photography. The world's stock exceeds 800 g. and the price, once \$135,000 per gram, has fallen to \$25,000 because of discovery of larger ore deposits.

One of the most important results of the research on radioactivity has been the discovery of the inner nature of the atom. In recent years this has led to "atom smashing" with such high-

velocity "bullets" as *alpha rays* from polonium, *protons* (hydrogen nuclei), *deuterons* (ionized deuterium), and *neutrons*. These high velocities are given by powerful X-ray tubes and by the Lawrence cyclotron (Fig. 35). The impact of such bullets energizes the atomic nucleus which throws off various types of particles leaving a different element. Even after the bombardment ceases, this artificial radioactivity may continue for hours or weeks.

*Artificial
radioactivity*

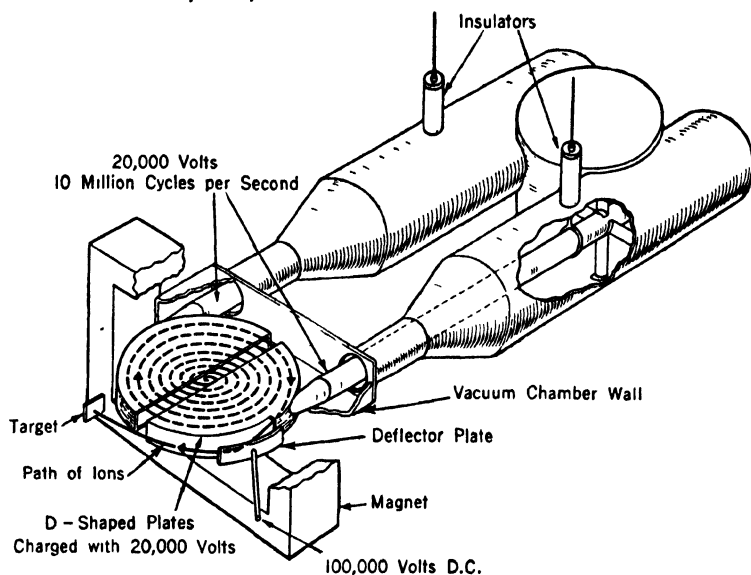


FIG. 35. CYCLOTRON PRODUCES ARTIFICIAL RADIOACTIVITY

Courtesy Life Magazine

Gaseous particles, ionized by a hot filament, are attracted by the two evacuated hollow electrodes, shaped like halves of a round pill box. The magnetic field deflects the ions and, with 20 million reversals of current per second, a succession of kicks is given them, increasing their speed until they go off at a tangent and strike the target where they may split atoms or produce artificial radioactivity.

In 1939 and 1940 the atomic fission (splitting) of the uranium isotope of mass 235 was recognized both in Europe and in America. Neutrons from radium bombardment of beryllium were slowed down by passage through such hydrogen compounds as water or paraffin so that they were effectively captured by the U-235 nucleus followed by its splitting into barium and krypton or into other elements not differing greatly in mass. This fission was accompanied by a tremendous release in energy, nearly 200,000,000 electron volts per atom of U-235.

*Uranium
fission*

There is an important chain of reactions in fission in the sense that secondary neutrons are released to cause other fissions and,

Chain reaction

theoretically, to continue until all the U-235 in a given sample is exhausted. However, the presence of U-238 and other impurities quickly breaks this *chain reaction*. Since U-235 is only one part in 140 of common uranium, the critical problem is separation to obtain pure U-235. The atomic bomb contained purified U-235 or the new element plutonium, No. 94, prepared by suitable bombardment of common U-238. Plutonium also can undergo chain fission. For further details read the note facing page 103.

THE ATMOSPHERE

Air, a mixture

About 99 per cent of air is composed of two gases, oxygen and nitrogen, and the proportions of these vary but slightly. Several other substances are present, some in proportions that vary considerably. The approximate composition of dry air by volume is

| | |
|--------------------------|----------------|
| Nitrogen | 78.00 per cent |
| Oxygen | 21.00 per cent |
| Argon | 0.93 per cent |
| Carbon dioxide | 0.03 per cent |
| | <u>99.96</u> |

In addition to the four named above there are found water vapor, helium, neon, krypton, xenon, dust, and sometimes ozone, hydrogen sulfide, and oxides of nitrogen in traces.

Not a compound

The evidence that these various constituents are present merely as a mixture, and not as one compound, is convincing.

1. Each constituent of the air retains its own properties irrespective of the presence of the others. The oxygen is just as soluble in water as if the nitrogen were missing.

2. The properties of air are the mean of the properties of its components, which is not true of a compound. The density, for example, is what might be calculated from such a mixture.

3. When liquid air evaporates, the more volatile nitrogen tends to escape faster than the oxygen. In pure compounds, water for example, the distillate has exactly the same composition as the original liquid, unless decomposition occurs. Therefore, air can be separated into its constituents by liquefaction and fractional distillation. Compounds cannot.

4. Nitrogen and oxygen can be mixed in the same proportion as found in air, without change in temperature or volume, producing a mixture like air. Evidently, no reaction occurs.

5. The proportions by weight cannot be represented by a chemical formula because the constituent gases are not found in exact multiples of atomic weights.

When air is saturated with water vapor at, say 26.6° C. or 80° F., we feel depressed and are incapable of much exertion. If there is only 50 per cent as much water vapor present as could be held at saturation, we feel brisk and alert. Hence the statement, "The most comfortable relative humidity is about 50 to 60 per cent." *Air conditioning*

Body cooling by evaporation of perspiration gives us comfort, but at high relative humidity this evaporation is slight. Even a fan stirring the air, blowing wet air away from the skin, is helpful. Air-conditioning apparatus for cooling, washing out dust, and maintaining a comfortable relative humidity makes life more pleasant on trains, in offices, homes, hotels. In factories of certain types conditioning of the air is essential.

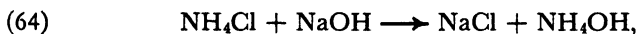
Nearly 1 per cent of the air is argon, a very "lazy" gas, valuable for its *inertness* in argon-filled electric-light bulbs. Only a trace of helium is present in air. We separate it from natural gas in the Amarillo district of Texas. Since its lifting power is about 90 per cent that of hydrogen, it is desirable as a filler for balloons and airships. Tracer and incendiary bullets cannot ignite inert helium. *Argon and helium*

NITROGEN AND AMMONIA

Nitrogen, making up four-fifths of the air, is cheap unless it must be entirely free from oxygen. Then air must be liquefied and the nitrogen separated by careful *fractional distillation*, at considerable cost.

This gas is lazy at room temperatures, but at high temperatures (electric arc) it may be forced to unite with oxygen, forming oxides of nitrogen. Persuaded by suitable catalysts and high pressure and temperature, it may react with hydrogen to form ammonia, a reaction of tremendous importance in war and in peace.

Ammonia gas is conveniently prepared in the laboratory by warming a dry ammonium salt with lime or with any other base. *Ammonia*

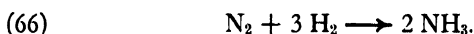


The ammonium hydroxide, NH_4OH , breaks down on heating into the gas and water. In fact, warming a concentrated solution of ammonium hydroxide is often found a most convenient method of securing the gas.

Soft coal contains 1 per cent, or more, of combined nitrogen. About one-fifth of this is recovered on distillation as ammonia. The mixed gases of distillation are passed through wash water which retains the ammonia and some impurities. Later, lime is added to this ammonia liquor and steam is passed through. The ammonia is thus driven off and may be neutralized with sulfuric acid to form ammonium sulfate (about 20 lb. from 1 ton of coal). In an average year the United States produces 900,000 tons of ammonium sulfate from coal.

*Coke
manufacture*

Up to thirty years ago coke was made in beehive coke ovens, allowing the ammonia and other gases to waste into the air. Now over 90 per cent of all American coke is made in by-product ovens with recovery of the ammonia, benzene, tars, coal gas, and other products. However, by far the larger part of all commercial ammonia is prepared by the catalyzed high-pressure, high-temperature union of nitrogen and hydrogen (Haber process).



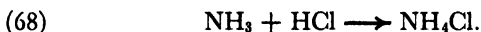
*Reactions of
ammonia*

Ammonia will "steal" oxygen from metallic oxides.

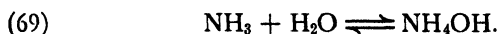


Ammonia reacts directly with some hot metals to form nitrides such as Mg_3N_2 . It is actually used to harden the surface of steel by formation of an iron nitride.

Acids react with the gas to form ammonium salts, but only in the presence of at least a trace of water.



One of the most important reactions of ammonia is with water to form ammonium hydroxide.



*Uses of
ammonia*

Ammonia is the raw material for the Ostwald catalytic process of making nitric acid. It is combined with acids to make ammonium salts used in enormous quantities for fertilizers. Its salts are used in soldering, galvanizing, in dry batteries, and the gas itself is the chief refrigerant in commercial use.

*Nitrogen
fixation*

In 1913 the German chemist Haber developed the extremely difficult process of *forcing a reaction* between nitrogen and hydrogen.



Haber discovered that at 200 atmospheres pressure and 500° C. the formation of ammonia proceeds satisfactorily if osmium or uranium is present as a catalyst. Later, Claude, in France, improved the process by increasing the pressure to 1000 atmospheres or more. Working at such high temperatures and pressures is difficult, yet success was obtained. Chrome steel free from carbon must be used for the apparatus, as otherwise the hydrogen removes the carbon from ordinary steel, leaving "rotten steel." The catalyst now used is largely iron with some substances such as the oxides of aluminum and potassium present to promote the catalytic action.

World capacity for fixed nitrogen of various types was 4,000,000 tons annually in peacetime. The demand for explosives (derived from ammonia) makes this total very inadequate for war. In peace most of the ammonia so formed is converted into ammonium sulfate fertilizer. In war much of the ammonia is catalytically oxidized by air to yield nitric acid, the basis of most high explosives.

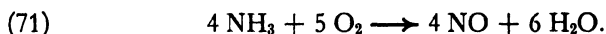
*World
production*

OXIDES OF NITROGEN. NITRIC ACID

There are five oxides of nitrogen, including laughing gas or nitrous oxide (N_2O) the anesthetic, but the engineer is mainly interested in two, nitric oxide, NO , and nitrogen dioxide, NO_2 .

The first reaction in the catalytic (over platinum gauze at 700° C.) oxidation of ammonia by air is:

*Nitric oxide,
NO*



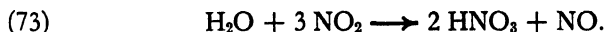
Nitric oxide (colorless gas, insoluble in water) is also made by the action of dilute nitric acid on copper and some other metals (in absence of air).

This very poisonous, red-brown gas is made quickly by direct union of nitric oxide with the oxygen of the air, below 620° C.

*Nitrogen
dioxide, NO₂*



When this NO_2 meets warm water it reacts at once.



This useful acid (dangerous to skin and clothing) was once made by action of concentrated sulfuric acid on any nitrate.

*Nitric acid,
HNO₃*



Then on warming, the volatile nitric acid was distilled and caught in condensers. Today it is all made by the catalytic oxida-

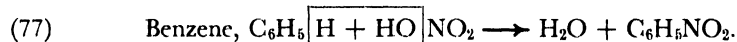
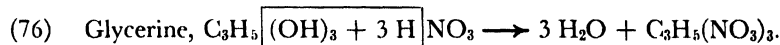
tion of ammonia and passage of the NO_2 (from $\text{NO} + \text{O}_2$) into water, as described above.

The commercial acid, density 1.14, boiling at 120.5°C ., is 68 per cent HNO_3 , the rest water. As stated before, it attacks many metals — even copper which is below hydrogen in the activity series. In this case this strong oxidizing acid probably first forms copper oxide, CuO , and then reacts with it (as with any metallic oxide).



Explosives

Cellulose (of cotton or wood) is attacked by nitric acid with formation of cellulose nitrates, some of which are high explosives. With glycerine a similar product called *nitroglycerine* is produced. Toluene yields the famous TNT (trinitrotoluene), and even starch is nitrated to form a powerful explosive. Concentrated nitric acid reacts with organic compounds in two ways, as is well illustrated by its reaction with glycerine.



Dynamite is essentially a solid mixture of nitroglycerine with wood dust, but the mixture usually includes nitrates of sodium or ammonium. Gelatin dynamite contains both nitroglycerine and “nitrocotton.”

Problems

1. How many grams of nitric acid can be made by catalytic oxidation from 1500 g. of NH_3 ? Ans. 5549.8 g.

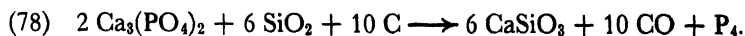
2. To make 10 kg. of nitroglycerine how many kilograms of glycerine are required? Ans. 4.05 kg.

THE PHOSPHORUS GROUP

Phosphorus, in the white or yellow form, is extremely active, igniting in warm air at about 35°C ., so it is always kept under water. Phosphorus burns are very painful and slow to heal. *White phosphorus is very poisonous*, a fatal dose being only 0.15 g. Red phosphorus, obtained by heating the white variety to $230^\circ\text{--}300^\circ \text{C}$. in absence of air is not poisonous and does not ignite below 240°C .

Phosphorus

Phosphorus is obtained from phosphate rock either in a blast furnace or in an electric furnace.



The phosphorus volatilizes (no air present), leaving the calcium silicate behind. The vapors condense to a liquid and finally solidify on cooling. With admission of air the element burns to the pentoxide, P_2O_5 , a white powder which reacts with water to form the most useful of the three phosphoric acids, H_3PO_4 . Its salts have wide industrial uses. "Superphosphate," for fertilizers, is a water-soluble calcium hydrogen phosphate, made by treating the insoluble phosphate rock with the proper amount of sulfuric acid.

Arsenic occurs as a sulfide in various ores. In fact most of it is obtained as a by-product in roasting copper sulfide ores. The arsenic burns to the trioxide, As_2O_3 , which is removed as a white dust from flue gases by the Cottrell electric precipitator. The element is black and brittle. It has a small use in industry as a hardener for some alloys, notably lead shot. The trioxide is for the greater part converted into lead arsenate, calcium arsenate, and Paris green, as weapons in the terrific battle with insects. *Arsenic*

Antimony occurs as the sulfide, Sb_2S_3 , stibnite. Formerly nearly all of it came from China but now Mexico supplies us. It is a silver-white, brittle element, melting at $680^\circ C$. *Antimony*

Its chief value is in making alloys, such as type metal, lead-antimony grids for storage batteries, bearing metals, and others. Its property of expanding on cooling makes it essential to type metal, for sharp edges result when the metal expands against the type forms.

Bismuth is found both free and in the sulfide. It is now obtained chiefly from anode slimes in the electrolytic refining of copper, lead, and tin. It is strictly metallic, brittle, and crystalline. In color it is silver-white with a reddish tint. It melts at $271^\circ C$. Bismuth may be electroplated, forming a film capable of taking a high polish. There is little use for the element alone, but in very low-melting alloys it is rather important. *Bismuth*

ORGANIC CHEMISTRY — HYDROCARBONS

There are several hundred *hydrocarbons* (compounds of hydrogen and carbon), most of them found in petroleum, natural gas, asphalt, and coal-tar products. These are divided into a few series, with some very definite relationships between the members. Rubber is a hydrocarbon, $(C_5H_8)_x$, in which possibly 2000 units of C_5H_8 are *chained together*. Natural and synthetic rubbers are discussed on pages 115 and 116. *Homologous series*

HYDROCARBON SERIES

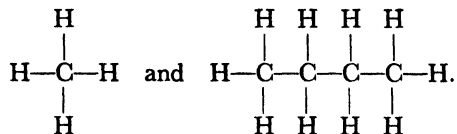
| Methane Series C_nH_{2n+2} | | Ethylene Series C_nH_{2n} | | Acetylene Series C_nH_{2n-2} | | Benzene Series C_nH_{2n-6} | |
|---------------------------------|-------------|--------------------------------|-------------|-----------------------------------|-------------|---------------------------------|-------------|
| Methane | CH_4 | { Ethene or | | { Ethine or | | | |
| Ethane | C_2H_6 | { Ethylene | C_2H_4 | { Acetylene | C_2H_2 | | |
| Propane | C_3H_8 | Propylene | C_3H_6 | Propine | C_3H_4 | | |
| Butane | C_4H_{10} | Butylene | C_4H_8 | Butine | C_4H_6 | | |
| Pentane | C_5H_{12} | Pentylene | C_5H_{10} | Pentine | C_5H_8 | | |
| Hexane | C_6H_{14} | Hexylene | C_6H_{12} | Hexine | C_6H_{10} | Benzene | C_6H_6 |
| Heptane | C_7H_{16} | Heptylene | C_7H_{14} | Heptine | C_7H_{12} | Toluene | C_7H_8 |
| Octane, etc. | C_8H_{18} | Octylene | C_8H_{16} | Octine | C_8H_{14} | Xylene | C_8H_{10} |

The CH_2
variation

In each hydrocarbon series any one member differs from its neighbor by one carbon and two hydrogen atoms, that is, by a CH_2 unit. For example, in the methane series (so called from the name of its simplest member) we know that a hydrocarbon with five carbon atoms must have twelve hydrogen atoms in the molecule (C_5H_{12}) because in the series formula (C_nH_{n+2}), the value of n is 5 in this case, and $2n + 2 = 12$.

The methane and benzene series are by far the most important in the field of organic chemistry, if their derivatives are considered. The first four members of the methane series are gases at ordinary temperatures; from C_5H_{12} to $C_{15}H_{32}$ they are liquids; beyond $C_{16}H_{34}$ they are solids.

The existence of so many compounds of two elements depends wholly on the *tetravalence of carbon* and the ability of carbon atoms to hold other carbon atoms in *long chains* or in *rings*. For example, we may write



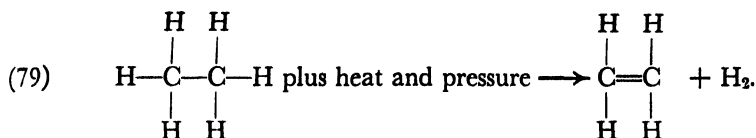
From the hydrocarbon table we conclude that these structural formulas represent methane and butane.

Methane

Methane, CH_4 , occurs in natural gas up to 90 per cent or more. It is the pressure of methane that forces petroleum up from deep wells.

Ethylene

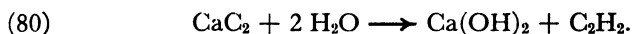
Ethylene, C_2H_4 , is the simplest member of a series of unsaturated hydrocarbons. Note the conversion of ethane, C_2H_6 , by "cracking" with heat into ethylene and free hydrogen.



The symbol, =, does not mean "equals" but indicates that the two connected carbon atoms are unsaturated, "hungry" for addition of other elements, bromine, for example. This makes possible ethylene bromide, $\text{C}_2\text{H}_4\text{Br}_2$, used in high-octane gasoline to convert the lead of tetraethyl lead into lead bromide, less harmful than lead oxide to ignition points. Ethylene glycol, $\text{C}_2\text{H}_4(\text{OH})_2$, a similar grouping, is the leading "antifreeze" liquid.

A rubberlike product called Thiokol is made commercially by converting ethylene into ethylene dichloride which is then treated with a sodium sulfide complex. *Polymerization* (joining of like molecules to form larger molecules) follows. *Thiokol*

Acetylene, C_2H_2 , still more unsaturated than ethylene, is made by reaction of water with calcium carbide.



Acetylene, fed with oxygen in the acetylene torch, produces an intensely hot flame. With this flame a 6-in. steel shaft can be cut through in less than a minute. As soon as the steel melts, it burns in air and is blown away as oxide, in sparks. *Acetylene*

Great quantities of acetic acid are made from acetylene by catalytic hydration and controlled oxidation in the second stage. The new synthetic rubber, Neoprene, is derived from acetylene, which itself is based upon limestone and coke in the manufacture of calcium carbide.

Normally the United States produces two-thirds of the world's 2,000,000,000 barrels of petroleum. Since mere distillation of crude oil yields only about 25 per cent of "straight run" gasoline, it is evident that the process of *cracking* (not yet universal) which more than doubles this yield has saved for the future nearly one-half the crude oil otherwise needed for our 30,000,000 automobiles. Even with every device in use for increasing gasoline yield, we are now beginning to face a serious shortage. The solution will be production of light oils from coal by *hydrogenation* as will be explained below. *Petroleum hydrocarbon mixtures*

Although kerosene was once the chief fraction separated (by distillation), gasoline now leads. Such other fractions as lubricants,

fuel oil, gas oil, paraffin wax, coke, and petroleum asphalt are also very important.

Gasoline

Gasoline is a mixture of hydrocarbons ranging from C_6H_{14} up to about $C_{12}H_{26}$ while kerosene hydrocarbons may range from $C_{12}H_{26}$ to $C_{16}H_{34}$; and lubricants range as high as $C_{36}H_{72}$. It is a remarkable fact that gasoline yields six times the power of an equal amount of nitroglycerine — but more slowly. This is because nitroglycerine contains its oxygen for combustion while gasoline uses air.

Cracking

Cracking is already familiar in thermal decomposition, remember $H_2S \longrightarrow H_2 + S$ and $C_2H_6 \longrightarrow C_2H_4 + H_2$. The apparatus used for cracking petroleum is indicated in Fig. 36. In the high-temperature cracking chamber, larger molecules are split, or cracked, into smaller and more volatile molecules. *Vapor-phase cracking*, operating under somewhat higher temperatures, produces large quantities of unsaturated hydrocarbons and “ring” compounds that give a high antiknock value to the gasoline formed. High temperatures, great pressures, and catalysts are now employed in “cracking.”

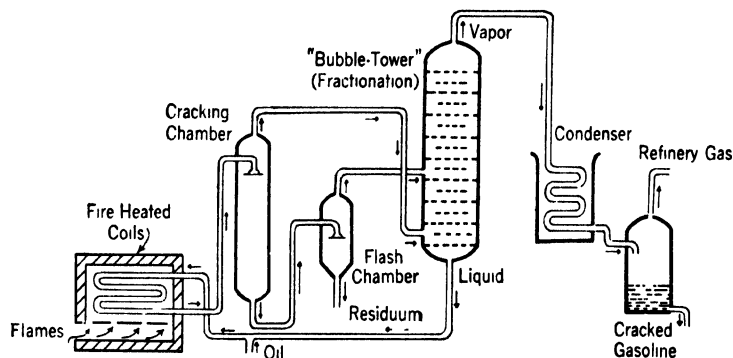


FIG. 36. PETROLEUM CRACKING PRODUCES GASOLINE FROM HEAVIER OILS

Hydrogenation

The success of hydrogenation of coal in Germany in recent years led to *catalytic hydrogenation* of petroleum in this country. The gasoline resulting is remarkably low in sulfur compounds and rich in benzene ring molecules of excellent antiknock value. In Germany and in England, where there is no local petroleum, lignite and bituminous coal are being hydrogenated to produce huge quantities of motor fuel.

Chemicals from petroleum

A large number of compounds are now being prepared, not merely separated, from petroleum by chemical reactions. The pentanes, C_5H_{12} , may be chlorinated, converted into amyl alcohol and finally into amyl acetate, a solvent used in nitrocellulose lac-

quers. Even common ethyl alcohol (intoxicating) can be prepared from petroleum fractions. Another petroleum chemical, isopropyl alcohol (C_3H_7OH), is a formidable rival to ethyl alcohol. One of the butyl alcohols, similarly prepared in large quantity, is converted into butyl acetate, an important solvent for the lacquer industry. Benzene, toluene (for TNT), glycerine, and butadiene (for synthetic rubber) are also triumphs of the petroleum chemist.

Pure isooctane (C_8H_{18}) is given an octane number of 100 (high antiknock value) while pure heptane (C_7H_{16}) shows no antiknock value whatever. Any fuel that matched a mixture of 70 per cent isooctane and 30 per cent heptane would be said to have an octane number of 70. (See Fig. 37.) The usual octane numbers of gasolines for automobiles range from 60 to 78. Aviation gas has a 100-octane rating, obtained by use of tetraethyl lead, isooctane, tryptane, neo-hexane, and benzene.

Propane, (C_3H_8), and butane (or isobutane), (C_4H_{10}), have recently become important by-products of oil refining, whereas formerly they were burned to produce steam. Now these gases are liquefied by pressure in steel cylinders or even in tank cars and are sold as a convenient economical source of gaseous fuel for rural homes.

Natural rubber occurs in the latex of many trees and plants, the most important being *Hevea brasiliensis*, grown in plantations in the Dutch-British East Indies and growing wild in Brazil. We have experimented with the guayule shrubs in California, the Russian dandelion, and a vine from the West Indies. In normal years we use 600,000 tons of rubber in the United States.

The milky latex, or sap from the tapped tree, is coagulated with acid, the raw rubber washed, mixed with carbon (carbon black) and zinc oxide to give wearing qualities to tires, and with sulfur and reaction accelerators to make vulcanization possible.

Synthetic rubbers are really not substitutes, for some of them have superior oil resistance as well as better air and light resistance than natural rubbers.

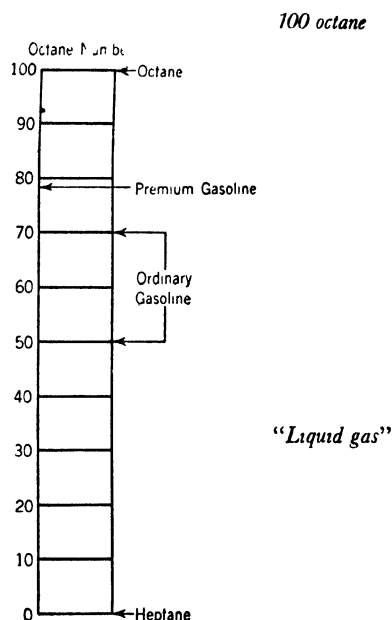


FIG. 37. OCTANE NUMBERS

Rubber,
(C_5H_8)_x

Synthetic
rubber

Natural rubber is a polymer (long-chain molecule) of isoprene, $\text{CH}_2=\text{C}(\text{CH}_3)-\text{CH}=\text{CH}_2$ which nature has joined in elastic chains. Vulcanization involves the reaction with sulfur at these double bonds, the sulfur atom serving to tie parallel chains together giving increased strength. The double bond sign ($=$) indicates unsaturation of carbon, that is, its ability to add sulfur or other atoms.

Monomers

This topic naturally follows a study of petroleum refining, because a number of these synthetic rubbers use as the building brick or unit (*the monomer*) butadiene ($\text{CH}_2=\text{CH}-\text{CH}=\text{CH}_2$) which may be prepared from petroleum derivatives or from common alcohol.

SYNTHETIC RUBBERS

| <i>Commercial Name</i> | <i>Monomer (Unit)</i> |
|------------------------|--|
| Natural rubber | $\text{CH}_2=\text{C}(\text{CH}_3)-\text{CH}=\text{CH}_2$, isoprene |
| Neoprene | $\text{CH}_2=\text{CCl}-\text{CH}=\text{CH}_2$, chloroprene |
| Buna S | $\text{CH}_2=\text{CH}-\text{CH}=\text{CH}_2$, butadiene, and styrene |
| Butyl rubber | Petroleum units. Not yet disclosed |
| Ameripol | Butadiene + copolymer secret as yet |
| Koroseal | $\text{CH}_2=\text{CHCl}$, vinyl chloride, plasticized with equal weight of tricresyl phosphate |
| Styrene | $\text{CH}_2=\text{CH}(\text{C}_6\text{H}_5)$ |

By 1945 we were making nearly 1,000,000 tons of synthetic rubber yearly, most of it Buna S, although neoprene, butyl, and other rubbers were included. In a competitive market after the war the relative proportions of natural and synthetic rubber sold will be determined by relative manufacturing costs, quality, and policies regarding foreign trade.

*Benzene,
 C_6H_6*

When bituminous coal is distilled (for the coke residue) many volatile products escape and are captured as by-products. Ammonia, coal gas, benzene, and toluene are notable products plus coal tar, a mixture that in turn is distilled to yield many valuable products. Our 130,000,000 gallons of benzene (benzol, as the public calls it) are consumed as an antiknock addition to our gasoline and as raw materials for manufacture of dyes and medicines.

Saturation

Chemically, benzene must be classed as an unsaturated hydrocarbon, since it is capable of adding on six hydrogen atoms or six bromine atoms per molecule. It reacts with nitric acid to form nitrobenzene, $\text{C}_6\text{H}_5\text{NO}_2$, and this in turn is reduced by active hydrogen to *aniline*, $\text{C}_6\text{H}_5\text{NH}_2$, the mother substance of a host of

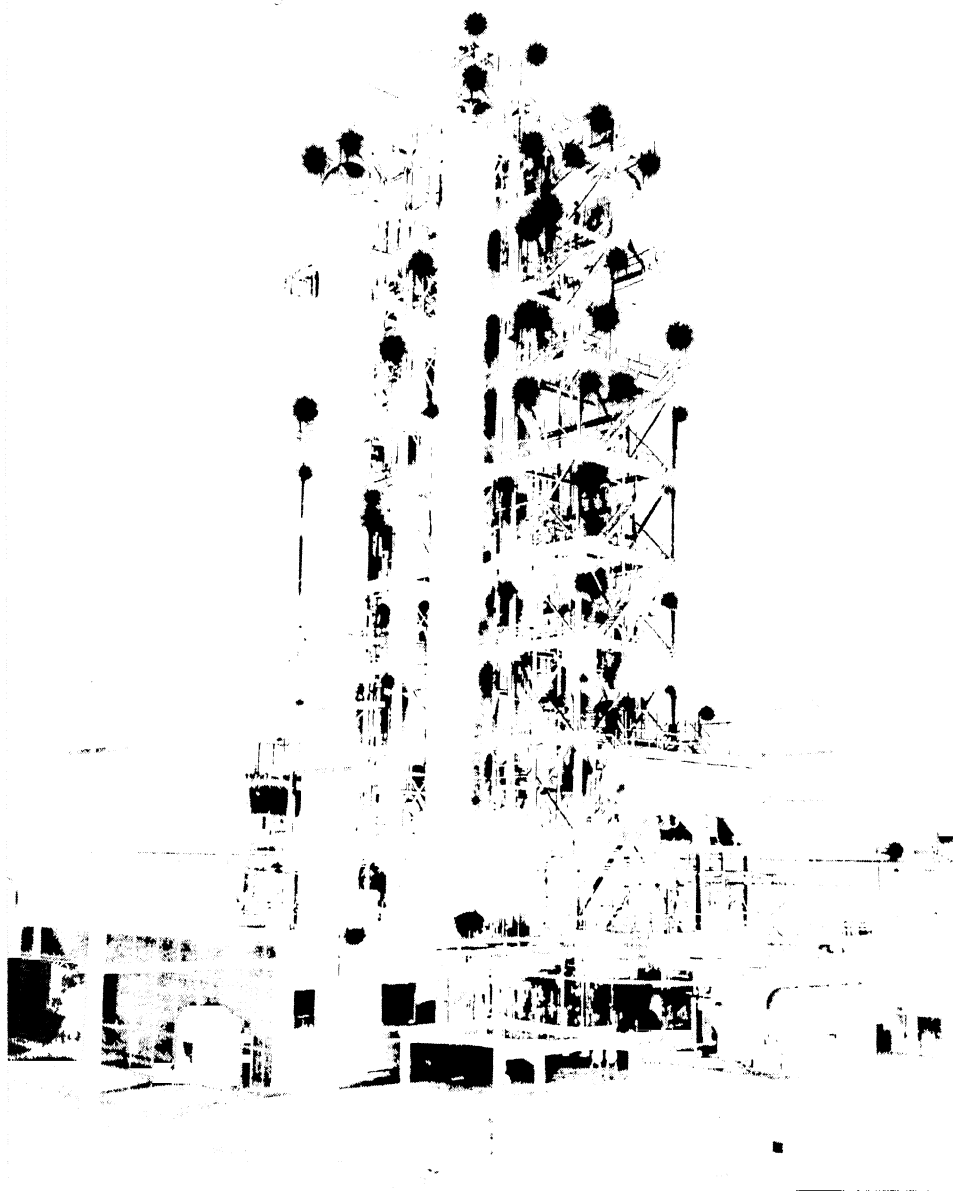
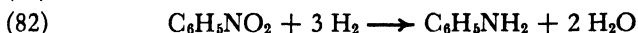


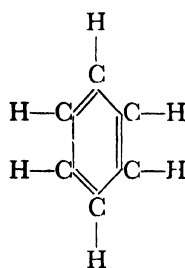
FIG. 38. STYRENE PLANT FOR PRODUCING BUNA-S RUBBER

beautiful "aniline dyes." It can also be attacked by hot concentrated sulfuric acid and other reagents. In fact, over one-half of all carbon compounds are derivatives of benzene.

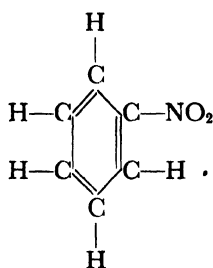


Ring molecules

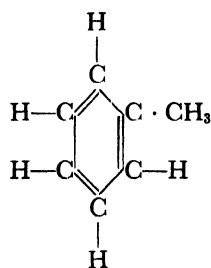
A very interesting line of evidence shows us that the six atoms of benzene are tied together in a ring, very different from the chains of carbons in the methane series. A comparison of a few structural formulas of benzene and its derivatives will be worth while.



Benzene



Nitrobenzene



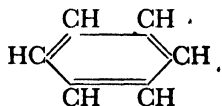
Toluene

Toluene,
 $\text{C}_6\text{H}_6 \cdot \text{CH}_3$

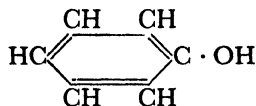
Toluene, a colorless liquid boiling at 110°C , is closely associated with benzene in its manufacture from bituminous coal and resembles benzene in many properties. Its special interest to the public is because it is the source of the world's greatest explosive, TNT (trinitrotoluene). Our supply of toluene from coal distillation is quite inadequate for a major war, so it is fortunate that we have learned how to prepare it in large amounts from petroleum.

Phenol,
 $\text{C}_6\text{H}_6 \cdot \text{OH}$

Phenol is important since it reacts with formaldehyde to yield the important plastic, bakelite. We use 80,000,000 lb. yearly in the United States.



Benzene



Phenol

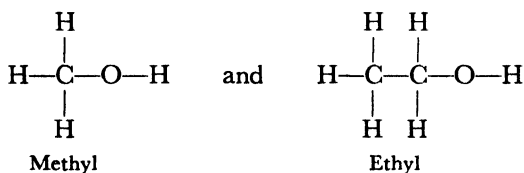
Picric acid, one of the chief high explosives, is produced by nitration of phenol. *Creosote* is a crude mixture of phenol-like compounds separated from coal tar. In a normal year we use 200,000,000 gallons to preserve railroad ties and other timbers.

Coal gas and water gas ($\text{CO} + \text{H}_2$, from hot coke and steam) *Fuel gases* have already been mentioned as has natural gas. *Producer gas* is one of the cheapest to manufacture. In a bed of coal five or six feet deep in a furnace, the lower layers may be red-hot while the upper layers are not burning at all, possibly only suffering distillation of their volatile matter. If no flame plays over the surface, the escaping gas has good fuel value. Carbon dioxide, CO_2 , formed in the lower layers is reduced by the excess of hot carbon above to the valuable carbon monoxide, CO . All the nitrogen of the air used is found in the producer gas, making up nearly one-half its volume; hence the heating value is low — but the cost is also low. The best economy is to use this gas while it is still hot, thus saving all its heat.

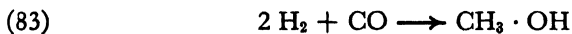
A "semiwater gas" is made by blowing some steam with the air through the deep fuel bed in the producer-gas process. Reaction with the steam adds hydrogen from the water and more carbon monoxide.

ALCOHOLS, ACIDS, FATS, AND SOAPS

Alcohols in general contain OH groups linked directly to carbon atoms as in methyl alcohol, CH_3OH , common ethyl alcohol, $\text{C}_2\text{H}_5\text{OH}$, and so on.



In commercial practice methyl alcohol is made by the retort distillation of wood, hence its common name, *wood alcohol*. In 1924 there was developed a catalytic process for preparing methyl alcohol by the direct union of carbon monoxide and hydrogen under considerable pressure in the presence of zinc oxide and copper, or other catalysts. *Methyl alcohol, CH₃OH*



A very pure product is prepared in commercial quantities at a cost lower than by wood distillation. The wood distillation industry would cease to exist were it not for the superior charcoal produced by it.

Methyl alcohol, CH_3OH , a colorless liquid boiling at 67.4°C. , mixes with water in all proportions, is intoxicating, and extremely poisonous. It is much used as a solvent in varnishes, and as the starting point in the manufacture of formaldehyde.

Ethyl alcohol,
 $\text{C}_2\text{H}_5\text{OH}$

Ethyl alcohol, commonly known as "grain alcohol," a colorless, inflammable liquid boiling at 78°C. , mixes with water in all proportions.

It is intoxicating but not poisonous. It is one of the best solvents in the laboratory and the starting point in the manufacture of many useful compounds. Internal-combustion motors can use it, with certain adjustments, and when our coal and oil deposits are exhausted the world will probably turn to cheap alcohol as a power fuel. Luckily our coal supplies are good for hundreds of years.

Alcoholic fermentation of sugars supplies us with a great part of the ethyl alcohol used. The fermented liquid must be concentrated by a column still (Fig. 39) to obtain the 95 per cent alcohol of industry. Glucose is the only common fermentable sugar; hence, cane sugar, malt sugar, starch, and cellulose must all be converted into glucose before they can yield alcohol. The reactions involved are given later (p. 124).

Absolute alcohol, when listed for taxation by our government, is termed *200° proof* and the ordinary 95 per

cent alcohol is *190° proof*. The standard is *100° proof* or 50 per cent alcohol by volume.

Glycerine (glycerol, to be exact) is an alcohol, although we are inclined to forget that fact, since it is nonintoxicating. Its structure shows the presence of alcoholic hydroxyl groups, three of them in each molecule. Glycerol is a colorless, syrupy liquid of sweetish taste. Since it is *hygroscopic* (attracts moisture from the air), it is often put in mixtures because of this property. Both glycerol and

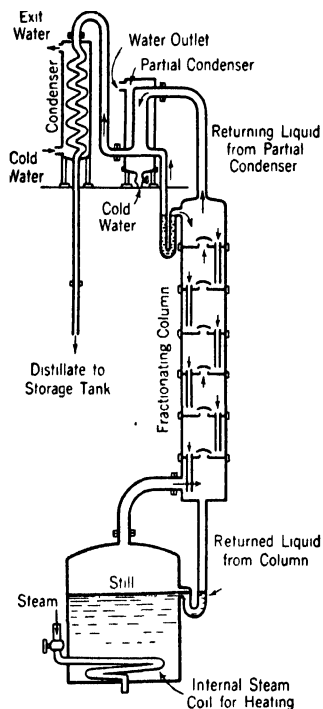


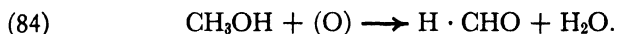
FIG. 39. ALCOHOL COLUMN STILL

The more readily condensed water vapor returns to the still while the more volatile alcohol vapor reaches the condenser.

Glycerine

glycol, $C_2H_4(OH)_2$, are much used in antifreeze solutions for automobiles. When fats are broken down in soapmaking, glycerol is a by-product. It is now possible to make great quantities of glycerol from certain products of the petroleum refinery.

The general formula for all aldehydes may be written $R \cdot CHO$, *Aldehydes,*
 $R \cdot CHO$
 in which R represents any radical such as methyl, CH_3 , or ethyl, C_2H_5 . Wood alcohol can be oxidized moderately by air and copper oxide, to yield formaldehyde, $H \cdot CHO$. Other aldehydes are made in similar fashion. They are all reducing agents.

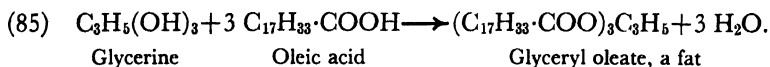


Formaldehyde is used as a 40 per cent solution called *formalin*. It is a valuable germicide and antiseptic. A large amount, 50,000,000 lb. annually, is used to react with phenol (carbolic acid) in the preparation of bakelite, perhaps the most common plastic. *Formaldehyde,*
 $H \cdot CHO$

The simplest organic acid is formic acid, $H \cdot COOH$, but a far more important one is acetic acid. Fruit acids (citric acid and so on) and such fatty acids as stearic acid are well known. The general formula for any organic acid is $R \cdot COOH$.

Acetic acid, $CH_3 \cdot COOH$, found in vinegar and made from acetylene, is used to the extent of 200,000,000 lb. yearly in this country. Much of it is converted into acetic anhydride, $(CH_3CO)_2O$, for the manufacture of cellulose acetate (one form of rayon). *Acetic acid,*
 $CH_3 \cdot COOH$

When an alcohol, $R \cdot OH$, reacts with an organic acid, $R' \cdot COOH$, water and an "ester," $R' \cdot COOR$, are formed. The esters derived from glycerine (as the alcohol) and the higher fatty acids, such as palmitic and stearic (and oleic from a related series), are called fats. This means that fats are a particular kind of ester. We do not have to synthesize the fats — plants and animals have done that for us — but if necessary it could be done. *Fats and oils*



There are very few simple fats in animal and plant tissue. Olive oil is largely glyceryl oleate (*olein*) but lard is a mixture of 40 per cent glyceryl palmitate (*palmitin*) and stearate (*stearin*) with 60 per cent olein. Beef suet contains the same fats but is harder because it contains only 25 per cent olein. Olive oil and cottonseed oil are liquids because they contain fully 75 per cent olein.

FATS USED IN THE UNITED STATES, 1939

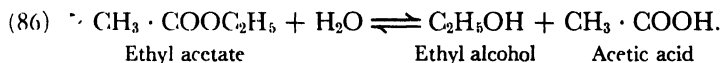
| | | | |
|----------------------------|-------------------|-------------------------------------|-----------------|
| Cottonseed oil | 2,630,000,000 lb. | Fish oils | 222,000,000 lb. |
| Tallow, inedible | 878,000,000 | China wood oil (tung oil) | 91,000,000 |
| Coconut oil | 800,000,000 | Corn oil | 220,000,000 |
| Soybean oil | 710,000,000 | Castor oil | 41,000,000 |
| Linseed oil | 345,000,000 | Peanut oil | 132,000,000 |
| Palm oil | 383,000,000 | Babassu | 24,000,000 |

In 1943 the United States production of soybean oil was three times the 1939 amount.

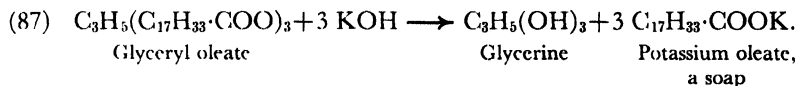
The only suitable drying oils to rival linseed oil at present as a paint vehicle are tung oil, soya oil, fish oils, and perilla oil. Such paint oils do not really dry, but are *oxidized* to form tough solids.

Esters

Esters (compounds of alcohols with organic acids) are hydrolyzed by water, especially hot water. The equilibrium reaction below indicates this.



This *hydrolysis* takes place faster if a base is present. Of course, a salt of the fatty acid must be formed when the particular ester is a fat. For this purpose a base such as KOH is used.



This reaction is called *saponification* of the ester, and the salt of the fatty acid is a soap.

Soap

Soapmaking on a commercial scale is usually a matter of boiling cheap mixed fats with caustic soda, NaOH, if the firm, "hard" soaps are desired, or with caustic potash, KOH, if "soft" soaps are wanted.

THE 'CARBOHYDRATES

This important class of compounds, which includes starch, the sugars, and cellulose, contains carbon and just twice as many hydrogen atoms as oxygen atoms in each molecule. This means that the two latter elements are found in the proportion to form water. Hence the name *carbohydrates*.

Cellulose,
(C₆H₁₀O₅)_x

Cellulose, the world's greatest crop, makes up half the weight of dry wood. Pure linen and cotton fibers, as well as the stringy parts of many vegetables and grasses, represent cellulose.

One of the reasons our forest resources of cellulose are rapidly being exhausted is the fact that every American citizen uses over 200 lb. of paper annually. Spruce, hemlock, and balsam are among the woods favored for papermaking. Interlacing fibers give paper its strength.

Starch is stored in roots like potatoes, in grains, in chestnuts, and in the trunk of the sago palm. Rice contains 75 per cent starch, corn 50 per cent, and potatoes 20 per cent. It is a white substance of very high molecular weight $(C_6H_{10}O_5)_x$, insoluble in cold water, but swelling in hot water to a pasty translucent mass, useful as an adhesive. Having alcoholic hydroxyl groups, it can, of course, be nitrated. *Nitrostarch* is a very safe explosive and is used to the extent of thousands of tons in filling hand grenades. In the laundry, starch is indispensable. It is also one of the chief sources of ethyl alcohol. When treated with water and a trace of acid as a catalyst, it is hydrolyzed into fermentable glucose.



FIG. 40. THE SUGAR MOLECULE

The formula for sugar is $C_{12}H_{22}O_{11}$. In the model the black balls represent carbon atoms, the grey balls represent oxygen atoms, and the white balls are the hydrogen atoms. Notice the arrangement.

Sugars of the formula $C_{12}H_{22}O_{11}$ (such as cane, malt, and milk sugars) can be broken down into simpler sugars of the formula $C_6H_{12}O_6$ (like glucose and levulose). (See Fig. 40.)

Cane sugar, beet sugar, and maple sugar are all of the same composition when purified. The cane is crushed to extract the juice, which is then treated with lime to coagulate impurities that would otherwise ferment. The juice (12 to 18 per cent sugar) is next treated with carbon dioxide to precipitate the lime as calcium carbonate, or with phosphoric acid to precipitate calcium phosphate. The solution is boiled down in vacuum pans at as low a temperature as $50^{\circ}C.$ because if boiled at atmospheric pressure and above $100^{\circ}C.$, the sugar would be decomposed. Concentra-

Starch,
 $(C_6H_{10}O_5)_x$

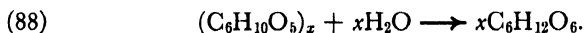
Sugars

Sucrose,
 $C_{12}H_{22}O_{11}$

tions of 65 per cent are possible. When the sugar crystallizes it is whirled free from molasses in *centrifugals* and shipped to refineries for further purification. When we can get it we use 7,000,000 tons yearly.

Glucose
(dextrose)

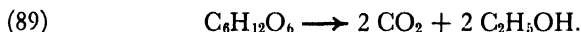
Glucose, $C_6H_{12}O_6$, is made commercially from starch which is heated with water and a little hydrochloric acid as a catalyst.



Glucose is used in candies, soda fountain syrups, baked goods, and even as a filler for cheap soaps and leather. As an intermediate compound in the manufacture of alcohol (as well as whisky and beer) from the starch of cereals, glucose is of vast importance.

Enzymes and
fermentation

The yeast plant, a very lowly but useful organism made up of microscopic cells, secretes two chemicals known as *invertase* (or *sucrase*) and *zymase*. These are true catalysts. Sucrase splits sugar (in the presence of water) into glucose and fructose, both fermentable, although fructose is less so than glucose. Zymase finishes the work begun by the sucrase and converts the simple sugars into alcohol and carbon dioxide.



Nature's
catalysts

It is an interesting fact that a germinating seed (barley, for example) develops enzymes as it sprouts. These catalysts, *amylase* and *maltase*, convert the insoluble starch of the seed into soluble glucose capable of being carried by the sap of the growing plant. The brewer borrows this clever device from the sprouting grain in order to make alcoholic drinks from starchy cereals. He allows barley to sprout — of course amylase and maltase are developed in the process — then heats the sprouts to kill all life. The catalysts, mere chemicals, are uninjured. This “malt” is then mixed with the warm, moist, crushed grain and the starch is hydrolyzed to maltose (malt sugar), $C_{12}H_{22}O_{11}$. Yeast is then added and the malt sugar is further hydrolyzed to form glucose. The zymase completes the change by fermenting this glucose into alcohol. By distillation a 45 per cent alcoholic solution well known as “whisky” is secured.

Starch (by amylase) \longrightarrow maltose,
Maltose (by maltase) \longrightarrow glucose,
Glucose (by zymase) \longrightarrow alcohol.

PLASTICS AND SYNTHETIC FIBERS

We have already mentioned bakelite (made from formaldehyde and phenol) and celluloid (lower nitrated cellulose mixed with some alcohol and camphor). These were the first plastics, but now we have many, readily molded into solid forms. Some soften with heat and can be shaped readily. A waterproof plastics liquid is now used as a bond between the thin layers of plywood. With heat and pressure it sets to a powerful bonding solid used in the bodies of some aircraft, in light boats, etc. Among other plastics we should mention the following:

The protein (casein) of milk as well as the protein of soy beans reacts with formaldehyde, $H \cdot CHO$, to form *protein plastics*.

Urea, $CO(NH_2)_2$, which is now manufactured in this country as a possible fertilizer, a stabilizer in high explosives, and a reactant for plastics manufacture, condenses with formaldehyde to yield a new plastic called *beetle* or *plaskon*. It takes colors well, is not easily broken, and so finds use as tableware. The *urea-formaldehyde resins* or plastics are much used now in treating textiles for anticreasing, waterproofing, and fireproofing.

Vinyl resins, prepared indirectly from acetylene or ethylene, take transparent colors extremely well, are odorless and tasteless, and make useful varnishes or lacquers. Vinyl chloride, $HC_2=CHCl$, is polymerized to produce polyvinyl chloride fabrics.

The rather new *methacrylate resin*, "Plexiglas" or "Lucite," is transparent (even to ultraviolet light), flexible, and is easily scratched (and repolished). Heavy sheets serve as windows for some airplanes. The monomer or unit for polymerization is $CH_2=C(CH_3)-COOCH_3$.

Rayon is a term used to describe various fibers resembling silk in appearance (although chemically different). In every case the material is dissolved to yield a pasty liquid that may be spun (through the many holes of a "spinneret") and coagulated in a suitable bath. (See Fig. 41.)

Viscose is made by the reaction of sodium hydroxide, $NaOH$, and carbon disulfide, CS_2 , with cellulose. The liquid is spun and then decomposed into the original cellulose, but with the silken luster desired. Viscose leads all the rayons. Formed into tough transparent sheets known as *cellophane* it has great sale as a wrapping material. In waterproof form it is popular in food packaging.

Nylon

Nylon resembles silk and wool chemically for it is a nitrogen compound. Basically, it is derived from coal, air, and water. It is popular in hosiery and may be used in parachutes. Massive threads of *nylon* are substituted for catgut in tennis rackets.

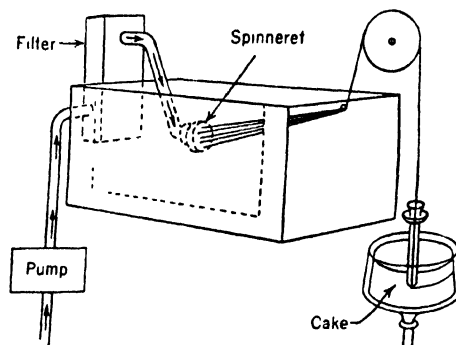


FIG. 41. RAYON SPINNING

Rayon solution is forced through many small holes in a platinum disk (spinneret), coagulated as thread in an acid bath, and spun into lustrous thread. (Colgate and Company)

SILICA AND GLASS

*Silicon and
Silica, SiO₂*

Silicon does not occur free, yet it is the commonest element after oxygen. Over one-fourth the earth's crust is combined silicon. Its oxide (SiO_2) is found everywhere as quartz or in sand and sandstone. Clay and all common rocks except limestone and dolomite are silicates. When a mixture of iron oxide, Fe_2O_3 , and silicon dioxide, SiO_2 , is reduced with carbon, ferrosilicon is produced. It is much used in steel manufacture to remove air bubbles.

Silicon steel, containing only a few per cent of silicon, has for the past twenty years been used for transformer cores and in that time has saved the electrical industry hundreds of millions of dollars. With the previous iron cores there was a considerable heat loss when the magnetic state of the transformer was changed. This and other losses were greatly reduced by silicon steel. Under low magnetizing forces it is far more magnetic than the best Swedish iron and this quality does not deteriorate.

Quartz

Silica, SiO_2 , called *quartz* as a mineral, is very common. Its melting point is not sharp due to the high viscosity of the melted material, but it is above 1700°C . Clear *fused quartz* (see Fig. 42), melted under alternate vacuum and pressure to get rid of air

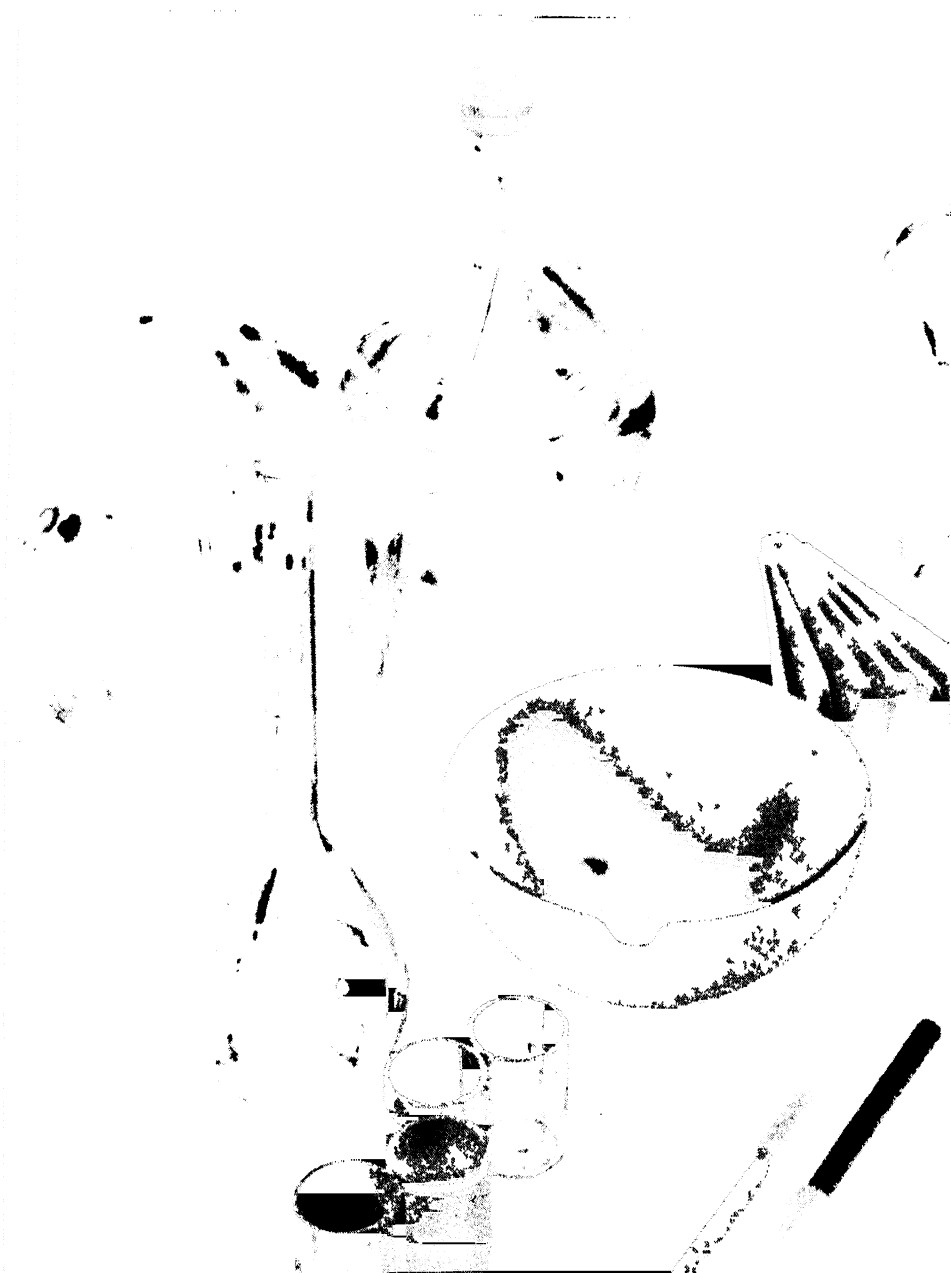


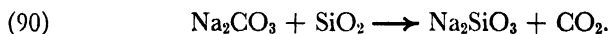
FIG 42 LABORATORY GLASSWARE OF FUSED QUARTZ

Courtesy General Electric Company

bubbles, is very useful. Its high transmission of ultraviolet light is an important property. Since its coefficient of expansion is only 6 per cent of that of ordinary glass, it can stand sudden changes of temperature without cracking. This property makes it suitable for clock pendulums, tuning forks, and thermometers, as well as for huge lenses and reflectors in astronomical instruments.

Silicic acid,
 H_2SiO_3

Since SiO_2 is acidic ($\text{H}_2\text{SiO}_3 \rightarrow \text{H}_2\text{O} + \text{SiO}_2$) it reacts with bases but not with acids (except hydrofluoric acid). With fused sodium carbonate it displaces carbon dioxide, as would any ordinary acid. The volatility of one of the products, carbon dioxide, prevents reversal of the reaction.



The sodium silicates (water glass) formed in this and other ways have enormous use as cheap adhesives in making heavy fiberboard for boxes.

Glass

Glass is a mixture of silicates, usually of sodium and calcium, although many variations are possible. Common glass might be considered as $\text{Na}_2\text{SiO}_3 \cdot \text{CaSiO}_3$, but in practice it contains more silica. The dot used in the formula above indicates close association or addition of molecules.

Glass is made by melting sand, SiO_2 , lime, CaO , and soda, Na_2CO_3 , in a gas-heated furnace. If iron is present as an impurity, the glass is green, due to ferrous silicate. Since ferric silicate, $\text{Fe}_2(\text{SiO}_3)_3$, has a very weak yellow color, it is an advantage to oxidize the green ferrous salt with manganese dioxide. Selenium is now replacing manganese dioxide as a color corrector. Soft or low-melting glass is of the lime-soda type. Hard glass, of higher melting point, is prepared by substituting potassium carbonate for the sodium carbonate.

Window
panes

Window glass is blown from a ball of hot glass into a cylinder, cut open while still hot, and allowed to flatten in the annealing oven. Bottles are made by blowing the glass in a mold. Where window-glass cylinders were formerly blown by men with strong lungs, they are now blown by machines. A hollow ring is lowered into a pool of molten glass, a bubble started, and then the ring is raised while air is blown into the bubble. As a result a beautiful glowing cylinder 40 ft. high and 2 ft. across is suspended in the dark factory like a pillar of light. This cylinder is lowered on to supports, cut open, and allowed to flatten.

Plate glass is cast on a wide iron table, rolled flat, and when *Plate glass* cold it is ground to remove unevenness and polished with rouge. In some factories the plate-glass "melt" is allowed to flow on a revolving drum and is then passed under a roller, cut into sheets, and annealed by slow cooling.

The automobile industry has created a demand for plate glass *Shatterproof glass* exceeding even its use for show windows. The possibility of accident and injury from flying glass has led to the use of nonshattering glass windshields and windows. Two layers of sheet glass and a middle layer of transparent cellulose acetate or vinylite plastic, with a suitable adhesive, are powerfully compressed. Under impact the flexible center layer holds broken fragments. A similar laminated glass (five layers) is in use for bulletproof windows in bank offices. Such a glass 1 in. thick turns aside machine-gun fire at 45 ft.

Tempered glass, a recent product, is important because it is five times stronger than plate glass. It is also more flexible, better able to stand sudden temperature changes, and although it crumbles it does not splinter when broken. Sudden cooling (by air) places the surface under compression and gives this glass its toughness.

Resistance glass of the pyrex type contains some boron trioxide *Pyrex* (B_2O_3) in place of part of the silica (SiO_2). It resists the attack of chemical reagents and is essential to the laboratory. Since it expands very little with heating, it is not likely to crack with sudden temperature changes. This property, with its unusual strength, has made pyrex glass popular for baking dishes.

Glass textiles have recently come into use wholly because the *Glass textiles* filaments are now drawn much smaller than ever before with resultant great increase in strength. A single filament has a tensile strength of 1,500,000 lb. per square inch of cross-section, but the filaments are so tiny that 60-strand thread still has a strength of only 17,000 lb. Two pounds of the finest filament would encircle the earth. This fine drawing that eliminates all internal flaws explains this tensile strength of ten times that of the best steel. A large market immediately developed for fibers woven into electric insulating tape, and another market for fluffy blanket heat insulation of houses, ships, and locomotives, and still another for dust removal from conditioned air. (See Fig. 43.)

Porcelain enamels are glasses varied somewhat in composition. They are made white by suspended tin oxide or calcium phosphate

or fluorspar. They may contain much Al_2O_3 , and B_2O_3 , and Sb_2O_3 as well as SiO_2 . Bath tubs, sinks, etc. may be made of iron or steel coated with enamel.

Boron compounds

Boron carbide, B_4C , a new product of the electric furnace, is next to the diamond in hardness. Boric acid, reduced at high temperature by pure coke, releases boron which then unites with carbon. Its extreme resistance to abrasion makes it suitable for sandblast nozzles, wire-drawing dies, and jeweled bearings. In powdered form it can be compressed in molds, electrically fused, and molded to desired shapes.

Boric acid, H_3BO_3 (a very weak acid and mild antiseptic), is deposited from certain hot springs in Italy in flaky white crystals.

Borax, $\text{Na}_2\text{B}_4\text{O}_7 \cdot 10 \text{H}_2\text{O}$, occurs in nature or can be readily prepared from another mineral, calcium borate. It has great use in making pyrex glass, as a flux in welding or soldering, in soaps, pottery, enamels, and as snow on indoor ski slides

Problems

1 What difference would it make to you if there were no glass? List many known uses of glass and give the preferred type or chemical composition

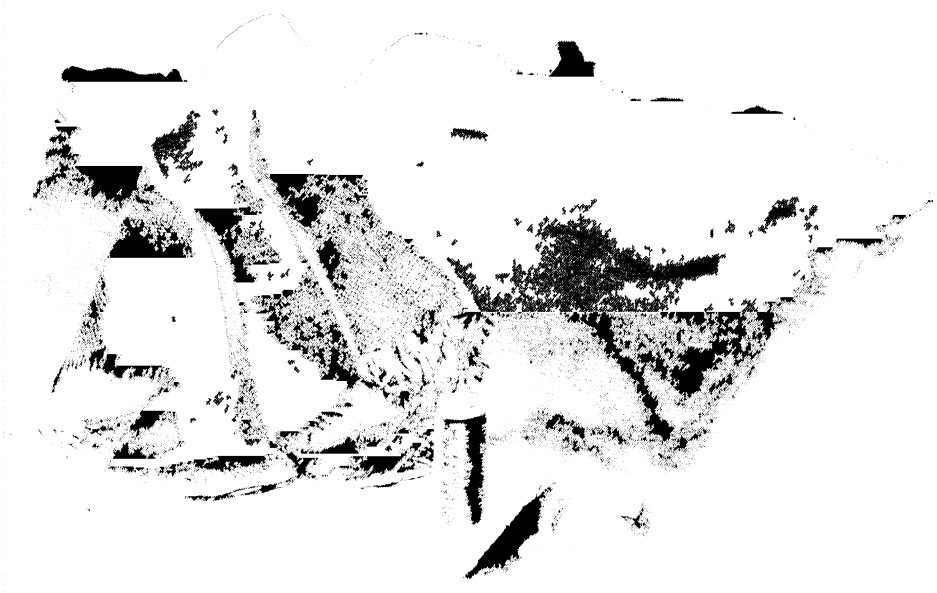


FIG 43 USES OF GLASS FIBER

Modern glass fibers in the form of cloth (resistant to insects and fire), tape (electrical insulation), fluffy blankets (heat insulation) and thread

2. Name three of the hardest substances used for cutting and grinding; give their formulas.

3. How many kilograms of Na_2CO_3 are required to react completely, on fusion, with 15 kg. of pure quartz? Ans. 26.45 kg.

PRIMARY CELLS AND STORAGE BATTERIES

When two strips of different metals are placed in a solution of an acid, base, or salt and connected outside by a wire, a current of electricity flows, while the more active of the two metals dissolves. If zinc dipping in a solution of zinc sulfate and copper dipping in a solution of copper sulfate (separated by a porous clay cup) are used, the current of electrons flows along the wire from zinc to copper and through the solution to the zinc.

This device (Fig. 44) is a primary cell (as distinguished from a secondary or storage cell) and converts chemical energy into electrical energy. The zinc throws atoms into solution as positive ions. This means that some zinc atoms give up electrons to the rest of the zinc plate, making it negative. These electrons flow along the wire (current of negative electricity) to the copper plate, where they are readily taken up by the adjoining positive copper

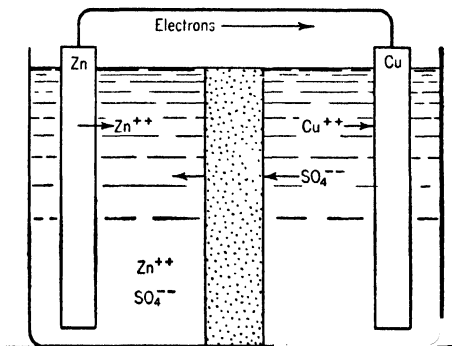


FIG. 44. A PRIMARY CELL

ions in solution. This action neutralizes the charge on the copper ions and causes a deposition of metallic copper on the plate. The zinc plate is finally dissolved or the copper ions all removed. The excess positive Zn^{++} ions attract negative SO_4^{--} ions from the other compartment, so all portions of the solutions are kept electrically neutral. Many types of such cells are possible. The one described is the Daniell cell.

The force that drives the electrons along a metallic circuit is termed *electromotive force* (e.m.f.) or *voltage* and its unit of measurement is the volt. The difference in this pressure or potential between the two poles of a Daniell cell is 1.10 volts. Other cells, such as the Clark and Weston, are better standard cells. If a cell is made of two metals close together in the electromotive series

(same as the activity series, p. 76), the voltage is very low. The farther apart they are in the series the higher the voltage.

A charged lead storage cell consists of a lead plate surfaced with lead dioxide as the positive pole and a second lead plate surfaced with spongy lead as the negative pole — both immersed in 20 per cent sulfuric acid. (See Fig. 45.)

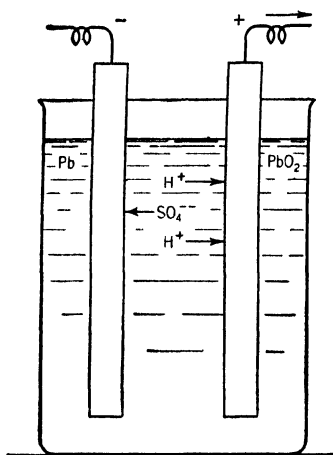
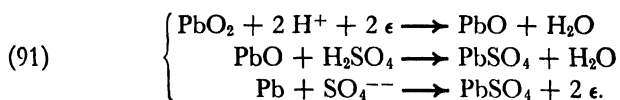


FIG. 45. A CHARGED LEAD STORAGE CELL WITH H_2SO_4 ELECTROLYTE

When the two poles are connected by a metallic circuit, some of the lead atoms on the surface of the lead plate give up electrons to the rest of the plate (making the plate negative) and go into solution as Pb^{++} ions. They do not go far, however, for they meet SO_4^{--} ions in the solution and are redeposited on the plate as insoluble PbSO_4 .

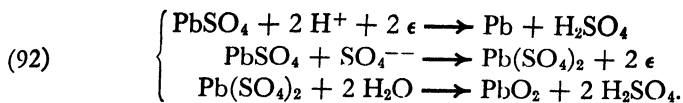
The two negative electrons given to the lead plate, when Pb^{++} ions are formed, travel along the metallic conductor to the brown lead dioxide plate, PbO_2 . There they unite with H^+ ions of the solution which on discharge promptly reduce the acidic PbO_2 to PbO , a basic oxide that readily reacts with sulfuric acid to form PbSO_4 .

DISCHARGE REACTIONS



On passing a current through the exhausted cell (as in electroplating), sulfuric acid is electrolyzed, its H^+ ions discharged on the PbSO_4 surface of the lead electrode reducing the lead sulfate again to the original spongy lead and forming more H_2SO_4 . At the other electrode (brown PbO_2 now surfaced with PbSO_4) the current discharges SO_4^{--} ions which then unite with the PbSO_4 to form $\text{Pb}(\text{SO}_4)_2$, lead disulfate. This, however, is quickly hydrolyzed to the original brown PbO_2 , lead dioxide, and to H_2SO_4 . The lead storage cell is very heavy and it is far from foolproof, but there is no better storage cell.

CHARGING REACTIONS



The action in charging is just the opposite of that in discharging. It is chemical energy that is stored up, not electrical. In practice it is unwise to convert too thick a layer of the plates into lead sulfate, for regeneration becomes difficult, probably due to a lack of permeability. Nor should a discharged cell stand long before charging. Otherwise the lead sulfate coating becomes more compact. The normal voltage of such a cell, fully charged, is about 2 volts, but this falls on discharge. Thus there are two ways of *Testing* learning the condition of the cell: measuring the density of the acid, or the potential fall between the poles. Since sulfuric acid is used up on discharge (the density of the liquid falling to 1.10), and since sulfuric acid is re-formed on charge (with a rise in density to 1.30) it is a simple matter to learn the cell condition.

PREPARATION OF THE METALS

The eight processes (broadly treated) used in process metallurgy *Extraction from ores* for obtaining relatively pure metals are:

- | | |
|------------------------------------|----------------------------------|
| 1. Reduction of oxides with carbon | 5. Roasting sulfides with carbon |
| 2. Reduction with aluminum | 6. Electrolysis of fused salts |
| 3. Reduction with hydrogen | 7. Amalgamation with mercury |
| 4. Roasting sulfides | 8. Cyanide extraction |

Sometimes a combination of two methods is applied to a single metal. Copper is prepared by first roasting the sulfide ore but the crude product must be purified for electrical uses by electrolysis of water solutions of a copper salt.

Carbonates of iron are converted into oxides on heating, and *Iron* oxides are reduced by heating in contact with coke and limestone. American ores are already oxides, occurring in the Lake Superior region, Alabama, Colorado, and other areas.

The modern blast furnace is of giant proportions. It is built of heavy steel plates lined with firebrick. The lower part where the heat is more intense is usually bound with water-cooled bronze blocks set in the brick wall. At the top of the "stack," as the furnace is called, are double bell-shaped doors which admit the ore, coke, and limestone of the "charge" without undue loss of gases.

An enormous volume of air, the blast, is blown through the furnace to burn the coke. It may seem that hot carbon is the active reducing agent in the blast furnace, but the carbon monoxide formed is even more effective because it is a gas and hence comes in better contact with the ore.

Slag

Impurities such as alumina and silica in the ore react with lime formed by heating the limestone of the charge to yield a low-melting slag, easily separated from the pig iron. The pasty particles of free iron left by reduction of Fe_2O_3 do not melt until they have dissolved enough impurities, such as sulfur, silicon, and carbon, to lower the melting point sufficiently. The melted iron runs into the crucible, while the lighter fluid slag floats on top.

Pig iron

The product of the blast furnace is termed *pig iron*. It is only about 92 to 94 per cent iron, the usual impurities being about 3 per cent carbon along with some silicon, manganese, sulfur, and phosphorus. Much of this iron is cast into "pigs" for convenience in handling and then shipped to the various markets. It is remelted at the foundries (by coke in a cupola) and cast into such shapes as are needed for radiators, stoves, wheels, and a thousand cheap useful articles. Pig iron is cast iron, but there are two types, depending on the rate of cooling. If poured into a metallic mold so that the liquid is chilled suddenly, the carbon does not have time to separate as graphite, but remains uniformly distributed as cementite, Fe_3C . The product is *white cast iron*, which is very hard and very brittle. If cooled more slowly in sand molds, *gray cast iron* results. In this form, part of the carbon has had time to separate as graphite. This product is not nearly so hard as the white variety and can be given some ductility if annealed for a few days.

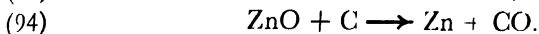
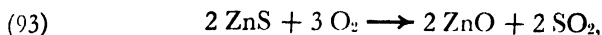
Melting point

Pig iron melts at about 1150°C ., while pure iron melts at about 1535°C . White cast iron is harder and less ductile than gray iron. Neither can be forged. Both are low in tensile strength, difficult to weld. Gray iron does not shrink from a mold as white iron or steel.

Cast iron has some useful properties not possessed by steel: its high fluidity and small contraction on cooling, which enable it at comparatively low temperatures to fill intricate molds; its ease of machining, due to the soft graphite flakes which lubricate the cutting tools; and its greater resistance to corrosion than steel. These qualities make it an irreplaceable metal. Admittedly, it has far less strength than steel and is rather brittle under shock. Pig iron may be turned into steel by burning out most of the carbon

(down to 0.2 per cent for mild steel) and also other impurities such as sulfur. Usually alloying elements are added.

Zinc oxides or carbonates need only heating with coal to reduce them to the metal. Sulfides must be heated with excess air to secure the oxide of zinc, which is then further heated with carbon to secure its reduction. In some smelters a sulfuric acid plant is attached to utilize the sulfur dioxide.

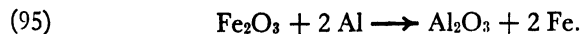


The zinc oxide and coal are heated in fire-clay retorts. The zinc is volatilized and then condensed in receiving vessels. Carbon monoxide burns at the mouth of each receiver. Since zinc boils at 907° C. and the temperature of the reduction occurs at 1200° C., it is evident that the vapor of the metal must be formed. When the vapors are led into cold receivers at any temperature below 419° C. (melting point of zinc), only zinc dust is obtained, but as the hot vapors heat the receivers above 419° C. only the liquid form is condensed. This is cast into slabs called *spelter* which may be purified by electrolysis.

Until the recent development of chrome plating, the free element was not prepared in great quantity, as its alloy with iron is cheaper and more suitable for steelmaking. Most of this alloy, "ferrochrome," is made by carbon reduction of chromite ore in the electric furnace. A small amount of the carbon-free alloy is made by the thermit process described below.

Manganese, at present, is prepared as ferromanganese by reduction of a mixture of ores of iron and manganese. If below 30 per cent manganese, this alloy is called *spiegeleisen*.

Powdered aluminum, when ignited, reduces many metallic oxides with evolution of a great amount of heat. Temperatures of 3000° C. are obtained, so the metal is molten and may be cast in a mold.



The heat of formation of Al_2O_3 is 392,600 cal. for the formula weight in grams, while the heat of decomposition of Fe_2O_3 is 197,000 cal.; hence, 195,600 cal. are freed. If a shaft on a ship at sea is broken, the ends can be welded together by surrounding them with a sand or clay mold and igniting a mixture of iron oxide and aluminum powder (thermit) in a funnel above. The melted iron runs

Zinc

Chromium and
manganeseThermit
process

into the mold, melts the broken ends of the shaft, and produces a good union. In actual practice the shaft ends are preheated.

Copper

Most copper ores (but not all) are sulfides. Concentration by grinding, washing, and flotation is a common preliminary to smelting. Roasting in air burns out 80 per cent of the sulfur, but some must be left in order to form a *matte* of melted sulfides later.

In a huge reverberatory furnace the roasted product is heated by a long flame of burning powdered coal, blown in with compressed air. Limestone, CaCO_3 , and magnesite, MgCO_3 , are present to form removable slag with much of the iron, silica, and alumina. A molten matte of mixed sulfides of copper and iron is then removed to a converter where a blast of air burns out (*oxidizes*) the remaining sulfur and oxidizes the iron and other impurities. Addition of a little raw ore causes formation of a removable slag with the iron oxide.

Such crude copper is too low in electrical conductivity for certain uses and must be refined by electrolysis. Crude copper slabs as anodes (positive pole) and thin sheets of pure copper as cathodes (negative pole) in a bath of copper sulfate make this possible. Copper dissolves (on passage of current) at the anode and deposits in pure form at the cathode.

Magnesium

The leading process of preparing magnesium is electrolysis of melted magnesium chloride, with some added chlorides of calcium and sodium to lower the melting point. Magnesium ions in the melt are discharged at the cathode. The Pigeon process, very convenient for small plants, uses furnace reduction of magnesium oxide with silicon, or rather the cheaper ferrosilicon.

Aluminum

We owe the present process of making aluminum to Charles M. Hall who, as a student at Oberlin College, invented the process in 1886. After preliminary removal of iron oxide from the bauxite ore, the roasted hydroxide (now alumina, Al_2O_3) is added to the cryolite in the electrolytic cell. These cells are iron boxes lined with "compact" (retort coke). Several carbon anodes dip into melted cryolite, $\text{AlF}_3 \cdot 3\text{NaF}$, which is kept molten (about 1200°C.) merely by the heat of resistance. The carbon-lined iron cell is the cathode. The aluminum metal is discharged at the cathode and sinks to the bottom, to be tapped off from time to time. Oxygen is evolved at the carbon anodes slowly converting them into carbon dioxide. Since it is the dissolved alumina that is decomposed, the cryolite needs renewal less regularly. A current of 20,000 amperes,

with a 5- to 7-volt drop in each cell, produces 250 lb. of metal per day in each cell, equivalent to 12 kw-hr. (kilowatt hours) per pound.

To make 1 lb. of the metal, we must use 2 lb. of aluminum oxide, 0.8 lb. of carbon anodes, 0.1 lb. of cryolite, 0.1 lb. of fluorspar, and 12 kw-hr. of electricity.

Metals of high melting point, such as tungsten, molybdenum, and tantalum, which cannot be readily melted, are prepared in small quantities by *powder metallurgy* with the aid of heat and pressure. The process is one of *sintering*, but not melting, the powder. Even with metals of lower melting point there are a number of advantages such as avoiding the usual final machining and grinding of small and complicated articles. The metal powders are prepared by reduction of oxides by electric deposition, or by atomization. Metals which are *immiscible* (do not mix) in the melted state, or which are widely different in volatility, may readily be alloyed by sintering their mixed powders.

PROPERTIES AND USES OF SOME METALS

Some metals are useful because they are light and others because they are heavy; some offer chemical activity and others provide resistance to corrosion; some are hard and others are malleable or ductile; some give us low melting points and others are useful because of their high melting points.

Magnesium is a silvery white metal melting at $651^{\circ}\text{C}.$ and boiling at $1110^{\circ}\text{C}.$ Its low density of 1.74 is interesting, for there is only one other commercial possibility, aluminum, among the light and strong metals to compete with it as an engineering material; and aluminum has a density of 2.70. (Beryllium, density of 1.84, is far too expensive for any but special uses.) It is important commercially that magnesium does not rust in dry air. A thin film of hydroxide, turning into the basic carbonate, does form in moist air, but it is coherent enough to *protect the metal* beneath. This film, however, is not adequate to prevent corrosion. At a temperature just below its melting point the metal can be drawn into wire.

Magnesium alloys are now used in aircraft, in auto pistons, crank cases, forged propellers, amplifying apparatus, in the refining of nickel, and in the de-gassing of radio tubes. An alloy of 92 per cent magnesium with 8 per cent aluminum can be hardened by repeated heating to $420^{\circ}\text{C}.$, then tempering four to seven days at $150^{\circ}\text{C}.$

Aluminum

The metal aluminum is the lightest of the very common metals, its density being only 2.7, while iron is three times as heavy. It melts at 660° C. and boils at 1800° C. It is rolled or hammered very easily at well below 150° C., yet thin sheets in the presence of a lubricant can be ground to a powder. It is an excellent conductor of heat and electricity. In wire of the same cross section and equal length it conducts electricity only two thirds as well as copper, but weight for weight it greatly excels copper in this respect. More than 450,000 miles of steel-cored aluminum cables are in use. The steel core is needed for tensile strength.

*Corrosion,
insulation*

Chemically, aluminum is very active, yet it is tarnished only slightly in air. This is due to the formation of a thin coherent film of the hard, white oxide which protects the metal beneath. The smooth, bright surface of aluminum is a splendid reflector of radiant heat and consequently the metal in the form of foil is finding growing use as a heat insulator for locomotives, milk tanks, houses, and naval vessels. On a locomotive, 13 lb. of foil replaced 1200 lb. of other insulating material.

Aluminum is much used for auto parts, the rigid structure of airships, and structural work of various kinds. The powdered metal is mixed with oil or a volatile liquid and used as a paint for radiators and other metallic articles. Aluminum paint is largely used as a covering for oil-storage tanks in order to reflect much of the sun's heat radiation and thus lessen the usual losses of volatile constituents such as gasoline. In the form of cooking utensils it has a merited popularity. A golden bronze containing 92 per cent copper and 8 per cent aluminum is now much used for gears in automobile trucks and for other purposes where toughness and resistance to abrasion and corrosion are important.

*Aluminum
alloys*

Duralumin, the most important of all aluminum alloys, containing 95 per cent aluminum, 4 per cent copper, 0.5 per cent manganese, and 0.5 per cent magnesium, is much used for airplane, airship, and automobile structures. It has nearly the strength of steel with about one-third of its weight. Heat treatment of certain aluminum alloys greatly improves their strengths.

Saving weight

Railway coaches using aluminum or its strong alloys for all but a few supporting parts (steel) are so much lighter that fuel costs go down and speeds go up. An aluminum Pullman sleeper weighs only one-half as much as the regular steel car of the same capacity. Aluminum dump cars save 11 tons of weight per car.

Zinc is a blue-white metal whose properties vary with its treatment. Cast zinc is moderately hard, but if heated to 100° – 150° C., it becomes so malleable that it can be rolled to thin sheets which remain soft and malleable on cooling to ordinary temperatures. Galvanizing is done by cleaning the rust or scale from iron with dilute sulfuric acid and dipping the clean iron into melted zinc. On cooling, the zinc crystallizes in the familiar spangled design. About two ounces per square foot of surface are needed to give protection. *Zinc*

Rolled zinc is used for roofing, gutters, washing-machine parts, and battery elements. Alloyed with copper it forms brass, one of the most useful of all alloys. Zinc of 99.99 per cent purity is remarkably ductile and brilliant. It also yields better *die castings* than the less pure metal.

Tin is most malleable at 100° C., while at 200° C. it is so brittle that it can be ground to a powder. *Tin*

Tin is placed on thin sheets of iron or steel by mere dipping of the clean steel into melted tin. The sheet must first be pickled in acid to free it from all oxides of iron. The modern ideal in tin-plating is to secure as thin a coating of tin as possible, 1.5 per cent of the total weight, without holes or cracks. Electroplating of tin on sheet steel requires less tin than the dipping process.

Lead, a heavy metal of specific gravity 11.4, is soft and easily cut with a knife. The metal melts at 327.5° C. and boils at 1620° C. Just below its ordinary melting point (lowered by pressure) it can be squeezed into tubes. When freshly cut, its luster is bright, but the surface quickly tarnishes with formation of oxide, PbO , and basic carbonate, $2\text{PbCO}_3 \cdot \text{Pb(OH)}_2$. The coating protects the metal from further *oxidation*, although it detracts sadly from the real beauty of lead. *Lead*

Solutions of lead salts are extremely poisonous, so that water conducted through lead pipes may produce lead poisoning. This is particularly true of soft water which, with air, forms the somewhat soluble hydroxide, Pb(OH)_2 . Hard water is much less dangerous because it soon forms a coating of insoluble carbonate and sulfate on the lead. Although lead is attacked by alkali, a very little sodium silicate in solution protects lead pipes so exposed. *Lead poisoning*

All lead paints begin with lead as raw material. Red lead is Pb_3O_4 and white lead, $2\text{PbCO}_3 \cdot \text{Pb(OH)}_2$.

Terneplate for roofing is made by dipping clean steel sheets into a molten lead containing 18 to 25 per cent tin. Without the tin, lead fails to coat steel on dipping.

Tungsten

Tungsten (W) has the highest melting point of all the metals, 3370° C. Tungsten is steel-gray, very hard, and very heavy, with a density of 19.3. The metal is obtained as a powder, on reduction of the oxide, and is worked into strong filaments or rods. Pure tungsten, after elongation by rolling and drawing while hot so as to develop long threadlike crystals, is soft and ductile.

Lamp filaments

Tungsten lamp filaments are saving the people of this country several hundred million dollars each year. The old carbon filaments used 3.25 watts per candle power, while the modern tungsten lamp uses less than 1.25 watts for the same illumination. We use over 900,000,000 of these lamps annually.



FIG. 46. HIGH-SPEED CUTTING TOOLS

Machine-shop work is greatly speeded up, in war for increased production and in peace for lower costs, by use of tungsten-molybdenum steel cutting tools that hold their sharp edges even when red-hot from friction. Cooling oils are needed nevertheless.

High-speed steel cutting tools contain 18 to 20 per cent tungsten. Since these tools can be worked on the lathe until red-hot without losing the necessary hardness, their use means an enormous saving in machine-shop time and expense (Fig. 46). Tungsten carbide (W_2C), with some cobalt, was developed in 1928 as the hardest cutting-tool material yet manufactured ("carbology"). *Tungsten tools*

IMPORTANT ALLOYS

Alloys were invented in attempts to improve upon the properties of the metals. The most used alloy, of course, is steel. *Steel as an alloy*

Pure iron (not very common) is rather soft and tough, pig iron (crude) is hard and brittle, but steel is hard, elastic, and strong.

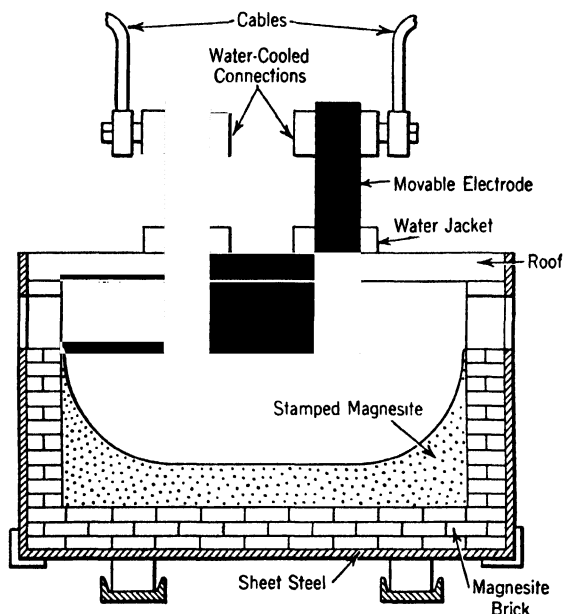


FIG. 47. ELECTRIC ARC OR RESISTANCE FURNACE WITH GRAPHITE ELECTRODES

Essentially the simplest steel is iron with 1 per cent or less of carbon, but most steels contain *alloying elements* such as chromium, manganese, nickel, tungsten, vanadium, to name a few. Many of the best alloy steels are made in an electric furnace (Fig. 47). Many alloys (brass, bronze, and so on) are formed by melting the metals together, others by reduction of mixed ores.

The bearing metals which surround revolving shafts are interesting examples of the introduction of harder particles in a softer *Bearing metals*

matrix. These *babbitts*, as they are often called, are alloys of tin and antimony, with some lead and copper.

High-speed, high-compression motors make severe demands upon bearing metals. Cadmium-nickel alloys containing from 0.75 to 3.00 per cent nickel have recently come into use in this field. Today, high-speed bearings may be lined with cadmium-silver-copper, cadmium-zinc, or cadmium-nickel, possibly electroplated with indium to increase resistance to oil corrosion.

Corrosion-resistant alloys

Monel metal (Cu-Ni) and the 18-8 chrome-nickel stainless steels of low carbon content are much used to resist corrosion. Manganese-bronze propeller blades for steamships contain Mn, Cu, Sn and Zn. This alloy is resistant to salt-water corrosion. Stellite, a Co-Cr-W-C alloy of great hardness, is so noncorrodible that it is used in surgical instruments and other fine products.

Low-melting alloys

The low-melting alloys, such as Wood's metal or Rose's metal, have an important application. The automatic sprinklers dotting the ceilings of great department stores or factories are merely water pipes closed by a plug of such an alloy. When a fire starts, these plugs melt very soon and a spray of water is released. Wood's metal melts at 158° F. or 70° C. Solder is another low-melting alloy of value. Each metal in the solid solution lowers the melting point or freezing point of the other; hence the mixture melts at a lower temperature than the constituents. This is the same effect as that produced by salt in water.

Plumbers' solder (62 per cent lead and 38 per cent tin) is useful because on cooling a large mass of lead-rich crystals separates, forming a mushy mass that can be "wiped" on a pipe joint. It begins to solidify at 240° C. but is not entirely solid until 180° C. is reached.

Some hard alloys

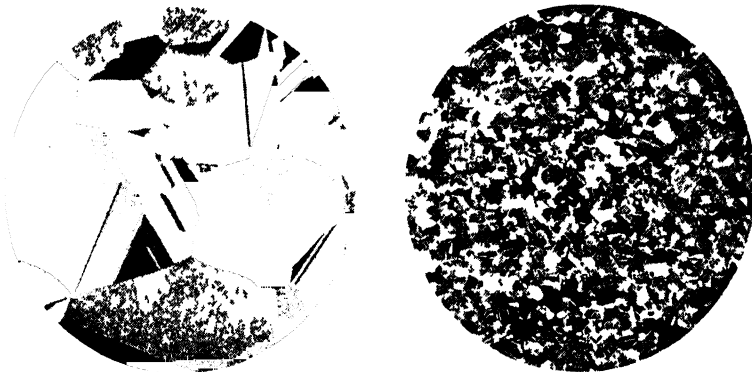
Cast iron (not pure iron) and the bronzes are very hard. One of the best bronze castings contains 10 per cent tin, 2 per cent zinc, and 88 per cent copper. Bronze (Cu-Sn) is the most ancient of the hard alloys. Copper is a soft metal and yet its alloy with a very little silver and chromium, "cupaloy," is harder than ordinary steel. Special heat-treatments are necessary in developing this great hardness. Lead containing several per cent of calcium is too hard to be cut with a hack saw. High-manganese steels make excellent jaws for rock crushers.

Beryllium alloys

Only 2 per cent beryllium in nickel gives it the tensile strength of steel. Beryllium is to copper what carbon is to iron: a powerful

agent for increasing hardness, elasticity, and strength. The alloy steels are among the strongest alloys known.

Beryllium has other useful alloying properties. Be-Ni valve springs are almost necessary for the highest speed airplanes because of their vibration resistance. The nonsparking beryllium-alloy tools have had some use where explosive vapors were present. The effect of nickel on the structure of a beryllium-copper alloy is shown by the photomicrograph in Fig. 48. *Microscopic structure*



Courtesy American Brass Company

FIG 48. MICROSCOPIC STRUCTURE OF ANNEALED ALLOY

Coarse grain beryllium-copper alloy (left) annealed from 800° C. Same alloy (right) with 0.25 per cent nickel annealed from 800° C shows finer grain. Magnification, 75x.

Steel is essentially iron containing from 0.1 to 1.0 per cent carbon, although most steels contain other alloying elements to give special properties. Steel contains no more than 2 per cent carbon because of brittleness. Aside from its great strength and elasticity certain steels can be magnetized. *Steel*

By the Bessemer process (actually discovered by William Kelly, an American, in 1852) many impurities are burned out of molten pig iron by a blast of air. At each "blow," with its brilliant flame, 15 tons of steel are made in 15 minutes. As the converter pours the liquid into molds or ladles, carbon and the alloying elements are added. *Bessemer*

The open-hearth process is slower, less spectacular, but makes better steel. Fifty tons of steel are made every eight hours. In this process oxygen needed to burn out impurities is secured from the air, from iron ore and rusty scrap. Preheated air and fuel gas supply the heat, reflected from the arched ceiling of the open hearth. *Open hearth*

Alloy steels

We could not make automobiles for less than several times their present cost were it not for the modern development of alloy steels. Crankshafts, driving shafts, transmission shafts, axles, frames, and gears are stronger, tougher, lighter, and stand shocks and strains better because they are made of alloy steels.

Silicon, aluminum, manganese, and titanium are added to steels at the moment of casting as "scavengers," to remove oxygen. Steel so cleansed is as much as 40 per cent stronger. But some of these elements, and a number of others, are also added to form valuable alloys with the steel. Alloy steels are often made in the electric furnace.

Manganese steel (12 to 14 per cent Mn), if suddenly cooled, is *hard, and tougher than any other steel alloy*. It is used in grinding machinery, rock crushers, and burglarproof safes. With a little added molybdenum it is suitable for heavy-duty gears.

Tool steels

Tungsten steel (such as 18 per cent tungsten, 4 per cent chromium, and 1 per cent vanadium) is self-hardening; that is, a tool of this material hardens on cooling in air without being quenched in oil or water. It also holds its temper when the tool is worked until red-hot from friction. (See Fig. 46.)

Molybdenum (fortunately produced chiefly in our own country) is replacing part of the tungsten in high-speed tools.

Nickel steel (3.5 per cent Ni) resists corrosion, is hard, elastic, and very suitable for armor plate, wire cables, and propeller blades. The automobile industry is a leading user.

Stainless steel

Polished stainless steel (18 per cent chromium-8 nickel) is highly resistant to corrosion. It is much used on the exteriors of office buildings, for certain chemical equipment in industry, and for other products. The corrosion resistance is increased by the addition of from 1 to 3 per cent molybdenum.

The modern oil refinery, with its high-pressure tubes and stills, is a large user of special alloy steels. The automobile industry is the largest purchaser of steels in general.

Heat treatment of steel

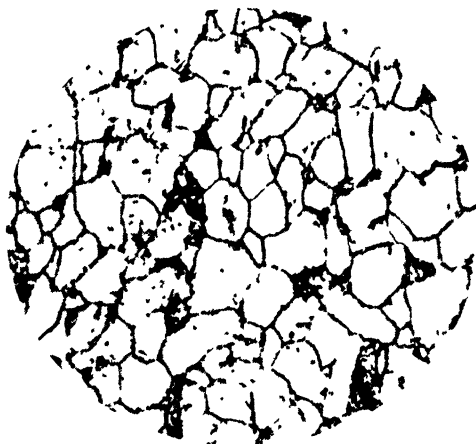
The two most important modifications of iron are *alpha-iron* (like pure wrought iron), which is soft, ductile, and magnetic; and *gamma-iron*, stable above 900° C. The latter is denser and only slightly magnetic.

Sudden cooling of white-hot steel gives a supersaturated solution of carbon in alpha-iron (*martensite*), which is hard and brittle. Slow cooling gives time for thin plates of Fe₃C to form, alternating

with alpha-iron masses, the two together forming *pearlite*. Because of the large amount of alpha-iron, this steel is soft and ductile. This structure is well illustrated in Fig. 49.

Annealing by reheating a quenched steel to 200° to 350° C. allows some crystal growth, some separation of hard cementite, Fe_3C , and some release from strains. Any intermediate degree of hardness can be secured by tempering. The effect of quenching and tempering becomes clear from a comparison of the two photomicrographs of Fig. 50.

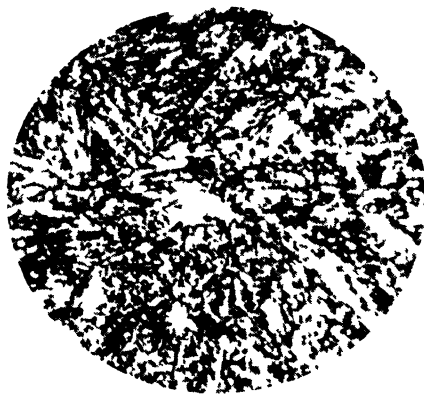
In *casehardening*, a low-carbon steel is held for several hours in charcoal and bone ash or suitable gas at 1000° C.; carbon is taken up by the surface, yielding a steel that is hard on the surface and tough in the interior. Gears must be casehardened.



Courtesy H. B. Pulsifer

FIG. 49. SIMPLE STEEL WIRE

Photomicrograph of a cold-drawn steel wire, 0.1 per cent carbon. Magnification, 250. The white areas of pure iron (ferrite) are surrounded by dark areas of pearlite containing the iron carbide, Fe_3C , and iron.



Courtesy J. F. Fillella

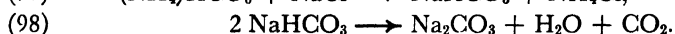
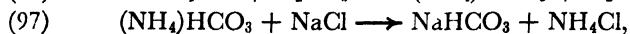
FIG. 50. COMPARISON OF ANNEALED AND QUENCHED STEELS

The effect of heat-treatment of steel is shown at the left. A coarse structure is obtained by cooling this 0.89 per cent carbon steel from 877° C. at the rate of 3° C. per minute. The finer structure of the same steel on the right is obtained by sudden water cooling from 877° C. and then tempering one hour at 550° C. Magnification of polished and etched surface, 2000.

SOME COMPOUNDS OF THE METALS

Sodium
carbonate,
 Na_2CO_3

By the Solvay process, concentrated sodium chloride solution is saturated with ammonia and carbon dioxide. At first, ammonium bicarbonate, $(\text{NH}_4)\text{HCO}_3$, is formed and this reacts with sodium chloride to form sodium bicarbonate. The latter salt, only sparingly soluble in the salt solution, precipitates, and after removal is heated to form sodium carbonate.



Part of the sodium bicarbonate, NaHCO_3 , produced in the early stage of the Solvay process is used in making baking powders. This is the "bicarbonate of soda" of the kitchen cabinet. Industry calls Na_2CO_3 *soda ash*.

Hydrates

Sodium carbonate, Na_2CO_3 , is a white solid, very soluble in water, yielding alkaline solutions by hydrolysis. On concentrating these solutions various hydrates crystallize out according to the temperature. On standing in the air the *decahydrate*, $\text{Na}_2\text{CO}_3 \cdot 10 \text{H}_2\text{O}$, effloresces, due to its high vapor tension, and crumbles to a white powder, which is the monohydrate, $\text{Na}_2\text{CO}_3 \cdot \text{H}_2\text{O}$. For some purposes the decahydrate is liked because it dissolves more rapidly. The common names for sodium carbonate are washing soda and sal soda.

The production in the United States in one recent year was 105,000 tons of natural soda and nearly 3,000,000 tons of Solvay soda. It is essential to many important industries, such as the manufacture of glass, soap, paper, leather, enamels.

Calcium
oxide, CaO

Calcium oxide, known as "quicklime," is usually obtained by heating limestone to about 900°C .



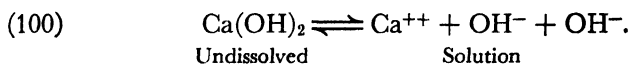
Calcium oxide is a white solid melting at about 2750°C . and boiling at higher temperatures. Pure calcium oxide takes up 31.1 per cent of its own weight of water, but the usual impurities lessen this amount. As the oxide hydrates to form the hydroxide, much heat is evolved.

On long exposure to the air, quicklime becomes "air slaked," changing into the hydroxide, $\text{Ca}(\text{OH})_2$, and the carbonate, CaCO_3 . Lime has uses to the extent of more than 4,000,000 tons in this

country in the preparation of paper, rubber, sand-lime bricks, calcium carbide and acetylene, lime-sulfur sprays, calcium arsenate dust for cotton crops, Bordeaux mixture, and in the clarification and softening of water.

Slaked lime, Ca(OH)_2 , is a white powder formed by careful addition of water to quicklime, CaO . It is slightly soluble, 100 cc. of water dissolving 0.17 g. of Ca(OH)_2 at 18°C . At higher temperatures it is less soluble. The little that is in solution ionizes rather well, but when considerable quantities of a base are required, it is customary to use a suspension of the solid in water known as "milk of lime." As fast as the little in solution is used up, some of the suspended hydroxide goes into solution.

Calcium hydroxide, Ca(OH)_2



This is the cheapest base known and has many industrial uses. It is important for making mortar, plaster, alkalies, bleaching powder, purifying illuminating gas, removing hair from hides, and clarifying sugar solutions. Naturally some of these uses duplicate those for the oxide.

In earlier days mortar, to the bricklayer, was a mixture of slaked lime, sand, and water which hardened slowly on drying. Now we add Portland cement for quick setting. The chemist knows that carbon dioxide of the air slowly penetrates the wet mortar and reacts with the calcium hydroxide to form solid *interlacing crystals of calcium carbonate*, $\text{Ca(OH)}_2 + \text{CO}_2 \longrightarrow \text{CaCO}_3 + \text{H}_2\text{O}$. The sand, preferably with sharp edges, merely gives bulk, rigidity, increases porosity, and reduces shrinkage. Possibly the colloidal particles of calcium hydroxide "set" in some way suggestive of the drying of glue or water glass, or interlacing crystals of calcium hydroxide may form. Wall plaster differs from mortar only in containing hair as a binder. It is porous enough to allow considerable circulation of air through the walls.

Mortar

Gypsum, $\text{CaSO}_4 \cdot 2\text{H}_2\text{O}$, is heated a short time at 200°C ., causing the loss of three-fourths of its water, leaving the *hemihydrate*, $2\text{CaSO}_4 \cdot \text{H}_2\text{O}$, or plaster of Paris. This residue, when mixed with the proper amount of water, sets to a white solid in from 5 to 15 minutes. The white finishing coat of common plastered walls contains plaster of Paris. Statuary is often made from it, because the material expands slightly on setting, thus taking a

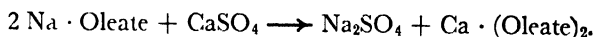
Plaster of Paris

very sharp impression of the mold. For commercial work, such as white plaster, stucco, interior wall bricks, and wallboard, the setting process is retarded by the addition of glue or a product made by digesting hair with sodium hydroxide.

Softening of water

In boilers a hard scale of salts (calcium sulfate, calcium carbonate) is left by the evaporation of feed water. This insoluble scale is a poor conductor of heat and causes a serious waste of fuel. A layer $\frac{1}{8}$ in. thick causes a loss of 10 to 12 per cent of fuel.

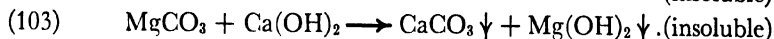
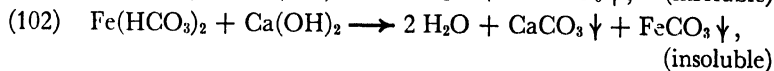
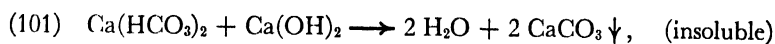
Hardness of water also causes an appalling waste of soap. No lather can form and consequently no cleansing can be achieved until all the hardness has been precipitated at the expense of the soap. Thus, with sodium oleate, representing a typical soap, we have this wasteful water-softening reaction.



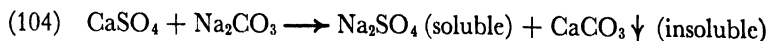
The insoluble calcium soaps form sticky curds that are only an annoyance.

Precipitating hardness

One of the cheapest softeners is slaked lime, $\text{Ca}(\text{OH})_2$. This sounds to the public like making matters worse, for it is often said that "lime causes the hardness of water." However, the following equations show the explanation of the paradox. In each case the $\text{Ca}(\text{OH})_2$ precipitates the salt (hardness) and leaves the pure water H_2O . The vertical arrow following a compound means that the compound is insoluble and "comes down" or precipitates.



To precipitate hardness due to sulfates, crude "soda ash," Na_2CO_3 , is used.



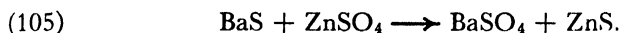
Magnesium compounds

Magnesium oxide, MgO , derived from the carbonate (magnesite) at high temperatures, is an important refractory lining for furnaces. Open-hearth furnaces require it. Magnesia pipe covering for heat insulation is the basic magnesium carbonate, $3 \text{MgCO}_3 \cdot \text{Mg}(\text{OH})_2 \cdot 3 \text{H}_2\text{O}$, formed by hot precipitation from any soluble magnesium salt with sodium carbonate. Asbestos, a natural competitor, is $\text{CaMg}_3(\text{SiO}_3)_4$. Its value in brake linings

and when mixed with Portland cement for asbestos shingles is great.

Magnesia cement is formed when a wet mixture of the oxide and chloride, MgCl_2 , hardens. Sawdust or wood-meal fillers may be added. Such compounds as this basic magnesium chloride make wall plaster, floorings, or stucco for outside walls.

The important pigment lithopone (a combination of barium sulfate and zinc sulfide) is made by mixing solutions of barium sulfide and zinc sulfate. *Lithopone*



Both products are white and insoluble. The precipitate first formed has no body or hiding power, but when washed, dried, heated to dull redness, suddenly quenched in cold water, ground wet, washed, and dried again, it yields an excellent pigment. Lithopone is a brilliant white with better hiding power than zinc oxide. It is unexcelled for interior painting, for first coats, linoleum, floor oil-cloth, and rubber compounding.

Research has improved the quality so much that we now use over 160,000 tons of lithopone per year for both outdoor and indoor paints, and even mixtures with white lead.

Kaolin or china clay is the weathered residue from feldspar. Its composition is represented by $\text{H}_2\text{Al}_2(\text{SiO}_4)_2 \cdot \text{H}_2\text{O}$, while that of feldspar is KAlSi_3O_8 . Common clay owes its color to impurities, usually oxides of iron. In burning brick or tile, the impurities fuse and act as a bond for the clay particles. Firebrick, with its great infusibility, must be made from rather pure clay. *Brick and pottery*

Porcelain is a fired mixture of kaolin and feldspar. The more fusible feldspar binds the kaolin particles and makes the mass translucent. When such ware is dipped into a water suspension of ground feldspar, dried, and burned again, it receives a glaze of feldspar.

The Romans ground together volcanic ashes and lime to form a cement that, when mixed with water, set to a solid. Some specimens of this *pozzuolana cement* are still in good condition. Today we do practically the same thing in grinding together blast-furnace slag (granulated) and lime. But the best of these hydraulic cements is the modern Portland cement in which the essential raw materials are clay and limestone. There must be present the proper equivalents in combined form of CaO , SiO_2 , and Al_2O_3 . The average composition of Portland cement is about as follows: *Portland cement*

| | | | |
|--|---------------|--|--------------|
| CaO | 63 per cent | Fe ₂ O ₃ | 2.5 per cent |
| Al ₂ O ₃ | 6.5 per cent | MgO | 2.5 per cent |
| SiO ₂ | 21.5 per cent | SO ₃ | 1.5 per cent |

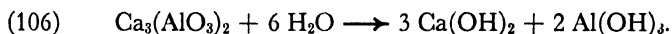
The raw materials, clay and limestone, are finely ground, fired in a rotary kiln to incipient fusion (the edges of lumps just melting), and the clinkered product is ground again. These rotary kilns are nearly horizontal, from 60 to 200 ft. long, and 6 to 9 ft. in diameter. They are heated with powdered coal. Nearly 8,000,000 tons of powdered coal are used as fuel in the cement industry annually.

Cement manufacture

The materials enter at the cooler end of the rotating kiln, slowly working down to the end where the flame enters. The product is a calcium aluminate and calcium silicate. About 3 per cent of gypsum (CaSO₄) is added to retard setting. A little iron oxide and magnesium oxide are allowed, but there is a definite range of concentrations for all the constituents. The clinker is ground to produce the cement powder of 200-mesh fineness.

Hardening

Common cement is probably a mixture of calcium silicates, calcium aluminate, and an excess of lime; consequently it is highly basic. Hardening is due to *hydration*, *hydrolysis*, and *crystallization*. The calcium aluminate reacts with the mixing water to give the following reaction:



The calcium hydroxide formed as in the above equation slowly crystallizes and cements together the particles of calcium silicate. The aluminum hydroxide fills in. The tricalcium silicate present, 3 CaO · SiO₂, hydrates rapidly, giving early strength. CaSO₄ is added to delay this action.

White lead paint

A paint is prepared by grinding some heavy insoluble pigment of sufficient covering power with linseed oil or other *drying oils*. Finally thinners are added. The drying of paint is not really a loss of water, but oxidation of the oil with formation of a tough, solid film. This oxidation of the oil is hastened by catalytic dryers made by boiling manganese dioxide with linseed oil. Among the pigments used are red lead, Pb₃O₄, barium sulfate, BaSO₄, lead chromate, PbCrO₄, zinc oxide, ZnO, cadmium sulfide, CdS, lithopone, BaSO₄ · ZnS, iron oxides, and titanium oxide, TiO₂; but the one most used is white lead or basic lead carbonate, Pb(OH)₂ · 2 PbCO₃. In the United States alone nearly 200,000 tons of white lead are used annually.

1. Calculate the percentage by weight of magnesia (MgO) in magnesite (MgCO_3). *Problems*
Ans. 47.81%.
2. How much would 1 ton of quicklime weigh after being completely slaked with water? *Ans. 1.32 tons.*
3. Calculate the percentage of aluminum in white clay (kaolin). *Ans. 22.56%.*

STRATEGIC MATERIALS

Materials necessary to national strength of which we produce none or only a part within our continental boundaries are in the main the following: *Limited production*

STRATEGIC MATERIALS FOR INDUSTRY

| <i>Name</i> | <i>Imported from</i> | <i>Amount Used per Year</i> | <i>Uses</i> |
|-------------------|-------------------------------------|----------------------------------|---|
| Natural rubber | East Indies | 600,000 tons | Tires, hose, many articles; synthetic rubber is replacing it. |
| Tin | East Indies and Bolivia | 75,000 tons | Tin cans, alloys (bronzes, etc.) |
| Chromium | Cuba, Turkey, Philippines, Rhodesia | 500,000 tons of concentrates | Stainless steel, armor plate, etc. New concentration of Montana ore will soon give us independence. |
| Manganese | Cuba, Brazil | 1,500,000 tons of rich ore | Cleansing steel from sulfur; very hard steels. Our low-grade ores are helping. |
| Linseed oil | Argentina | 350,000,000 lb. | Drying oil for most paints |
| Coconut oil | The Tropics | 800,000,000 lb. | Soapmaking |
| Nickel | Canada | | Stainless steel; plating |
| Quinine | East Indies | | For malaria; synthetic atabrine is helping. |
| Silk | Japan | Formerly \$200,000,000 | Clothing, parachutes, bags for smokeless powder; nylon substitutes well. |
| Quartz crystal | Brazil | All we could get; but not enough | Optical uses; thin vibrating plates for radio equipment; new sources found in the United States. |
| Tanning materials | South America | | Leather |
| Kapok | Tropics | | Life jackets, mattresses, etc.; milkweed floss may help. |
| Hemp | Philippines | | Rope |
| Jute | India | 600,000,000 lb. | Burlap bags and string |

In addition in 1941 and 1942 war priorities were placed on copper, steel, aluminum, magnesium, sugar, coffee, tungsten, antimony, plastics, nylon, and a host of common materials. We are large producers of some of these. *Changes in war*

Technical Drawing

THE LANGUAGE OF ENGINEERING

Early drawings

At the dawn of history man demonstrated a natural urge to draw by scratching upon the walls of his caves rude drawings of animals and other subjects of which he liked to be reminded. Later a mastery of crude tools made it possible for him to erect simple structures and devise elementary articles which he needed in his daily life. At some point in this development he realized that drawings could be very useful in planning and recording his ideas concerning things he wished to build. We do not have examples of these earliest drawings, made undoubtedly upon stone, but later developments show us that drawings of some character must have been used.

Temples of stone

The Bible states that Solomon's Temple was "built of stone made ready before it was brought thither," indicating that drawings of some kind showing the exact sizes and shapes of each stone or timber must have been used during construction. Perhaps the earliest known construction drawing in existence is the plan view of a fortress drawn by the Chaldean engineer, Gudea, and engraved upon a stone tablet as shown in Fig. 52. Obviously the ancient Greek temples, so complex in arrangement and refined in detail as, for example, the Parthenon, could not have been constructed without the most accurate drawings to convey the designer's ideas to the builders.

Technical drawing as a science

In 30 B.C. Vitruvius wrote a treatise on architecture in which he referred to projection drawings for structures. But the theory of projections of objects upon imaginary planes of projection to obtain *views* was not developed until the early part of the fifteenth century, by the Italian architects, Alberti, Brunelleschi, and others.

Near the end of the eighteenth century the French mathematician, Gaspard Monge, introduced the two planes of projection at right angles to each other, and the new science became an established academic study. This provided the basis for *descriptive geometry*, which embraces all of the theoretical background to modern technical drawing.

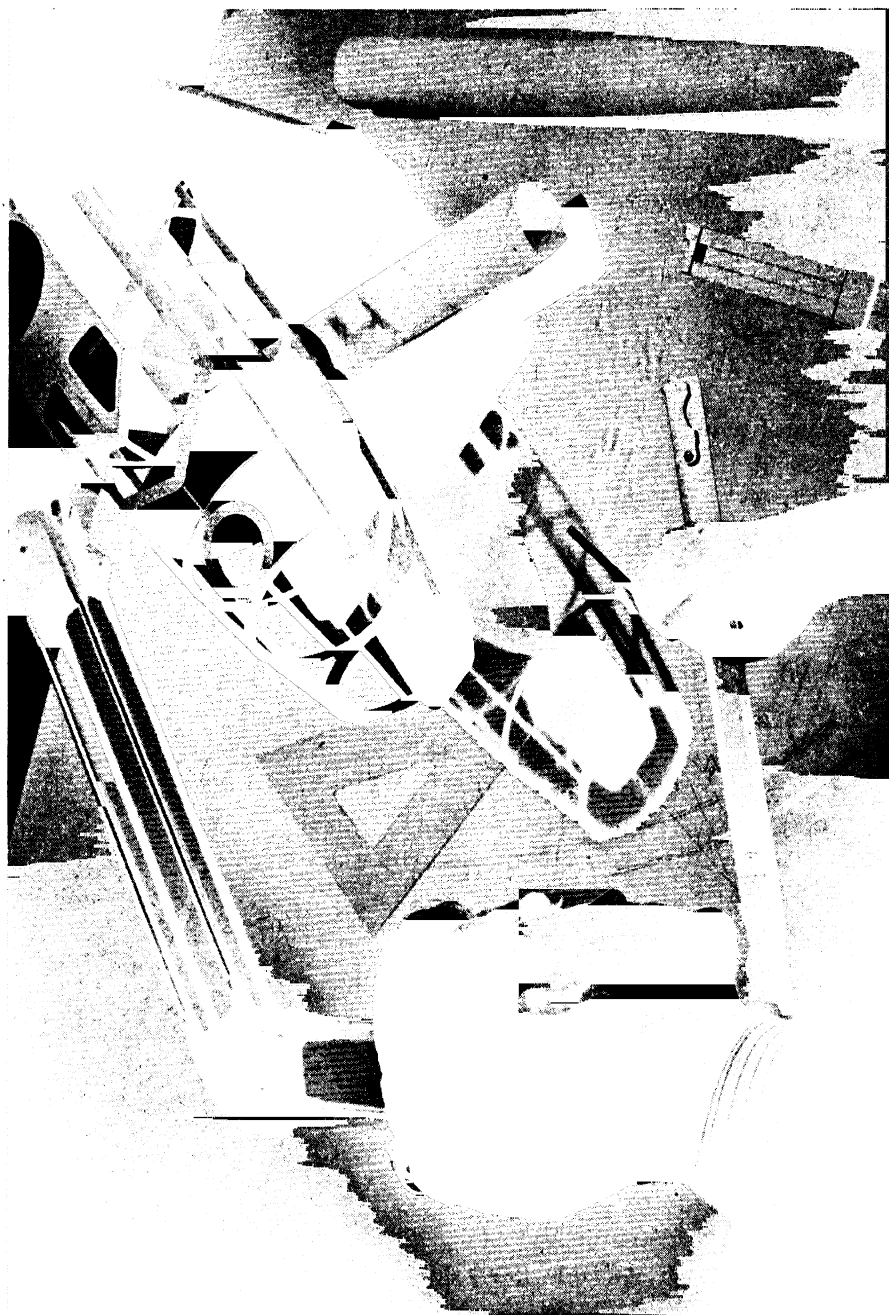
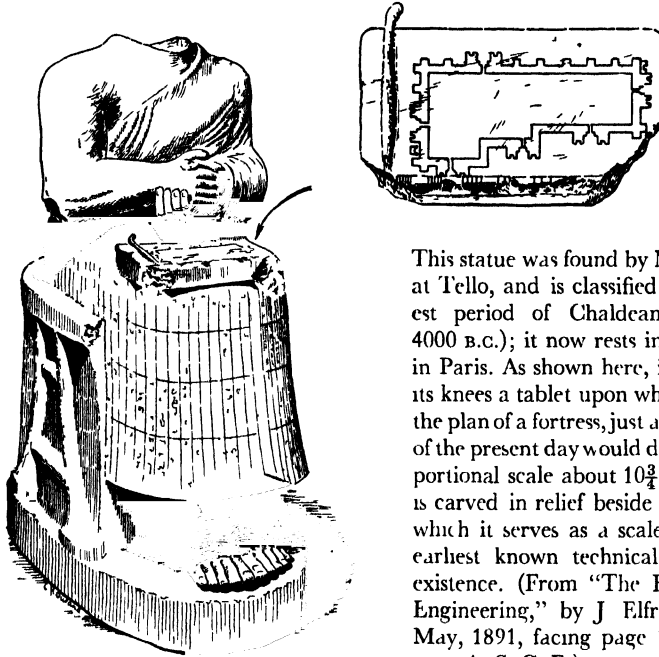


FIG. 51. ENGINEERING DRAFTSMAN AT WORK WITH MODERN EQUIPMENT

With the advent of this present industrial age came an increased need for exact methods of representation, by means of drawings, of all of the enormously complicated and extensive details of our machines and structures.

*Modern
technical
drawing*

Technical drawing is the universal language of engineering, and is as indispensable today as it will always be if technology continues in the activities of men. By means of it, the shape, size, finish, color, and construction of any object, no matter how complex,



This statue was found by M. de Sarzec, at Tello, and is classified in the earliest period of Chaldean art (about 4000 B.C.); it now rests in the Louvre, in Paris. As shown here, it holds upon its knees a tablet upon which is carved the plan of a fortress, just as an architect of the present day would draw it. A proportional scale about $10\frac{3}{4}$ in. in length is carved in relief beside the plan, for which it serves as a scale. This is the earliest known technical drawing in existence. (From "The Beginnings of Engineering," by J. Elfreth Watkins, May, 1891, facing page 309. *Transactions*, A. S. C. E.)

FIG. 52. EARLIEST KNOWN TECHNICAL DRAWING

can be described accurately and clearly. It is the language used by engineers to develop and record their ideas and to transmit them to the workers who carry out the designs. Evidently, everyone connected in any way with engineering or construction must understand this universal language. A typical modern technical draftsman at work is shown in Fig. 51. Because of the enormous growth of the airplane industry among others, such draftsmen, thoroughly skilled in using the language of technical drawing, are much in demand. The worker needs only an understanding of the principles of technical drawing so that he can "read a blueprint."

*Drawing as a
Language*

The engineer must have a thorough understanding of the principles and in addition must be able actually to produce drawings himself when the need arises.

The engineer has need for the development of skill in two phases of technical drawing: first, he must be able to execute rapid, clean-cut freehand *technical sketches* (Figs. 191 and 192), by means of which he can quickly develop a constructive idea and transmit his thoughts to others; second, he must acquire a certain amount of skill in executing a *mechanical drawing* (Figs. 206 and 207) by means of which he can bring his ideas into exact focus and record them for the use of others. *What the engineer must know*

The section of this book devoted to technical drawing covers the minimum fundamentals of the subject, but college students of engineering will supplement this material with more extensive study and practice.

INSTRUMENTAL DRAWING

You have frequently heard the apology: "I can't even draw a straight line." This is really a stupid remark to make, for no one can make a straight line without the aid of a guiding edge. As we shall see later, even a good freehand technical sketch is not composed of rigidly straight lines. Mechanical drawings, of course, are made with the aid of precision drafting instruments, the uses of which are described below.

The principal items of drawing equipment used by the engineer are shown in Fig. 53. In this brief introduction to technical drawing, however, you may wish to economize in the purchase of equipment. The drawing instruments should, if possible, be of high grade; serviceable sets may be purchased in a wide range of prices. You should consult an instructor or some other experienced draftsman or a reliable dealer before purchasing drawing instruments. *Drawing equipment*

The complete list of equipment shown in Fig. 53 is as follows: *Equipment list*

- | | |
|---------------------------------------|--------------------------------------|
| 1. Drawing board (approx. 20" × 24") | 10. Drawing pencils (F, 2H, 4H) |
| 2. T-square (24", transparent edge) | 11. Sandpaper pad or file |
| 3. Drawing instruments | 12. Ruby pencil eraser and art gum |
| 4. 45° triangle (8" side) | 13. Erasing shield |
| 5. 30° × 60° triangle (10" long side) | 14. Art gum |
| 6. Lettering triangle | 15. Pen staff |
| 7. Triangular architect's scale | 16. Gillott's 303 and 404 pen points |
| 8. Irregular curve | 17. Black drawing ink |
| 9. Protractor | 18. Drawing paper |

Penciled mechanical drawings are made upon detail paper, tracing paper, or tracing cloth. In industry the latter two are generally used for pencil drawings from which blueprints can be made. Detail paper is used for drawings that may require considerable correcting and changing. The finished drawings are then "traced" in ink on tracing paper or tracing cloth if blueprints are needed. Although detail paper may be purchased in white or even in light green, the popular color is buff, a pleasing light tan.

The paper should be placed close enough to the left-hand edge of the board to minimize the error resulting from the slight "give" of the T-square blade, and far enough from the top to permit the T-square head to maintain complete contact with the edge of the board (see Fig. 54).

*Fastening
the paper*

You should fasten one upper corner and then pivot the sheet about this corner until the top edge of the sheet "lines up" with the top edge of the T-square; then fasten a second corner. If the paper is small and of stiff character, two fastenings at the top may be sufficient, otherwise four or more fastenings may be necessary

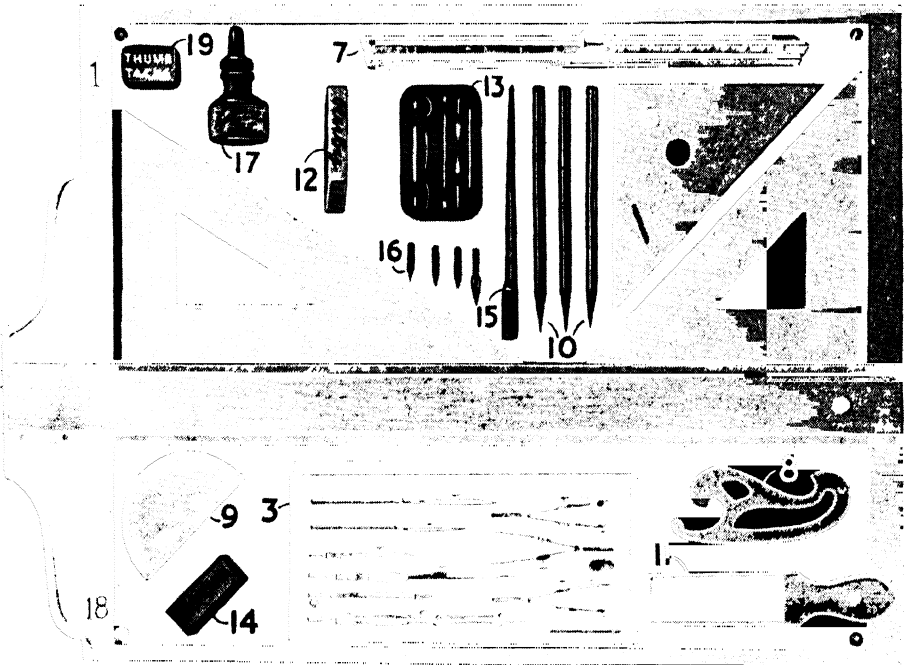


FIG. 53. EQUIPMENT USED IN TECHNICAL DRAWING

*Aligning
the paper*

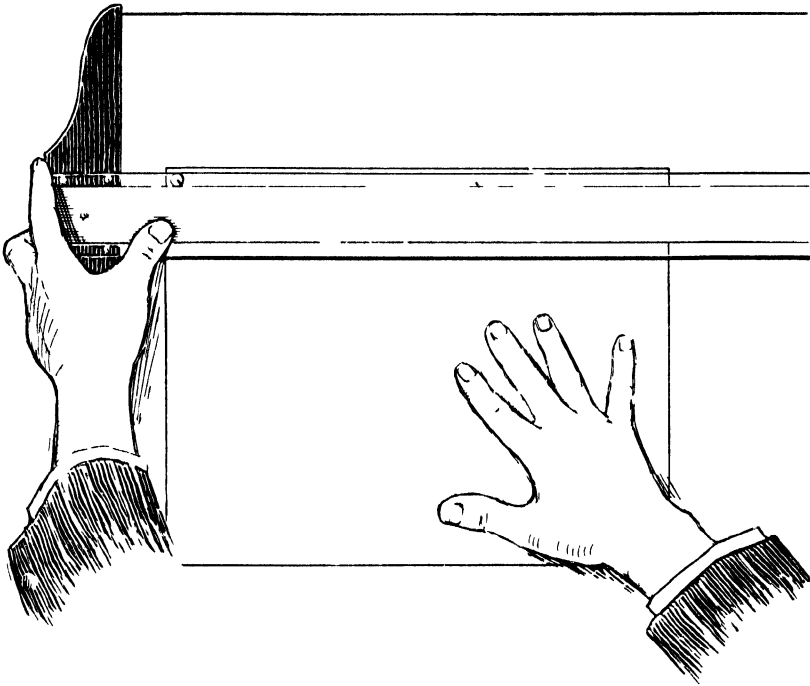
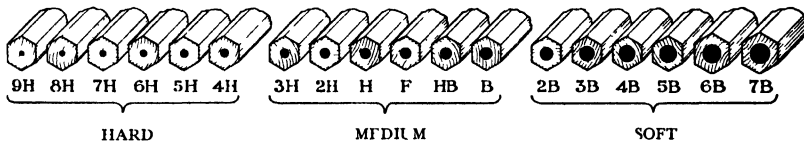


FIG. 54. FASTENING PAPER TO DIB BOARD

Drawing pencils are made in a wide range of grades: from 9H — the hardest, to 7B — the softest. The scale of numbers and the classification of pencils according to use are shown in Fig. 55. However,

*Grades of
drawing
pencils*



The harder pencils in this group (left) are used where extreme accuracy is required, as on graphical computations, charts and diagrams. The softer pencils in this group (right) are used by some for line work on engineering drawings, but their use is restricted because the lines are apt to be too light.

These grades are for general-purpose work in technical drawing. The softer grades (right) are used for technical sketching, for lettering, arrowheads, and other freehand work on mechanical drawings. The harder pencils (left) are used for line work on machine drawings and architectural drawings. The H and 2H pencils are widely used on pencil tracings for blueprinting.

These pencils are too soft to be useful in mechanical drafting. Their use for such work results in smudged, rough lines which are hard to erase, and the pencil must be sharpened continually. These grades are used for art work of various kinds, and for full-size details in architectural drawing.

FIG. 55. GRADES OF PENCILS

the hardness of lead alone should not determine the choice of a pencil to be used because there are other variables, including the

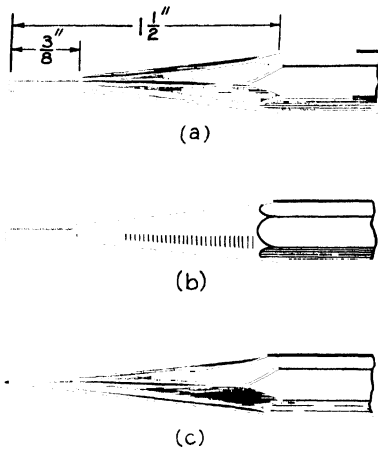


FIG. 56. PENCIL POINTS

pencil sharpener, so that the wood is removed, leaving about $\frac{3}{8}$ in. of the lead uncut. In either case you then “dress down” the lead to a sharp conical point as in Fig. 57, and wipe the lead clean with a cloth. While drawing, you should keep your pencil *sharp*. It is a good idea to keep the sandpaper pad or file close by for frequent pointing of the pencil.

LINEWORK

Always draw horizontal lines from left to right along the top edge of the T-square (Fig. 58). Left-handed persons should, in general, reverse all instructions in terms of “right” and “left,” which follow. Press the T-square firmly against the

“working edge” of the board, and lean the pencil at an angle of approximately 60° with the paper, with the little finger gliding lightly along the surface of the T-square blade. In order to produce lines of uniform width, rotate the pencil slowly, so that it will not wear off flat in one place.

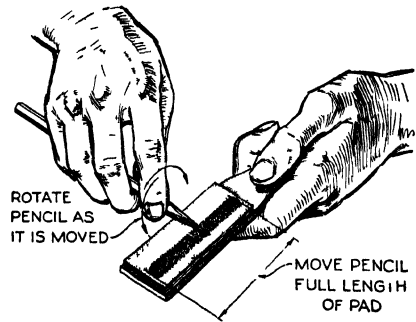


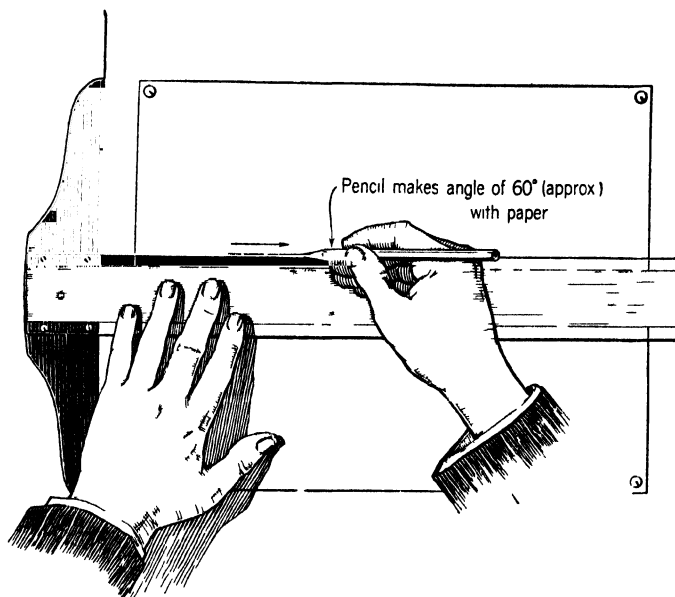
FIG. 57. POINTING THE PENCIL

*Sharpening
the pencil*

*Horizontal
lines*

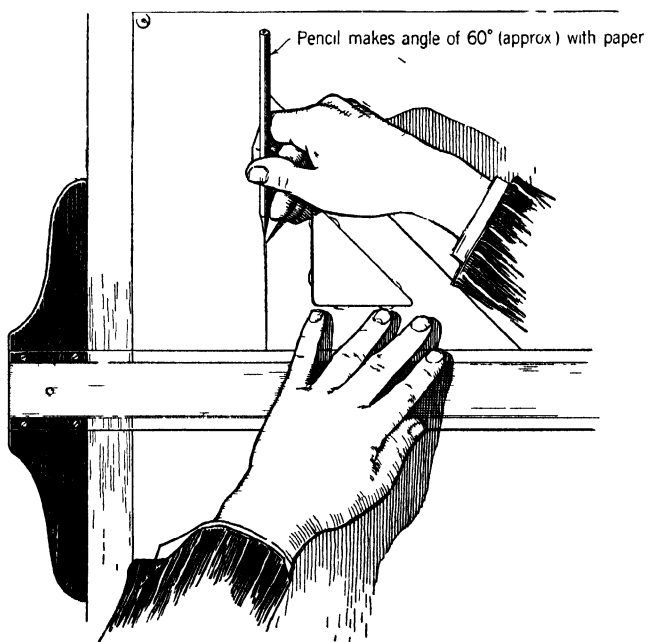
hardness or softness of the paper, the humidity of the air (which affects the paper), and the personal preference of the draftsman. In general, all construction lines (or “blocking-in” lines) should be drawn lightly with a fairly hard pencil, say 4H, 5H, or 6H; and all main required lines and lettering should be drawn *black* with a fairly soft pencil, such as F, H, or 2H.

Always sharpen the unlettered end of the pencil as in Fig. 56, either with a pocket knife or with a draftsman’s



Horizontal lines, drawn from left to right

FIG 58 DRAWING A HORIZONTAL LINE



Vertical lines, drawn upward

FIG 59 DRAWING A VERTICAL LINE

We use the 45° and $30^\circ \times 60^\circ$ triangles (Figs. 60, 61, and 62) to draw vertical and inclined lines. You will realize that a 180° angle can be divided into four equal 45° angles with the 45° triangle, or into six equal 30° angles with the $30^\circ \times 60^\circ$ triangle.

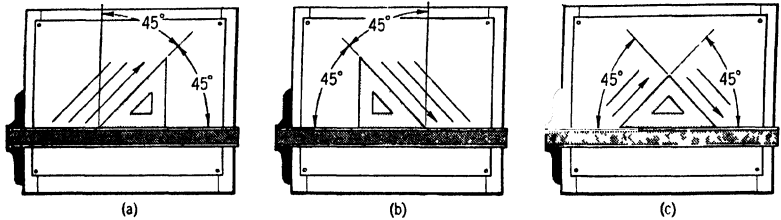


FIG. 60 DRAWING LINES WITH 45° TRIANGLE

Vertical lines

Vertical lines are drawn upward with either triangle along the left side, as shown in Fig. 59. Press the head of the T-square firmly against the drawing board, and then shift the left hand to the position shown in Fig. 59 where it holds both the T-square and triangle firmly in position. Here again, the pencil is leaned at an

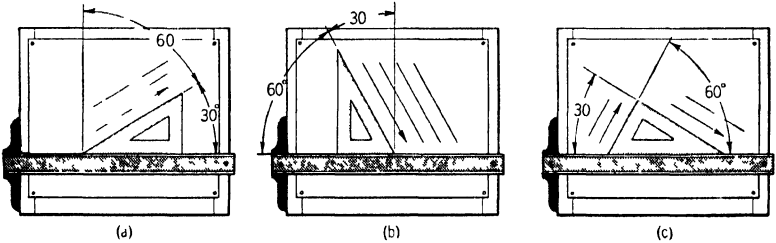


FIG. 61 DRAWING LINES WITH $30^\circ \times 60^\circ$ TRIANGLE

angle of approximately 60° with the paper, and the pencil is rotated slowly throughout the extent of the line.

Inclined lines

You will soon learn to draw inclined lines at a large variety of angles with the 45° and $30^\circ \times 60^\circ$ triangles, either singly or in combination. The method of drawing lines with the 45° triangle

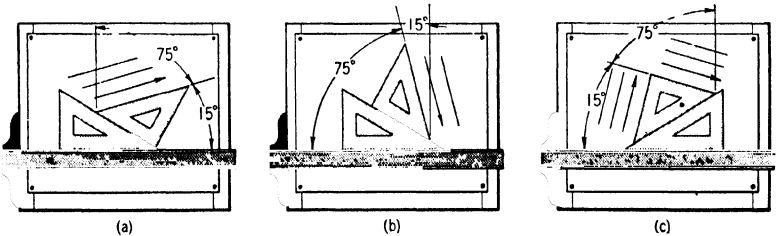


FIG. 62 DRAWING LINES WITH TRIANGLES IN COMBINATION

is shown in Fig. 60, the use of the $30^\circ \times 60^\circ$ triangle is shown in Fig. 61, and the use of both triangles in combination is illustrated in Fig. 62. The direction in which the lines should be drawn is clearly indicated in each case. Good draftsmen follow these procedures exactly. By using the 45° and $30^\circ \times 60^\circ$ triangles singly

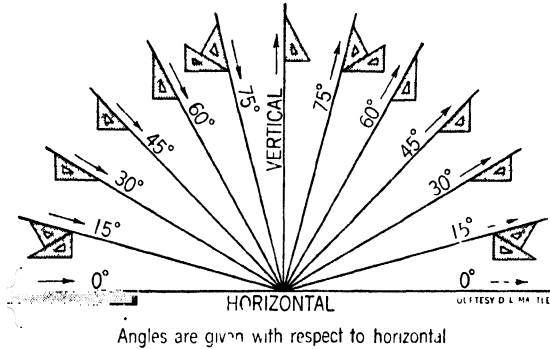


FIG. 63. STANDARD INCLINED LINES

and in combination, you can draw a complete set of radial lines with uniform 15° intervals, as shown in Fig. 63.

Parallel lines at the standard inclinations indicated above may be readily drawn by sliding the triangles along the T-square and repeating the lines, as shown in Figs. 60–62. If you wish to draw a line parallel to *any* given line (Fig. 64), move the T-square and triangle as a unit until the hypotenuse of the triangle lines up

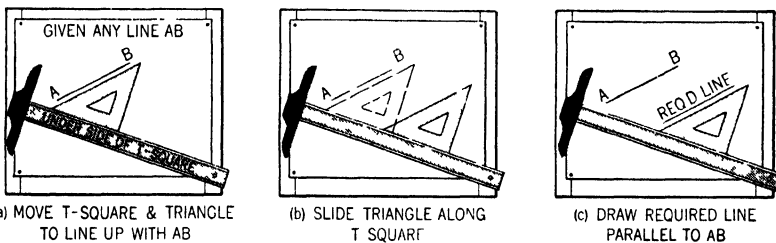


FIG. 64. DRAWING PARALLEL LINES

with the given line AB in *a*; then, with the T-square held firmly in position, move the triangle along the T-square away from the line as in *b*, and draw the required line as shown in *c*. Another triangle could be substituted for the T-square.

If a given line slopes at any standard angle, illustrated in Fig. 63, a line also may be drawn perpendicular to it with the triangles used as shown above. For example, in either Fig. 60, 61, or 62,

the lines in b are perpendicular to those in a , and the two sets of lines in c are mutually perpendicular.

To draw a line perpendicular to any given line by the *adjacent-sides method* (Fig. 65), move the T-square and triangle as a unit

*Using the
legs of a
triangle*

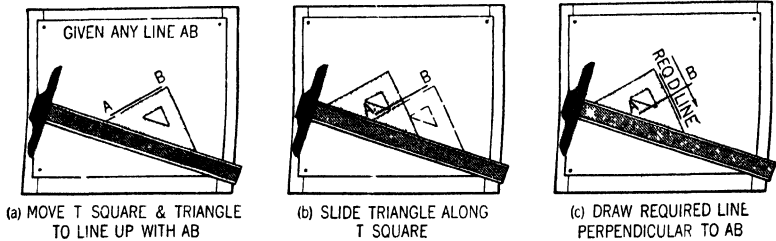


FIG. 65 DRAWING PERPENDICULAR LINES — ADJACENT-SIDES METHOD

until a side of the triangle “lines up” with the given line as in a ; then with the T-square held firmly in position, slide the triangle across the line as in b , and draw the perpendicular as in c . A perpendicular to a given line can also be drawn by the *revolved-triangle*

*Revolving a
45° triangle*

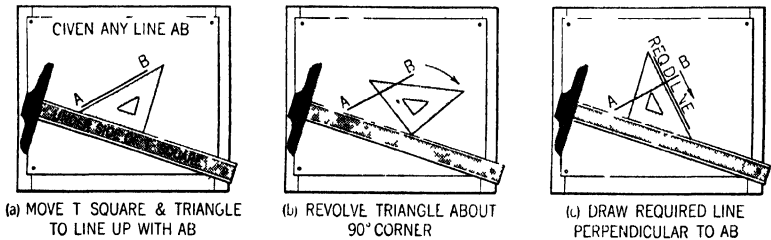


FIG. 66 DRAWING PERPENDICULAR LINES — REVOLVED-TRIANGLE METHOD (45° TRIANGLE)

method, the T-square and triangle being manipulated as shown in Fig. 66 and Fig. 67. We choose this method when the hypotenuse is needed to draw a longer line.

*Revolving a
30° × 60°
triangle*

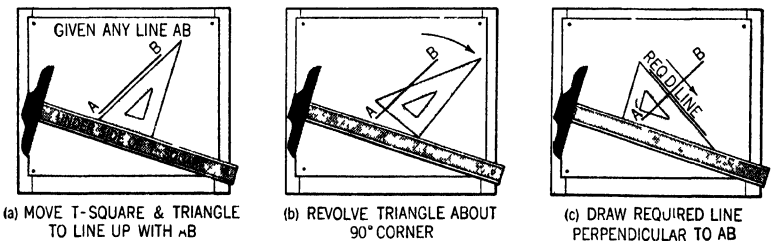


FIG. 67. DRAWING PERPENDICULAR LINES — REVOLVED-TRIANGLE METHOD (30° × 60° TRIANGLE)

SCALE DRAWING

In technical drawing we make all measurements with a scale (never called a ruler). A drawing of an object is made *full-size* if the size of the paper permits, or to a *reduced scale* if necessary, such as half-size, quarter-size, or eighth-size. For example, a machine part may be drawn half-size ($\frac{1}{2}$ in. = 1 in.); a building may be drawn $\frac{1}{16}$ size ($\frac{1}{4}$ in. = 1 ft.); or a map may be drawn $\frac{1}{1200}$ size (1 in. = 100 ft.).



FIG 68 ARCHITECT'S SCALE

The two most common scales are the *architect's scale* (Fig. 68) and the *engineer's scale* (Fig. 69). The former is used wherever the unit of measurement is the inch divided into sixteenths, and the latter is used wherever measurements are made in the decimal system. Inches are represented by the symbol " and feet by the symbol '. For example, eight feet three and five-eighths inches is written 8'-3 $\frac{5}{8}$ ", while eight feet and five-eighths of an inch is



FIG 69 ENGINEER'S SCALE

8'-0 $\frac{5}{8}$ ". On drawings where all measurements are in inches, they are understood to be in inches, and the symbol " is omitted.

Professional draftsmen usually prefer flat scales rather than triangular scales, but a set of flat scales is needed to provide all the scales that may be found on one triangular scale; hence the triangular scale is preferred where cost is a factor.

The architect's scale is used for architectural, structural, and machine drawings because those drawings are made by the inch-divided-into-sixteenths system, as distinguished from the decimal system. In machine drawing, however, decimal dimensions are also commonly used along with the regular measurements; therefore an engineer's scale or a conversion table will be convenient.

The architect's full-size scale is divided first into inches, then halves, quarters, eighths, and finally sixteenths, as in Fig. 70 a, the division lines diminishing in length with each decreasing divi-

sion. The use of the full-size scale is illustrated in the same figure, where it should be noted that the beginning of the scale is at zero and not at the end of the stick.

Half-size

To set off a measurement half-size, the full-size scale is used and each dimension is divided mentally by 2 as in Fig. 70 *b*. The $\frac{1}{2}$ -in.

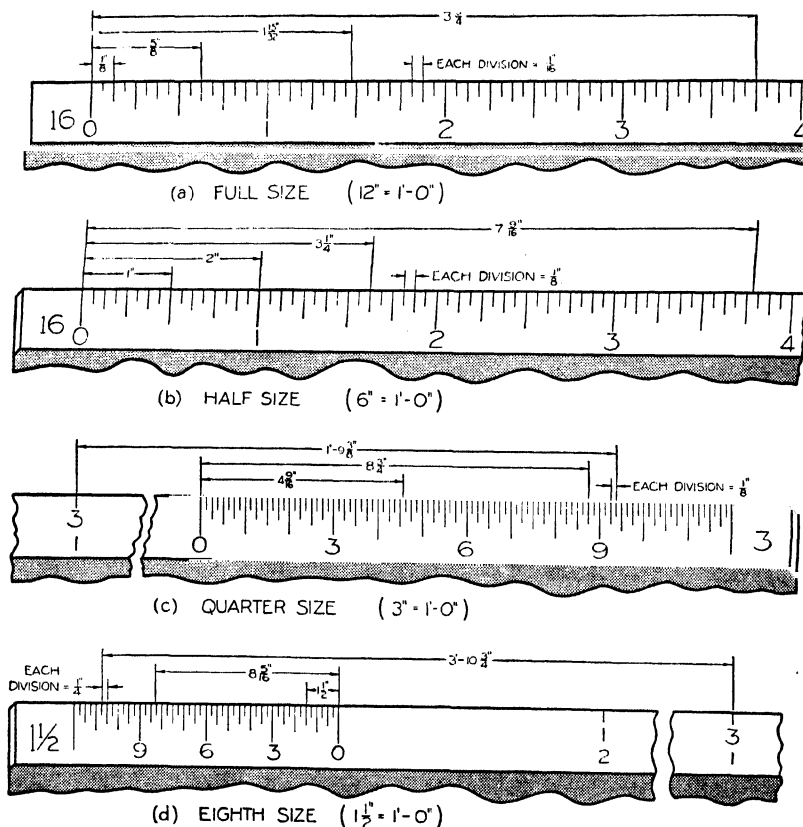


FIG. 70. READING THE ARCHITECT'S SCALE (SHOWN ACTUAL SIZE)

scale should not be used for half-size measurements, as this scale is intended for drawing to a scale of $\frac{1}{2}$ in. = 1 ft., or $\frac{1}{4}$ size.

Reduced scales

All of the remaining scales on the architect's scale are reduced scales, in which inches or fractions thereof stand for feet. The reduced scales, marked 3", 1 1/2", 1", 3/4", 1/2", 3/8", 1/4", 3/16", 1/8", and 3/32" are each intended to equal 1 ft. per main division.

Quarter-size

To set off a measurement to quarter-size the 3-in. scale is used in which 3 in. = 1 ft. (Fig. 70 *c*). The subdivided portion to the right of zero represents 1 ft., and is divided into inches, then half-

inches, then quarter-inches, and finally, eighth-inches. To set off a measurement of more than 1 ft., the *feet* are set off to the *left* of zero, and the *inches* to the *right* of zero, as for example, the dimension $1'-9\frac{3}{8}"$ in Fig. 70 *c*.

To set off a measurement to one-eighth size the $1\frac{1}{2}$ -in. scale is *Eighth-size* used, in which $1\frac{1}{2}$ in. = 1 ft. (Fig. 70 *d*). The subdivided portion to the left of zero represents $1'$, but is actually $1\frac{1}{2}"$; hence $1\frac{1}{2}$ in. = 1 ft., or one-eighth size. To set off more than 1 ft., for example, $3'-10\frac{3}{4}"$ (Fig. 70 *d*), the dimension $3'$ is set off to the right of zero, and the dimension $10\frac{3}{4}"$ to the left of zero; the required distance is that between these two extreme marks. This measurement is *one-eighth* of $3'-10\frac{3}{4}"$. The remaining scales are used in the same manner.

The habit of accuracy is an objective every engineering student should strive to attain. Accurate drafting depends largely upon the correct use of the scale. Measurements should not be taken directly off the scale with the dividers or compass. Always place the scale along the line to be measured, and with the eye directly above the correct graduation on the scale (Fig. 71) make a short dash with a sharp pencil at right angles to the scale as in Fig. 72 *a*. If extreme accuracy is required, make a tiny prick mark at the exact graduation with a needle point or with one leg of the dividers, as shown in Fig. 72 *b*.

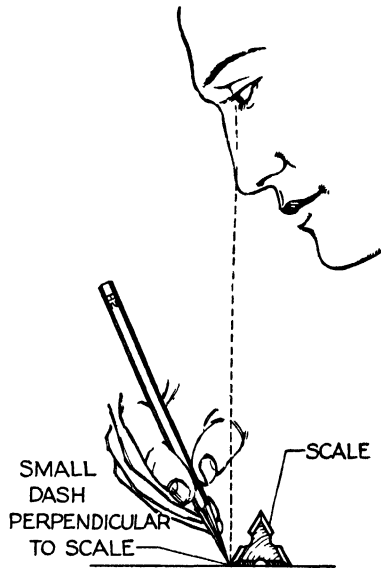


FIG. 71. ACCURACY IN USE OF SCALE

Accuracy



FIG. 72. ACCURATE MEASUREMENTS WITH THE SCALE

You must avoid *cumulative errors* in the use of the scale. If a series of measurements is to be made along one line, all should be set off at one setting of the scale, by adding each measurement to the

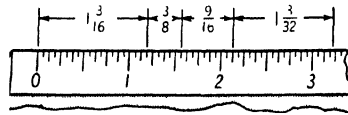


FIG. 73. SUCCESSIVE MEASUREMENTS

preceding one (Fig. 73). Such distances should never be set off individually by moving the scale to a new position each time, since even slight errors may accumulate and give rise to a large error.

USE OF INSTRUMENTS

Drawing instruments

A case of drawing instruments suitable for school or professional drafting is shown in Fig. 74. This set contains the following instruments:

- | | |
|--|-----------------------------|
| 1 Compass, with a pencil attachment, a pen attachment, and a lengthening bar | 1 Bow pencil |
| 1 Dividers | 1 Bow pen |
| 2 Ruling pens | 1 Bow dividers |
| | 1 Case of leads for compass |

In technical drawing, neatness, accuracy, and speed are essential; and high-grade instruments are always to be used if possible.

Use of compass

Circles and arcs with a radius of approximately 1" or over are drawn with the compass. You can and should handle the compass entirely with one hand. When the compass is lifted from the case, the legs are squeezed apart with the thumb and forefinger, the needle point set at the center of the circle, and the compass adjusted to the required radius previously set off with a scale on the center line — all with one hand as in Fig. 75 *a*. Start drawing the circle on the left-hand side (*b*), lean the compass slightly forward, and draw the arc in a clockwise direction with the handle held between the thumb and forefinger (*c*).

The compass point

If you want to excel in your draftsmanship, use great care to sharpen the compass lead properly on the sandpaper pad or file, as shown in Fig. 76 *a*, *b*, and *c*. Adjust the needle point carefully as shown in *d*; avoid deep holes in the paper by exerting only a light pressure on the compass. For ink work the legs of the compass should be "broken" or adjusted so that the lower sections

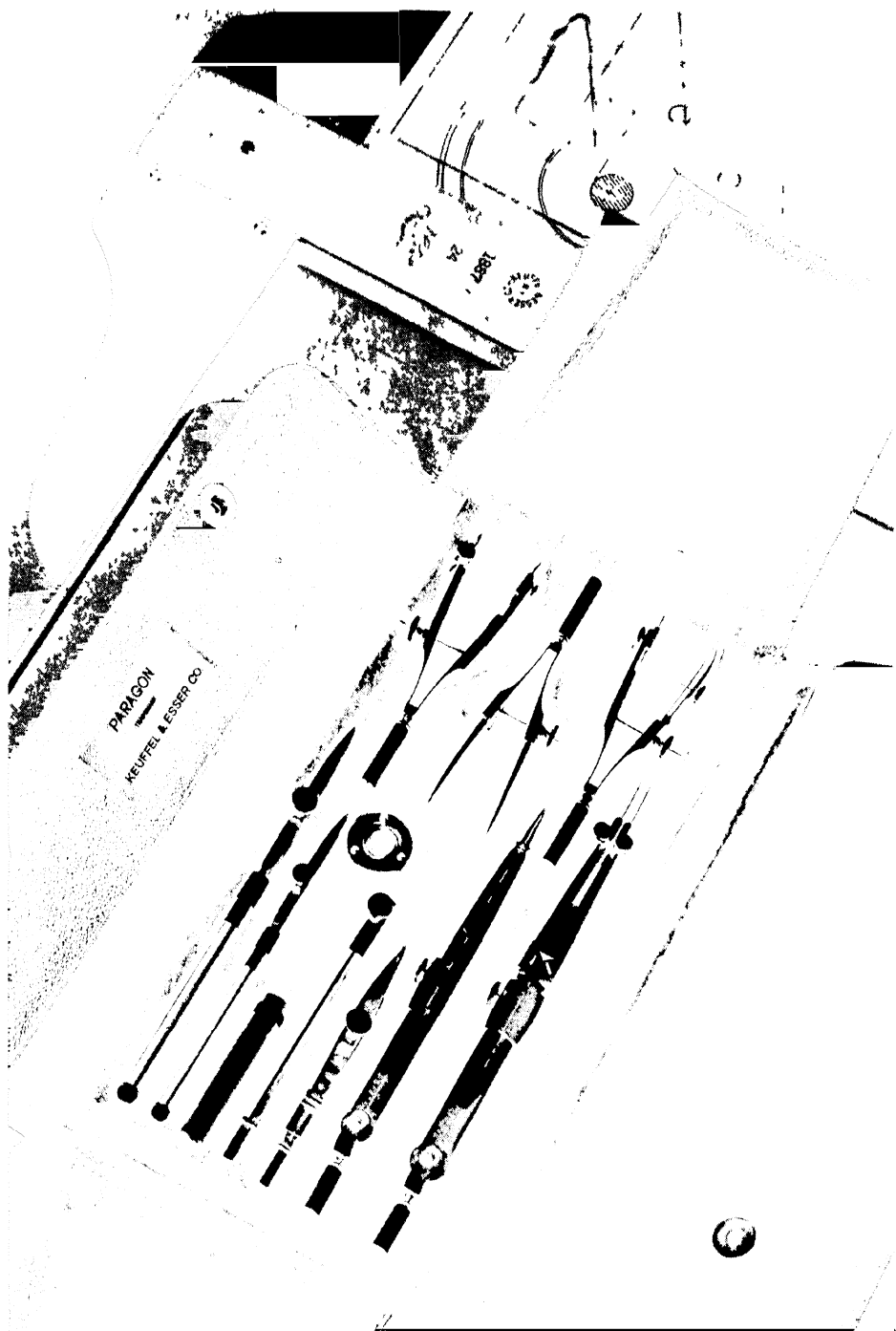


FIG. 74. SET OF DRAWING INSTRUMENTS

are parallel, in order to bring both nibs of the pen attachment in contact with the paper (Fig. 75 *d*).

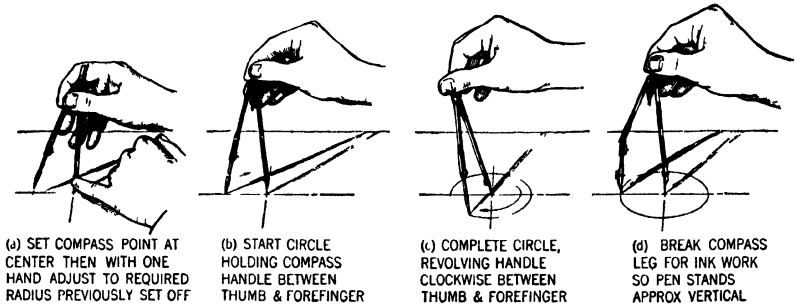


FIG. 75. USING THE COMPASS

Use of bow pencil or pen

Circles and arcs of about 1" radius or less are drawn with the bow pencil or bow pen (Fig. 77). First, the needle point is set at the exact center (*a*) and the radius adjusted by turning the thumb-

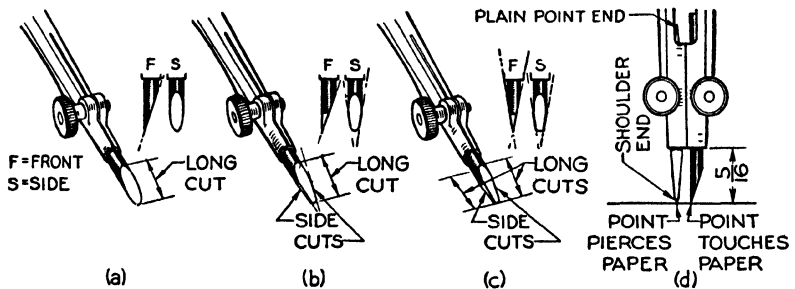


FIG. 76. COMPASS LEAD POINTS

screw between the thumb and third finger (*b*). Then the circle is started on the left-hand side and is drawn in a clockwise direction as shown in Fig. 77 *c*.

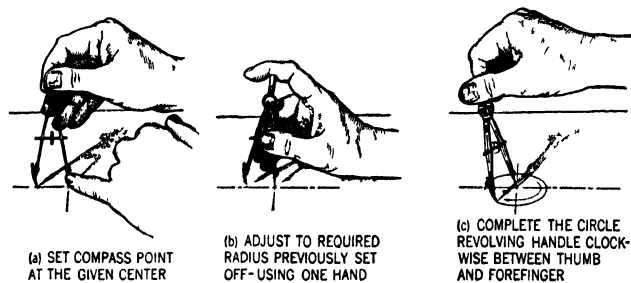


FIG. 77. USING THE BOW PENCIL

The dividers are used for subdividing lines into equal spaces and for transferring distances that are approximately 1" or over. To divide a line into equal parts, say three, as in Fig. 78, lift the dividers, squeeze the legs apart between the thumb and forefinger until the points are separated an estimated one-third of the total distance, and set the point at one end of the line as in *a*. Then the

Use of dividers

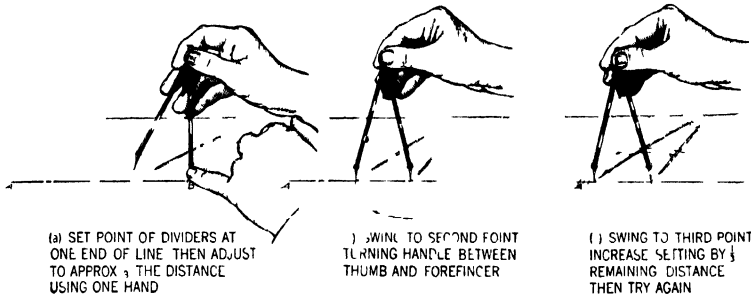


FIG 78 DIVIDING A LINE INTO EQUAL PARTS

dividers are twirled between the thumb and forefinger to the second point (*b*) and finally to the third point (*c*). Here the spacing of the dividers is decreased or increased by one-third the total error, large or small as the case may be, and the process is repeated until the spaces appear as equal thirds.

Where small divisions are needed, the bow dividers should always be used instead of the large dividers, because of their greater rigidity. The bow dividers are adjusted with one hand, as shown for the bow pencil in Fig. 77 *a*, and are used in a manner similar to that described for the large dividers (Fig. 78).

Use of bow dividers

The correct use of the irregular curve requires skill, especially if the lines are to be drawn in ink. First, sketch a light freehand line smoothly through the points; then match the various segments of the curve in succession to the freehand curve (Fig. 79).

Use of irregular curve

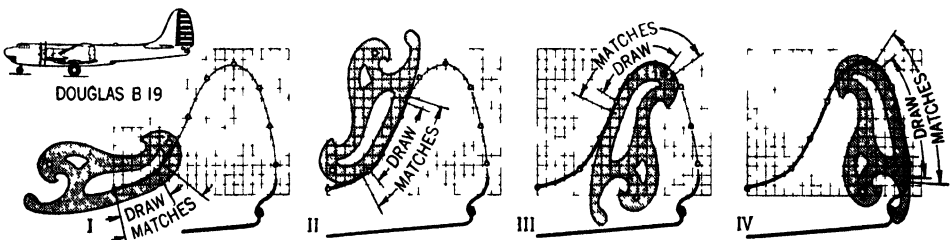


FIG 79. SETTINGS TO DRAW A SMOOTH CURVE

Care must be taken to *overlap* the matched portions at each setting; otherwise abrupt changes in direction are sure to result.

DRAFTING TECHNIQUES

*Keeping
drawings clean*

The utmost cleanliness in drawing should become habitual, for the draftsman can always keep his work clean if he really tries. First, your hands must obviously be kept clean, and second, your drawing tools must be frequently wiped and cleaned. Art gum is excellent for cleaning triangles and T-square; water is not recommended. Of course, you should never sharpen a pencil over a drawing, and your sleeves or arms should not be allowed to rest

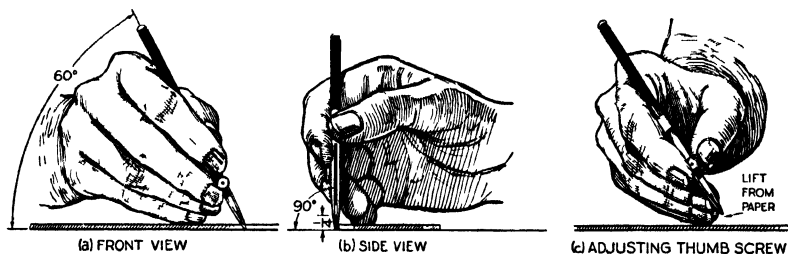


FIG. 80 USE OF RULING PEN

upon a penciled area, else smudging will result. So far as possible nothing should be allowed to come in contact with the penciled surface except the T-square and triangles, and they should be lifted to be moved, whenever possible.

Using ink

The ruling pen (Fig. 80) is used to ink ruled lines and is never used freehand. It is filled to a height of about $\frac{1}{4}$ in., as shown in *b*, with the *quill* of the ink stopper. The thumbscrew controls the width of the line and is adjusted with one hand as shown in *c*.

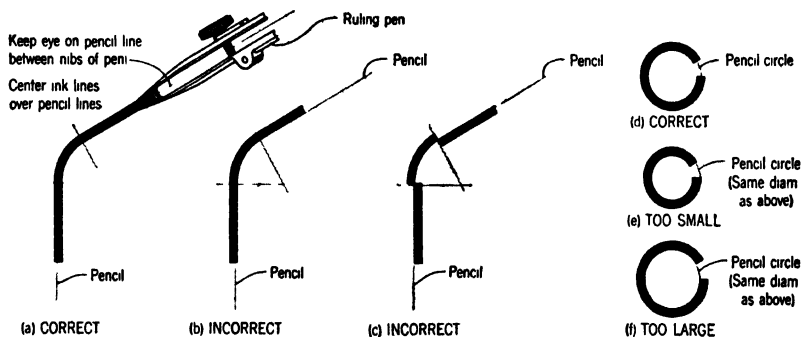


FIG. 81 INK LINES OVER PENCIL LINES

Always draw ink lines exactly centered over the pencil lines and not to one side (Fig. 81). You should lean the pen at an angle of about 60° with the paper (Fig. 80 *a*), keeping the thumbscrew on the side away from the T-square or triangle. The correct position of the pen and the resulting correct line are shown in Fig. 82 *a*;

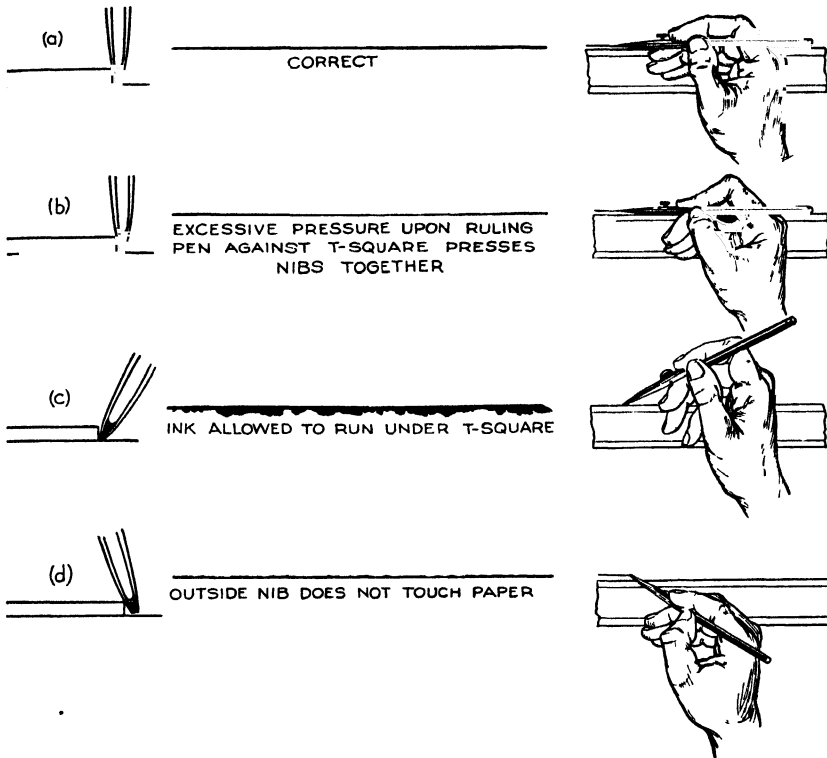


FIG. 82. POSITION OF RULING PEN

incorrect uses are illustrated in sketches *b*, *c*, and *d*. The ruling pen should be cleaned frequently by inserting a blotter or a piece of cloth between the nibs; *it should never be laid down with ink in it*. When you need to draw large circles in ink, fit the ink attachment to the large compass. The bow pen should be used for inking circles of less than about 1 in. radius.

When you make an error in inking, your first act should be to blot the area quickly with a blotter to absorb excess ink; then after the spot has dried, you can with patience erase the spot or error with the Ruby pencil eraser. Draftsmen have no use for a gritty "ink" eraser, which is apt to damage the paper.

Ink lines

Erasures

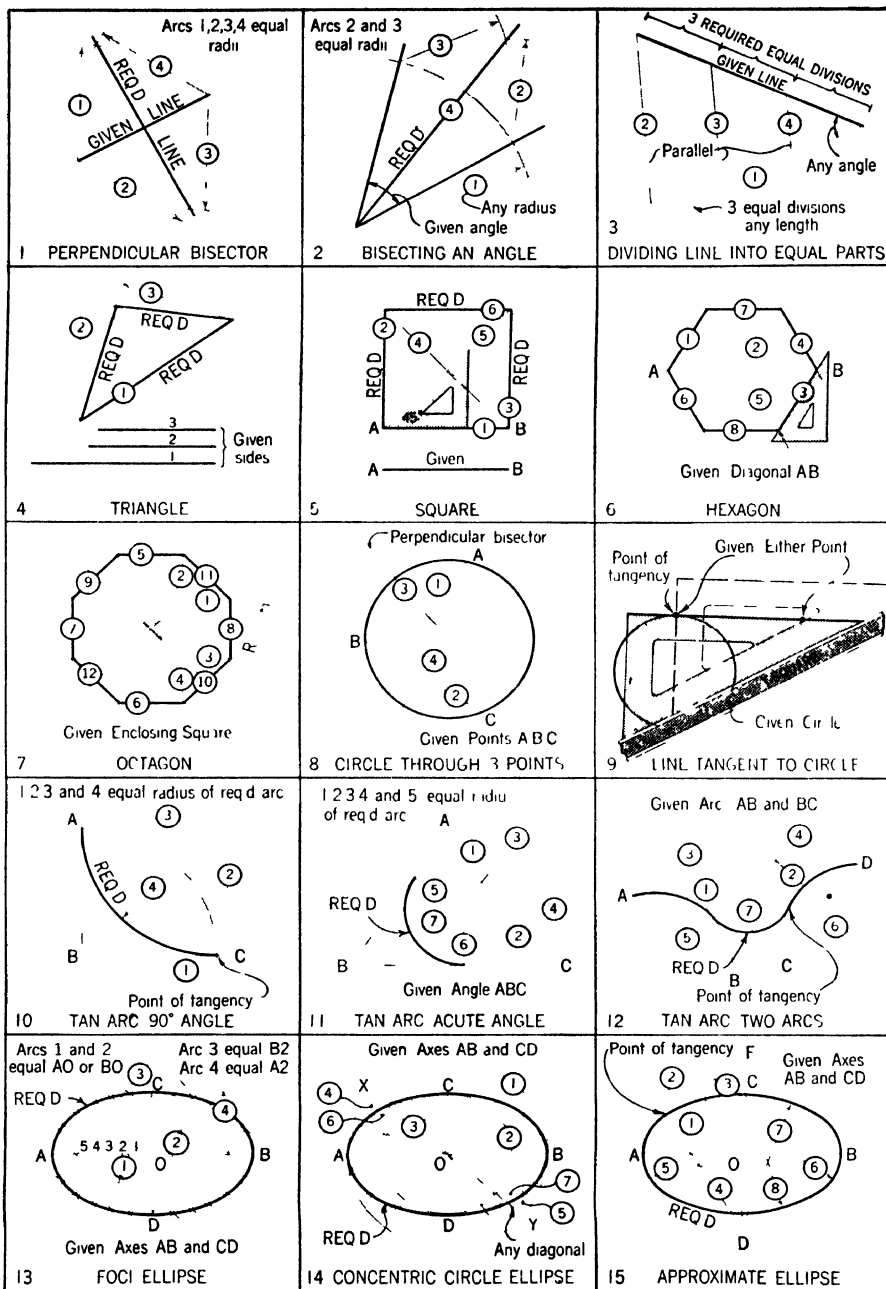


FIG. 83. COMMON GEOMETRICAL CONSTRUCTIONS

In technical drawing there are a number of applications of plane geometry which are useful and which are used repeatedly (Fig. 83). Some of these may be drawn with the compass and a straight edge as in pure geometry. However, the draftsman has at his disposal drafting instruments which can be used in many cases to better advantage, and these methods are shown where applicable.

Geometrical constructions

Given and *required* lines are generally heavy and should be drawn with a sharp medium-grade pencil so as to produce a black line. Construction lines should be extremely light and should be drawn with a sharp hard pencil. In Fig. 83 the small encircled numbers show the order in which the lines should be drawn in each case.

Weight of lines

Technical drawings are composed chiefly of symbolic lines (Fig. 84). These lines are designed to provide a reasonably natural appearance of the object represented, but they are strictly symbols and should be drawn to conform to fixed standards, as illustrated. A pencil drawing is composed roughly of only two line thicknesses, which may be described as "thin" and "medium," while an ink drawing also may have "heavy" lines and some variations between these. It is important on both pencil and ink drawings, and especially the latter, to obtain a clear *contrast* between the various weights of lines. Each line in its own class should be drawn uniformly throughout the drawing.

The graphic alphabet

The greater part of commercial drafting is executed in pencil. Most blueprints are made from pencil tracings, and all ink tracings are preceded by pencil drawings. It should therefore be evident that skill in drafting implies chiefly *skill in pencil drawing*. "Technique" is a style or quality of drawing imparted by the individual draftsman to his work. It is characterized by bright, clean linework and lettering. On pencil drawings all lines are equally *black* with the single exception of the construction line, which should be extremely light. The several types of black pencil lines should vary in width but not in color.

Pencil tracings

The proper order of penciling a working drawing is illustrated in Fig. 85. The drawing is carried through two main stages: first, the construction stage, embraced in spaces I and II, in which the drawing is "blocked-in" completely with a hard pencil, such as 4H to 6H, with extremely light lines; and second, the finishing stage, shown in spaces III and IV, in which the lines are heavied-in and the lettering added with a softer pencil, such as 2H or 3H.

Order of penciling

A definite order should be followed in inking a drawing or

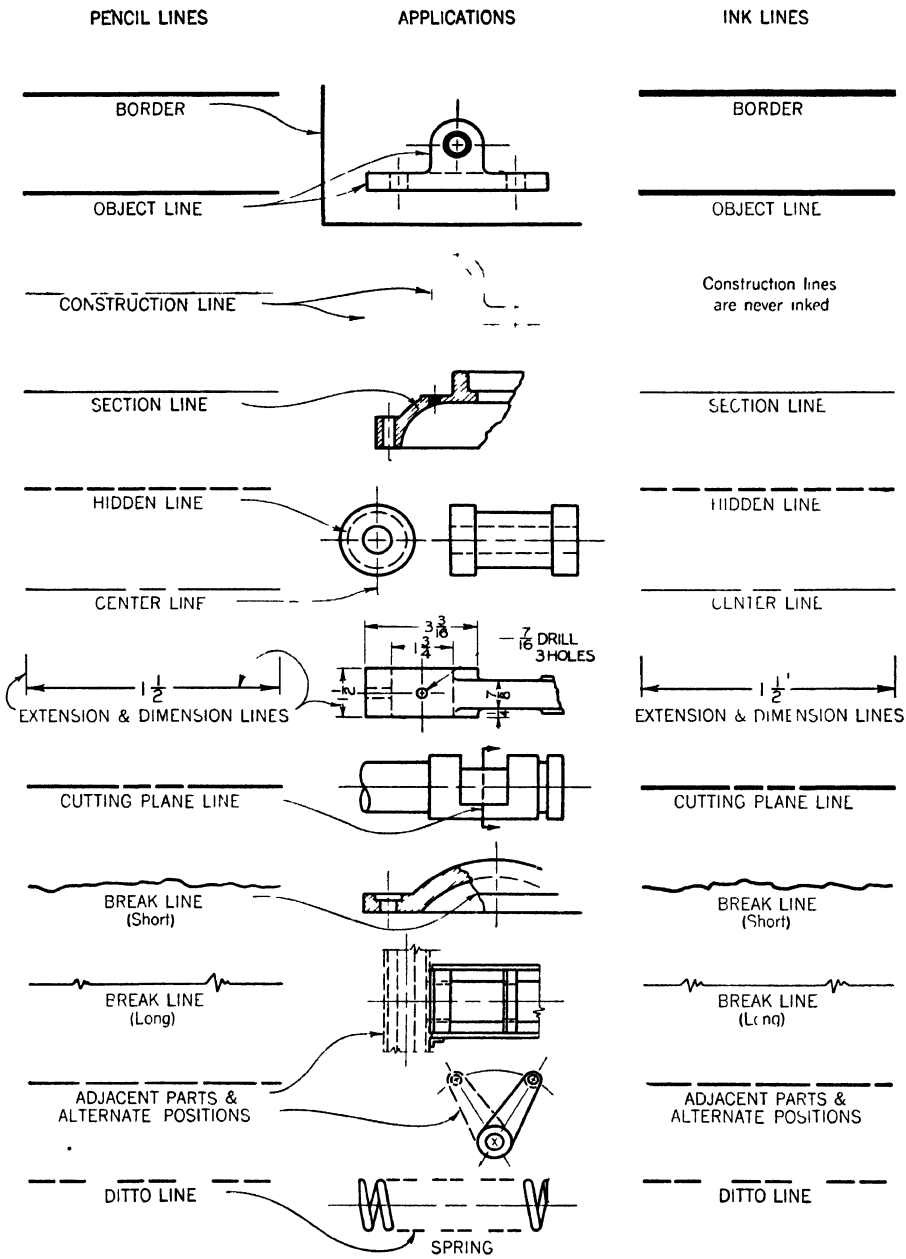


FIG 84. THE GRAPHIC ALPHABET

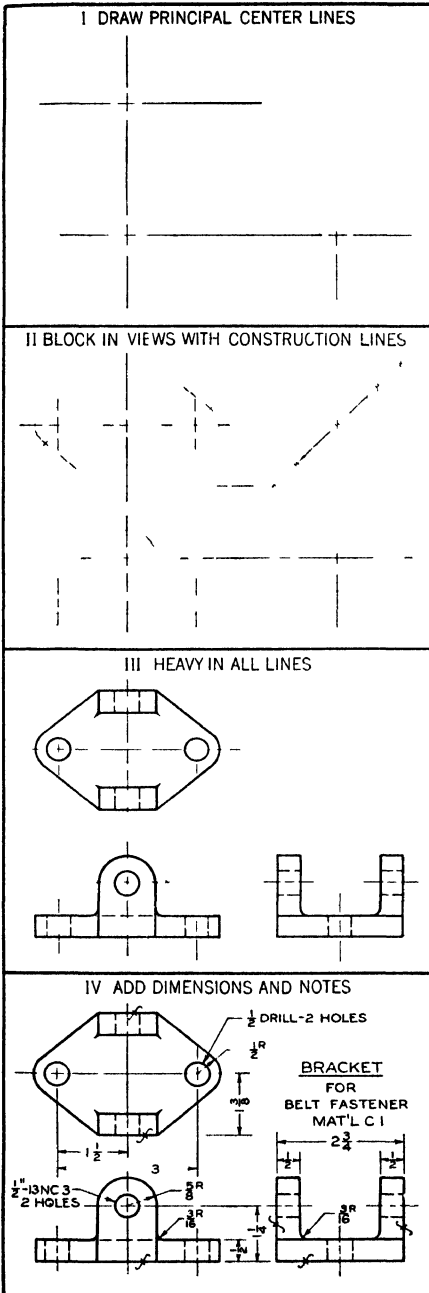


FIG. 85. ORDER OF PENCILING

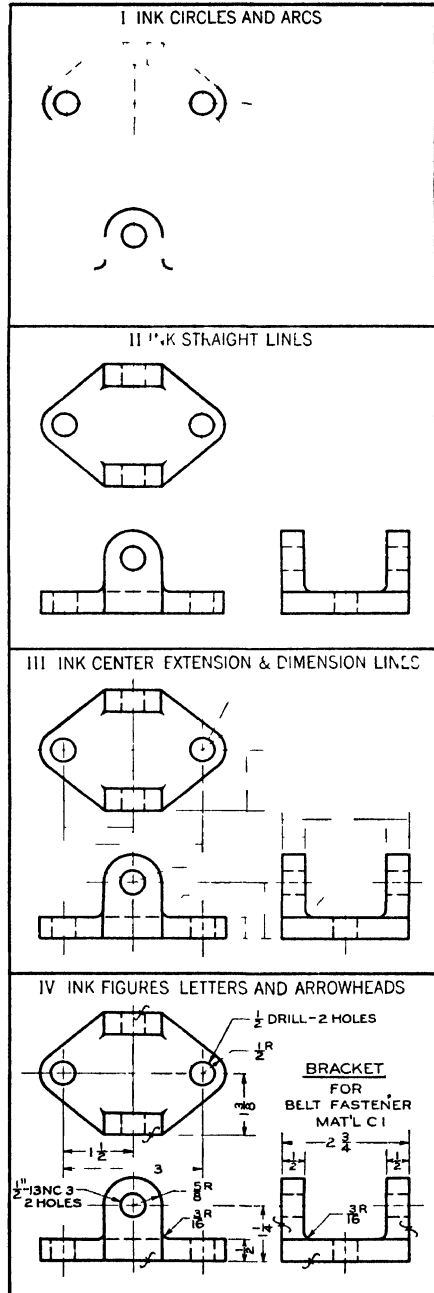


FIG. 86. ORDER OF INKING

tracing, as illustrated in Fig. 86. Before the inking is started, all points of tangency should be located, establishing the extent of all arcs. All arcs and curves are inked first, followed by the straight lines. Lettering and arrowheads are then added to complete the tracing.

TECHNICAL LETTERING

Technical drawing is universally recognized as a basic study in all branches of engineering, and as such is taught as a "tool subject" in virtually every technical school in the world. It occupies this position in technical curriculums because the fundamentals embraced are prerequisite to the engineering studies which follow and to the practice of engineering itself.

Importance of lettering

In a similar way *skill in lettering* is prerequisite to engineering training and practice, and more especially to technical drawing. The ability to letter is almost as important to the engineer as the ability to write longhand is to the average person.

History

Before the invention of printing by Gutenberg in the fifteenth century, all letters were made by hand, and ornamented according to the artistic taste of the letterer. Early type was modeled upon the accepted letter-forms in each country, but modern newsprint capital letters are essentially the same as the *Old Roman* letters, developed approximately two thousand years ago. The present small letters, called *lower-case* letters because they were found in the lower case of type by compositors, which are used in all of our books and newspapers, were developed in the Italian Renaissance from earlier capital letters.

Single-stroke gothic letters

During the latter part of the nineteenth century the development of engineering made evident a need for a simple legible letter which could be made with single strokes of an ordinary pen. Our present single-stroke gothic letter is due largely to the work of C. W. Reinhardt, chief draftsman in the 1890's for the *Engineering News*.

A complete working drawing must not only describe, by means of views, the *shape* of the object, but by means of figured dimensions it must give the *size* of the object. Also, information which is needed in the construction of the part will be given by notes and specifications. Well-executed lettering contributes more than any other single factor to the attractiveness and legibility of a drawing.

The position of the hand when you are lettering should be natural and uncramped, as in Fig. 87. The forearm should always rest upon the drawing board or table top. Pencil lettering should be made with a medium-soft pencil with a sharp point. You should frequently turn the pencil in order not to wear the point flat in any one place. For ink lettering the Gillott's 404 steel point, or its equivalent, is recommended for the usual $\frac{1}{8}$ " lettering; for smaller letters the Gillott's 303 is satisfactory; and for letters over $\frac{1}{8}$ " the various ball-pointed pens may be used to advantage. When you are doing ink lettering, you should wipe the pen point clean before each dip of fresh ink.

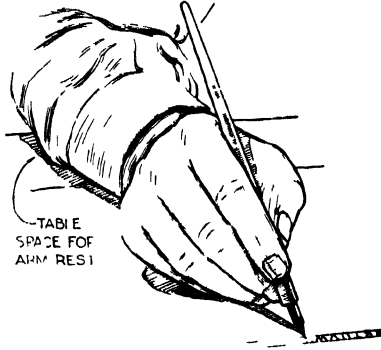


FIG. 87 POSITION OF HAND IN LETTERING

Any normal person can learn to do at least acceptable lettering if he is persistent in practice and in his will to improve. There are three steps in learning to letter: (1) learning the strokes and proportions of the letters, (2) learning correct spacing methods, (3) persistent practice.

Both vertical and inclined letters are used on technical drawings, according to the preferences of the various companies or of individual draftsmen. You will be wise to learn both styles and finally adopt the one which suits you better. Of the two, the vertical should be learned first.

VERTICAL LETTERING

The vertical capital letters are shown in Fig. 88. The letters are built up by a series of strokes, as shown below each of the large letters. The arrows indicate the direction in which each stroke should be drawn. With the exception of the letters I and W, all letters are either five or six units wide, such simple widths being easily remembered. The letters O, C, D, G, and Q are based upon the true circle. You must study and practice until you have developed a clear and accurate mental image of each letter; this is absolutely essential for success in lettering. It is especially suggested that you learn the forms of the letters by sketching them, six squares high, on cross-section paper.

How to start

Steps to follow

Vertical upper case letters

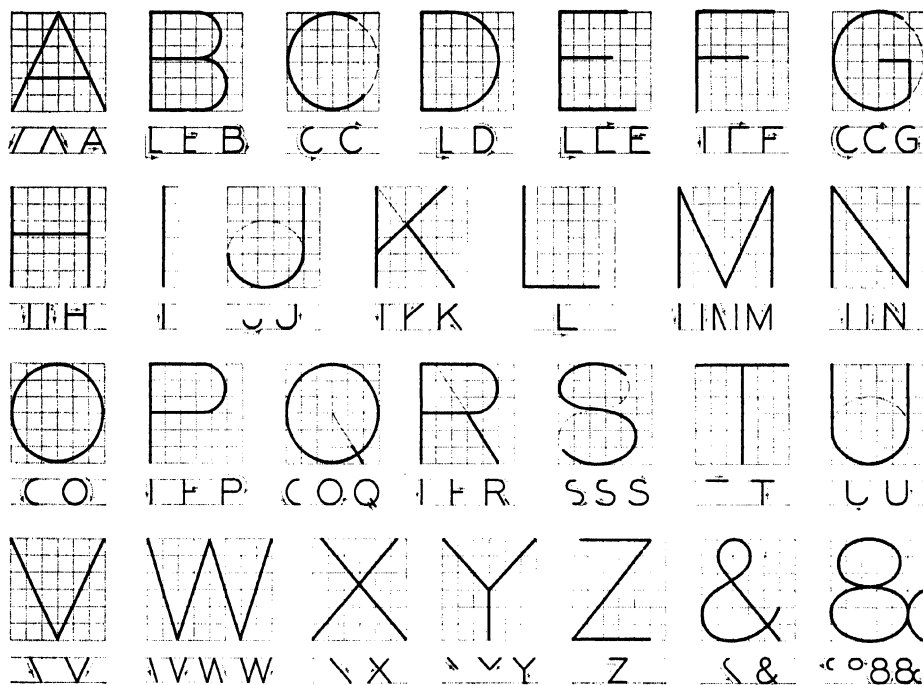


FIG. 88. VERTICAL CAPITAL LETTERS

Guide lines

Light horizontal and vertical guide lines, drawn with a sharp hard pencil, will assist you by limiting the height of the letters and by helping you to keep them vertical. (See Fig. 89.) The vertical guide lines help to keep the letters vertical but are not intended to assist in spacing. The clear space measured vertically between lines of capitals may vary from two-thirds of the height to the full height of the letters.

Guide-line spacing

Guide lines are spaced by means of the scale or by using bow dividers, as in Fig. 90. A convenient height-spacing for most letter-

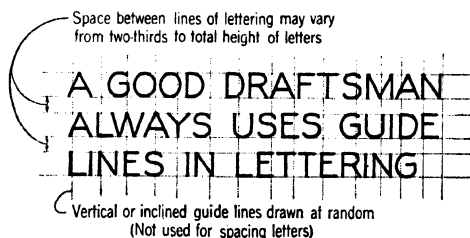
*Lines of lettering*

FIG. 89. GUIDE LINES FOR CAPS

ing is $\frac{1}{8}$ ", with spaces of $\frac{1}{8}$ " between lines, as shown in Fig. 90 *a* and *b*. Where the spacing between lines is equal to the height of the letters, you will also find the use of bow dividers very convenient, as only one setting of the points is necessary (*b*). The

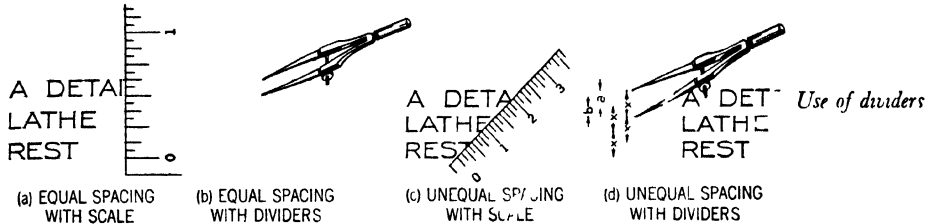


FIG. 90 SPACING OF GUIDE LINES

scale may be inclined to the guide lines to decrease vertical distances between guide lines, even though the spacing between lines does not equal the height of the letters (*c*). The bow dividers may still be used, with a single setting of the points, in cases where the spacing between lines does not equal the height of the letters (*d*).

Guide lines may be drawn conveniently with the aid of the Ames Lettering Instrument (Fig. 91 *a*) or the Braddock-Rowe Triangle (*b*). With either instrument, guide lines are drawn by inserting a sharp pencil in the systematically arranged holes and moving the instrument along the T-square. Numbers are provided to indicate the groups of holes that give various heights of letters in thirty-seconds of an inch.

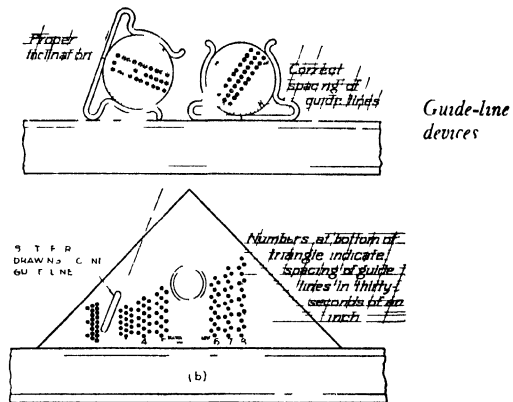


FIG. 91 GUIDE-LINE INSTRUMENTS

Uniformity in spacing letters and words is a matter of balancing spaces by eye. *The background areas between letters must appear approximately equal* (Fig. 92). Certain letters, such as the L and T, are of such shape that they may be overlapped to produce good spacing. Words should be spaced a distance apart equal to the letter O.

To letter to a stop-line as in Fig. 93 *a*, space off each letter from right to left by eye, and then letter forward to the stop-line.

Centering lines
of lettering

To space a line of letters symmetrically about a center line (Fig. 93 *b*), count the letters and spaces between words, and place the middle letter or space at the center; then letter the right half, followed by the left half.



FIG. 92. SPACING LETTERS AND WORDS

A strip of paper, as shown in Fig. 93 *c*, may also be used to letter a trial line, to be placed for reference immediately above the line to be lettered.

Vertical
numerals

The importance of correct formation of the numerals is understood when it is realized that all *numerical values on technical drawings*

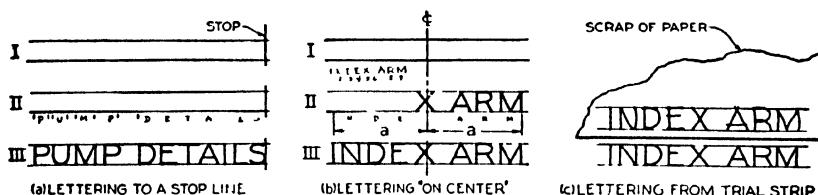


FIG. 93. SPACING

are expressed by various combinations of the ten basic numerals shown in Fig. 94. It is therefore worth your time to study carefully the proportion and spacing of these numerals. The background areas should be kept "open," and in fractions a clear space must be left between the numerals and the division line. All numerals

Fractions

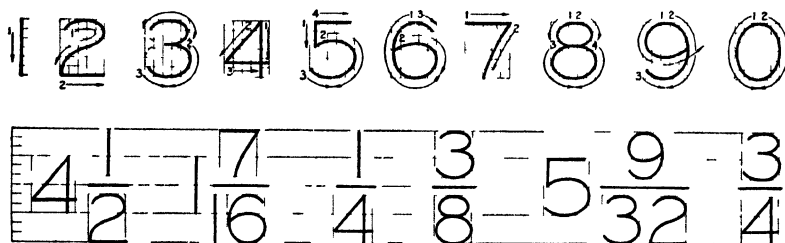


FIG. 94. VERTICAL NUMERALS AND FRACTIONS

are five units wide except the 1 and the 4. A *fraction* is one and two-thirds to twice the height of the whole number. The numerator and denominator are each two-thirds the height of the whole number.

A simple method of spacing guide lines for fractions is to use the bow dividers and set off spaces a equal to each other as illustrated by Fig. 95. The guide lines are then drawn through the points with the aid of the T-square or triangle.

If the upper portions of certain letters and numerals are as wide as the lower portions, the characters will appear to be top-heavy or to lack *stability*. To correct this optical illusion the upper portions are reduced in size, thus producing a more pleasing appearance (Fig. 96). Stability in the letters B, E, F, and H is improved by placing the horizontal cross-bars slightly above center.

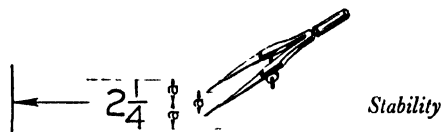


FIG. 95 SPACING WITH BOW DIVIDERS

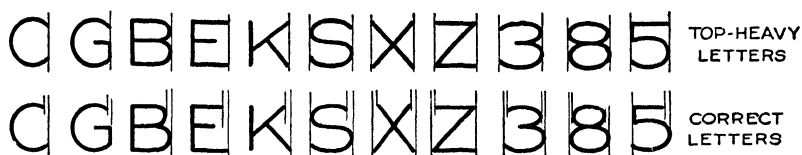


FIG. 96. STABILITY OF LETTERING

INCLINED LETTERING

The order and direction of the strokes and the proportions of inclined letters correspond to those for vertical letters. The circular parts of vertical letters, both upper case and lower case, become ellipses for inclined letters. Inclined capitals, or upper-case letters, are shown in Fig. 97 and the inclined lower-case letters are shown in Fig. 100. In making inclined letters, you should exercise care to make the inclined strokes slope uniformly and to keep the sides of the letters A, V, W, X, and Y sloping symmetrically about the inclined center lines. A comparison of correct and incorrect inclined capitals is shown in Fig. 98.

Light horizontal and inclined guide lines are used to regulate inclined letters (Fig. 99). Except for the inclined guide lines, all spacing is comparable to that for vertical capitals (page 178). The inclined guide lines are drawn at random (not used for spacing) at a slope of 2 on 5, which is approximately $67\frac{1}{2}^\circ$ with horizontal

(Fig. 99). The slot on the Braddock-Rowe Triangle shown in Fig 91 *b* is inclined at $67\frac{1}{2}^\circ$ with the horizontal and is convenient for making inclined guide lines.

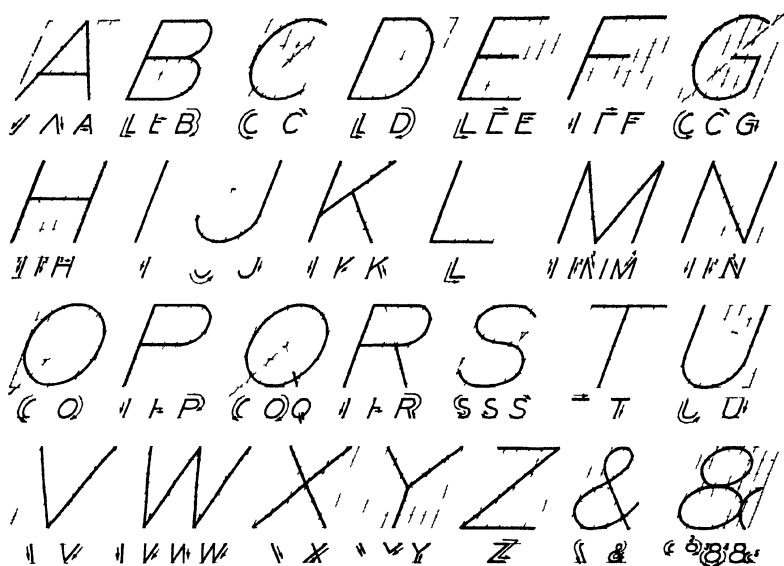


FIG 97 INCLINED CAPITAL LETTERS

*Reinhardt
letters*

Inclined lower-case letters, known as the "Reinhardt letters," are shown in Fig 100. This letter comes from the same letter forms as our handwritten script and is, therefore, the easiest letter for most people to make acceptably. It is easy to read because it

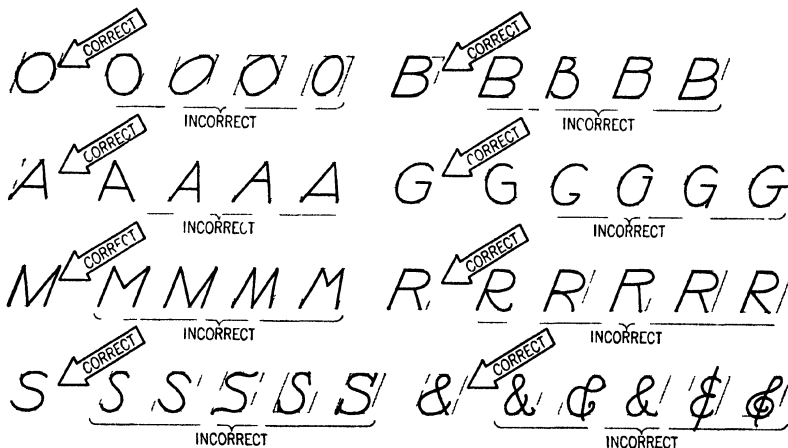


FIG 98 CORRECT AND INCORRECT INCLINED CAPITALS

closely resembles book and newspaper type, which is predominantly lower case, and it may be executed more rapidly than the other forms. Hence, this letter is widely used by engineers in all

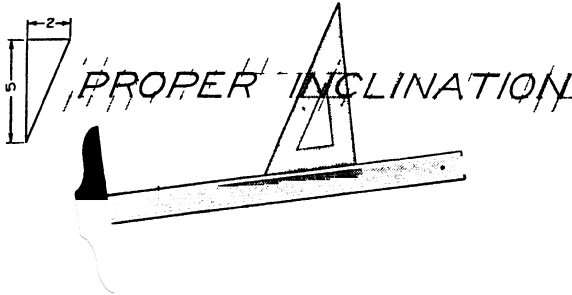


FIG 99 INCLINED GUIDE LINES

fields. The *bodies* of the letters are two-thirds the height of the corresponding capital letters, and the *ascenders* or *descenders* extend upward or downward one-third of this height (Fig. 100).

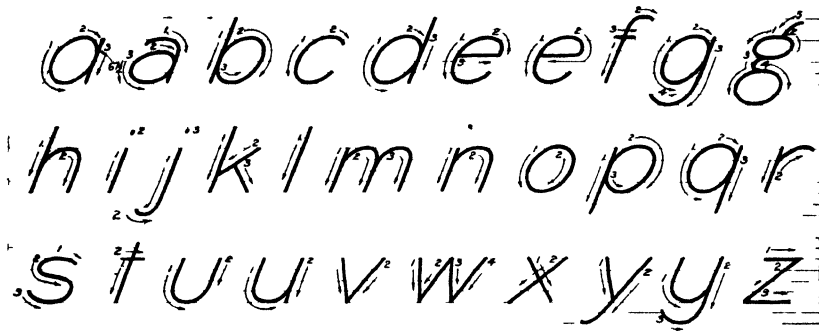


FIG 100 INCLINED LOWER-CASE LETTERS

This alphabet is composed of combinations of only two forms, the straight line and the ellipse, the long axes of the ellipses sloping *Amount of slope*

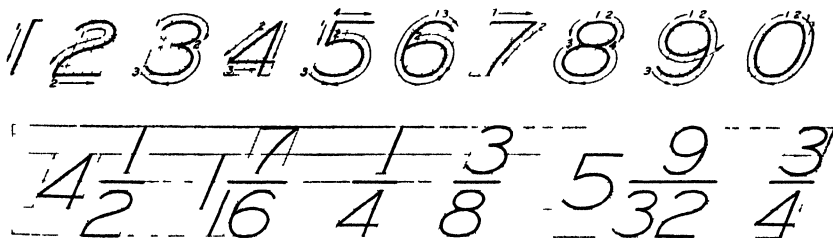


FIG 101 INCLINED WHOLE NUMBERS AND FRACTIONS

at 45° as in the inclined capital letters. The slope of the letters is $67\frac{1}{2}^\circ$ with the horizontal.

*Inclined
numerals*

The inclined numerals are shown in Fig. 101. They are different from the vertical numerals only in that they are inclined.

SHAPE DESCRIPTION

*Views and
line drawings*

A mechanical unit may be represented attractively, though superficially, by means of a photograph (Fig. 102). For technical purposes, such as the development of the original design and the conveying of the designer's ideas to the shop, *line drawings* are

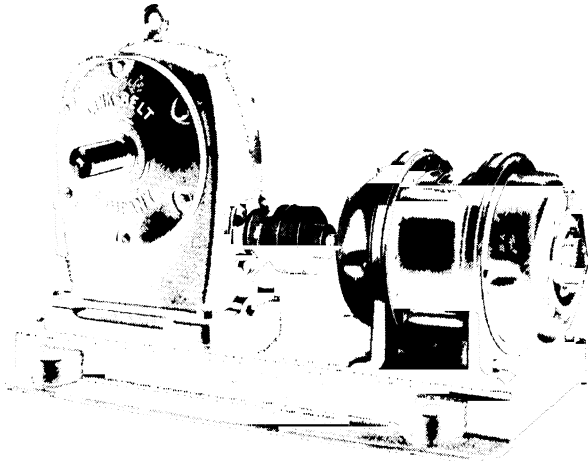
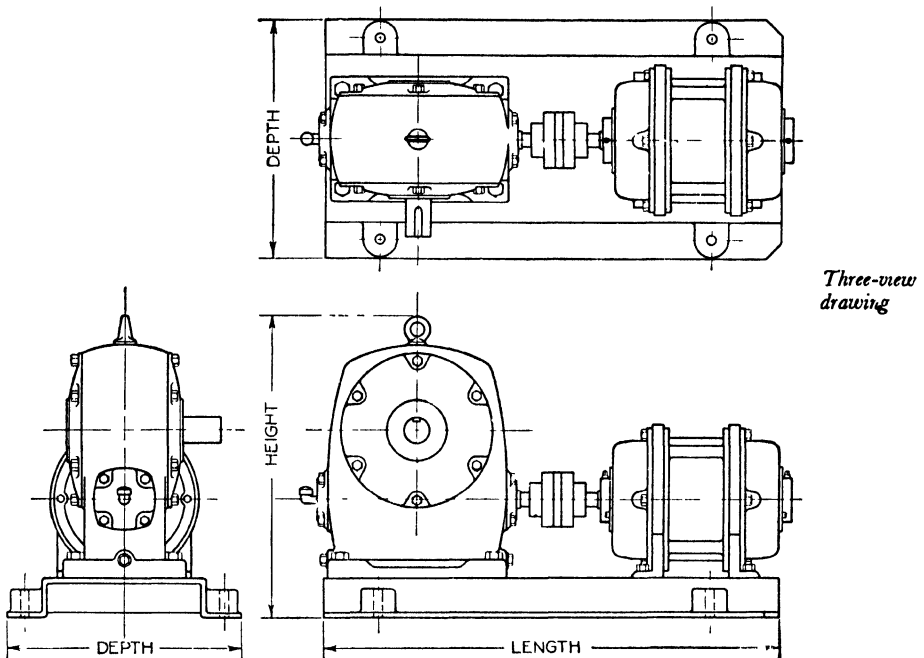


FIG. 102. REDUCTION GEAR DRIVE (PHOTOGRAPH)

necessary. The object must be represented by means of *views* showing the various sides, so that the true dimensions of the object will be shown. A three-view drawing of the "Reduction Gear Drive" is shown in Fig. 103, in which you should note that the three principal dimensions — *length*, *height*, and *depth* — appear in their true sizes (or proportionately, if drawn to a reduced scale). A white line print showing three views of a worker is shown in Fig. 104, which illustrates the front view, top view, and right side view. Notice, however, that the *right side view* of the man shows *his* left side.

*Viewing
the object*

Not only are assemblies represented by views, as in Fig. 103, but every individual part of a machine must be separately represented by means of its views, systematically arranged. For example, the "Holder for Timing Roller on an Offset Press" placed in an imaginary *glass box*, as shown in Fig. 105, is viewed by the observer

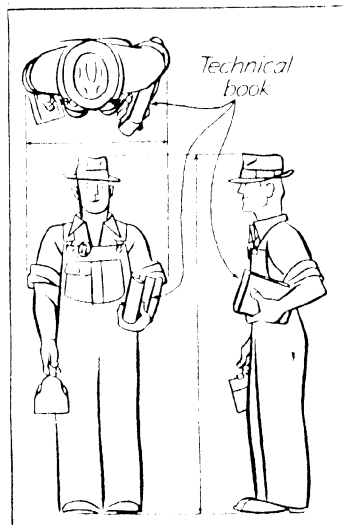


Three-view drawing

FIG. 103. THREE VIEWS OF A REDUCTION GEAR DRIVE

looking perpendicularly toward the front of the object to obtain the *front view*, perpendicularly toward the top of the object to obtain the *top view*, and perpendicularly toward the right side of the object to obtain the *right side view*. In Fig. 105 the arrows indicate the respective *lines of sight* for the three principal views, and these arrows are mutually perpendicular.

For purposes of graphical analysis, all objects can be assumed to have three principal dimensions — width, height, and depth. Two of these dimensions are shown true size in any view. As illustrated in Fig. 105, the front view always shows width and height,



Principal dimensions

By Russel Hamilton, courtesy Publishers' Weekly
FIG. 104. THREE VIEWS OF A TECHNICIAN

the top view shows width and depth, and the right side view shows height and depth. For each view the observer is, theoretically, at an infinite distance from the object, so that in a given view the lines of sight to the various points on the object will be mathematically parallel.

Possible views

There are, obviously, six principal directions in which an object may be viewed in this manner; hence six regular views of any object can be obtained, though three are usually adequate.

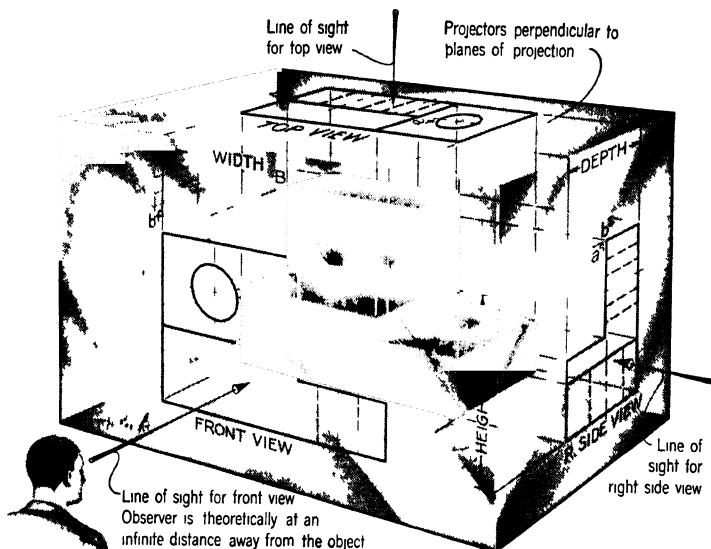


FIG. 105 THE GLASS BOX

Projections

A view of an object is known technically as a "projection," and we call the system of views described above *multiview projection*. A projection is a view of an object "projected" upon an imaginary *plane of projection*.

In Fig. 105 the *projectors*, or lines of sight, are parallel to each other and perpendicular to the plane of projection for that view. This is equivalent to saying that any view may be obtained by dropping perpendiculars from all *edges and contours** of the object to the corresponding plane of projection. The piercing points of these projectors, being infinite in number, form lines on the plane of projection. Thus, from a point *A* on the object in Fig. 105, perpendiculars strike the front plane of projection at *a'*, the top plane

* An edge is the intersection of two surfaces; a contour is an outline of a body seen from one position.

of projection at a' , and the right side plane of projection at a'' . Likewise, the projections of the point B are obtained at b' , b'' , and b''' ; and if all points between A and B are similarly projected, the front view of the line AB is seen to be $a'b'$, the top view $a''b''$, and the right side view $a'''b'''$, which happens to be a point. If this procedure is continued with all of the edges and contours of the object, six complete views may be obtained.

One of the most important advantages in the use of multiview projection is our ability to show edges and contours which normally would be invisible in a given view, by means of lines composed of dashes, or *hidden lines*. In Fig. 106 the edge AB would not

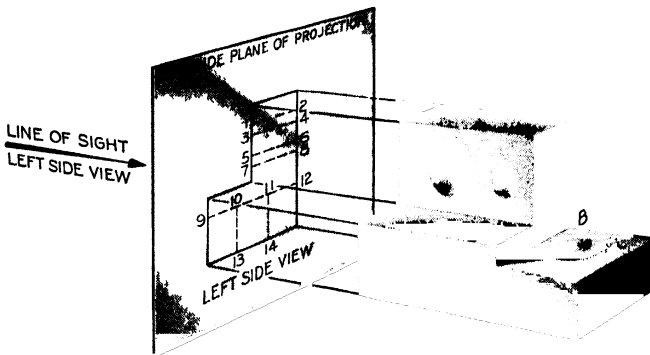


FIG. 106. HIDDEN EDGES AND CONTOURS

be visible to the observer for the left side view, and therefore its projection 9–12 is represented as a hidden line. Likewise, the contours of hole C are invisible, and are projected as hidden lines 10–13 and 11–14. (Note the T's formed by the dashes at 10 and 11.) Hole D is projected as hidden lines 1–2 and 7–8; hole E is projected as hidden lines 3–4 and 5–6. You should similarly visualize the projection of each of the other hidden lines in the top and side views of Fig. 105.

This unique system of projection has been devised to represent the mutually perpendicular views of a three-dimensional object on a single plane, such as a sheet of drawing paper. To accomplish this the glass box is unfolded until all planes coincide with the front plane of projection, as in Fig. 107. The top, bottom, and side planes are “hinged” to the front plane, which remains stationary, and the back plane is hinged to the left side plane. The edges about which the planes are revolved are called *axes* or *folding lines*.

Projection lines

The lines extending around the glass box from one view to another are the *projections of the perpendiculars* from points on the object to the planes of projection. For example, the projector *A-7* (Fig. 107) is projected on the top plane at 3-5 and on the side plane at 11-8. Since 3-5 and 11-8 are both equal in length to the projector *A-7*, the side view is seen to be the same distance from

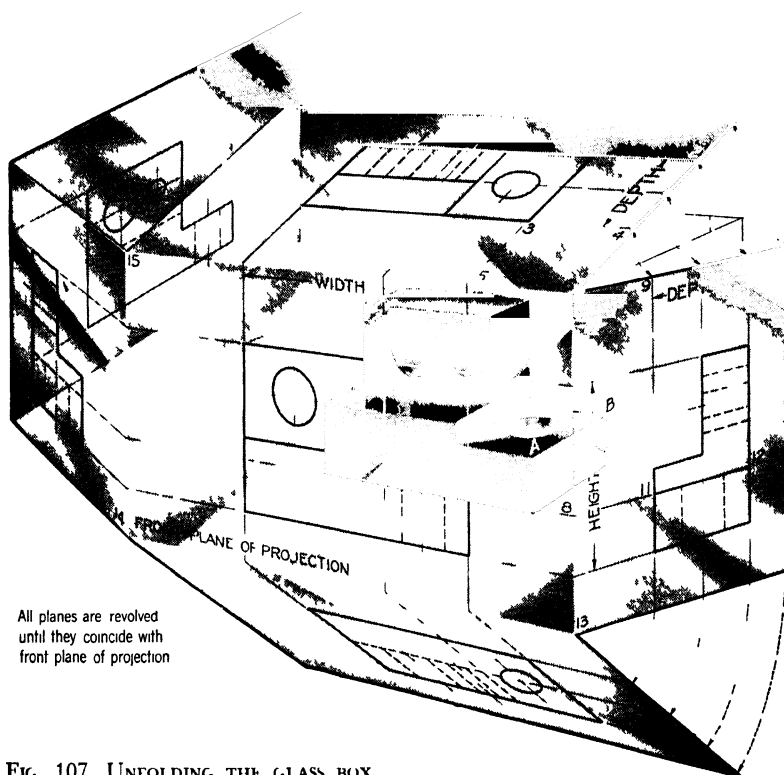


FIG. 107 UNFOLDING THE GLASS BOX

the front plane as is the top view. Likewise distance 6-4 equals 6-9 and 4-2 equals 9-10, which is to say that depth in the top view is the same as depth in the side view.

Positions of views

The positions of the six planes, after they have been revolved into one plane, are shown in Fig. 108. The top view is always directly over the front view, the bottom view is directly under the front view, and the width of the object is shown in all three. The right side view is always directly to the right of the front view. The left side view is directly to the left of the front view. The back view is directly to the left of the left side view, and the

height is shown in all four views. The depth is shown in the four views that surround the front view.

It is now evident (Fig. 108) that the separation of the top and right side planes produces arcs 4-9 and 2-10 which could be used on a drawing to transfer the dimension from the top to the side view and vice versa. This same relation occurs at all four corners of the front plane of projection (Fig. 108).

The draftsman must thoroughly understand the principles of projection, but he need not and should not show the planes of projection, *Construction lines*

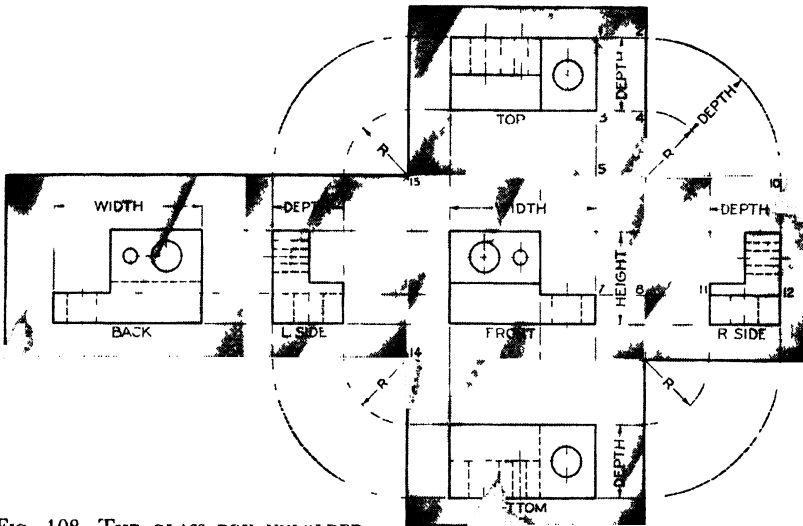


FIG. 108. THE GLASS BOX UNFOLDED

projection on a drawing. The six views, with the planes omitted, and drawn sufficiently near each other to look well, are shown in Fig. 109. The arcs of projection are omitted and the simpler 45° construction lines are substituted to transfer the depth dimension between the four views adjacent to the front view. Many draftsmen omit even this construction, preferring to transfer the depth dimensions with the dividers or scale.

Although the views must always “line up” as shown in Fig. 109, *Spacing the views* the distances between views are adjusted by the draftsman. *The distance between any two adjacent views is independent of the distance between other adjacent views.* For example, the distance between the front and top views is not necessarily equal to the distance between the front and right side views. However, a balanced appearance is highly desirable.

Relation of surfaces

In Fig. 109 it should also be observed that the back face of the object appears in the top view at 1-2, in the right side view at 3-4, in the bottom view at 6-5, and in the left side view at 8-7; in all four cases, on the side farthest from the front view. Conversely, the front face of the object appears in the top view at 9-10, in the

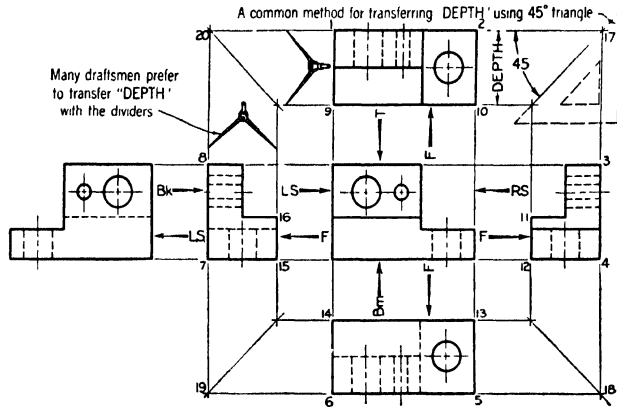


FIG. 109. THE SIX VIEWS ON PAPER

right side view at 11-12, in the bottom view at 14-13, and in the left side view at 16-15; in all four cases, on the side nearest to the front view.

Reciprocal views

Notice also in Fig. 109 that *adjacent views are reciprocal*. For example, if the front view is visualized as the object itself, the right side view is obtained by looking toward the right side of the front view in the direction of the arrow *RS*, which is the direction of sight to obtain the right side view.

Likewise, if the right side view is visualized as the object itself, the front view is obtained by looking toward the left side of the right side view in the direction of the arrow *F*.

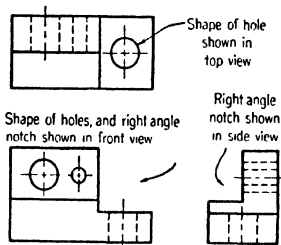
Needed views

FIG. 110. THE THREE VIEWS

Clearly, all of the six possible views are not needed to describe completely the shape of the object. The draftsman therefore draws only the *necessary views* which are those that together provide the clearest description of the shape of the object. The three *standard views* of the "Holder for Timing Roller" are shown in Fig. 110, and the features shown in each view are indicated. If a not-too-strict interpretation

of the rule of necessary views is followed, as is often the case in commercial drafting, we might omit the right side view, or the top view, on the assumption that shapes not shown completely are understood to be rectangular in form.

However, you should note carefully that *many different side views* *Possible side views* could be drawn to accompany rectangular top and front views, as in Fig. 111. If only two views of a simple rectangular block are

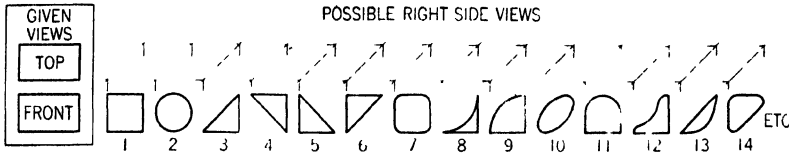
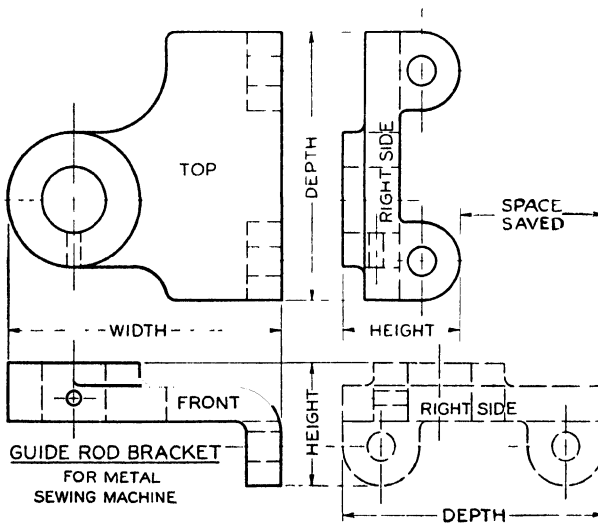


FIG. 111. POSSIBLE SIDE VIEWS

drawn, it is understood that the object is rectangular and not any of the odd shapes shown in this illustration. It becomes, therefore, a matter of judgment on your part as to the number of views that should be drawn for each given object, and the only rule to follow is to include those views which in your opinion show the shape of the object most clearly.

Complicated shapes may require more than three views, and in many cases special views, such as *partial views* (Fig. 206), *auxiliary views* (Fig. 160), or *sections* (Fig. 139). Always select those views *Other types of views*



Space saving arrangement

FIG. 112. SIDE VIEW BESIDE THE TOP VIEW

which show essential shapes or contours, and give preference to those with the least number of hidden lines. In Fig. 110 the right side view and the top view were chosen in preference to the left side and bottom views because of the numerous hidden lines in the latter.

In drawings of flat wide objects, such as the "Guide Rod Bracket" in Fig. 112, it is preferable to place the side view opposite the top view instead of opposite the front view in order to use the space on the paper to better advantage. Here the side plane is considered hinged to the top plane of projection instead of to the front plane of projection.

SURFACES AND EDGES

In studying multiview projection we find it useful to classify objects according to their various types of surfaces and edges and to examine each in turn.

Normal surfaces

A normal surface is a plane which is parallel to one of the planes of projection. It will appear in true size and shape on the plane to which it is parallel, and as a line (i.e., edgewise) on adjacent planes of projection. For purposes of analysis, we may consider the "breakdown," or steps in machining the block of cold-rolled steel, shown in Fig. 113, in which all surfaces are normal surfaces. In *space I*, normal surface *A* is parallel to the top plane and appears true size in the top view at 2-3-7-6, as the line 9-10 in the front view, and as the line 17-18 in the side view. Normal surface *B* is parallel to the side plane and appears true size in the side view at 17-18-20-19, as the line 3-7 in the top view, and as the line 10-13 in the front view. Normal surface *C* is parallel to the front plane and appears true size in the front view at 9-10-13-14-16-15-11-12, as the line 5-8 in the top view, and as the line 17-21 in the side view. All other surfaces in the block may be similarly visualized, and you should identify each individually. Observe carefully the changes of shape in the views produced by each machining operation, including the introduction of hidden lines, new visible edges, and the "dropping out" of certain lines as the result of a new cut.

Surfaces in projection

In the top view of *space I*, normal surfaces 1-2-6-5, 2-3-7-6, and 3-4-8-7 appear as adjacent rectangles. Lines 2-6 and 3-7 indicate that there must be three surfaces, but we must examine these surfaces in the front view also to determine the height of each. In the front view, surface 9-10 is seen to be the highest, and

surfaces 11-12 and 13-14 are at the same lower level. In the side view these two latter surfaces appear as *one line* 19-20. Surface 11-12 might appear as a hidden line 19-20 in the side view, but surface 13-14 appears as a visible line 19-20, which covers the hidden line and takes precedence over it.

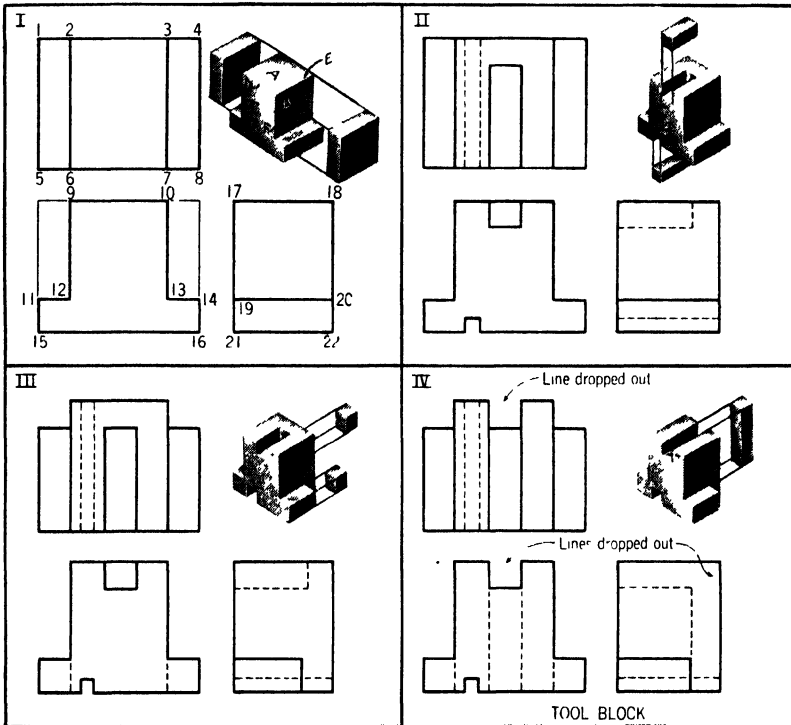


FIG. 113. MACHINING A "TOOL BLOCK" — NORMAL SURFACES AND EDGES

A *normal edge* is a line of intersection of two plane surfaces which is *Normal edges* perpendicular to a plane of projection. It will appear as a point on the plane of projection to which it is perpendicular and as a line in true length on adjacent planes of projection. In Fig. 113 I, edge *D* is perpendicular to the side plane of projection and is shown as the point 17 in the side view. It is parallel to the front and top planes of projection and is shown true length at 9-10 in the front view and as 6-7 in the top view. Edges *E* and *F* are perpendicular respectively to the front and top planes of projection, and their views may be similarly analyzed.

An *inclined surface* is a plane which is perpendicular to one plane of *Inclined surfaces* projection but not parallel to any plane of projection. It will show as a

*Steps in
machining
inclined
surfaces*

line on the plane to which it is perpendicular and will not show true size in any view. Inclined surfaces are represented by the steps in "Machining a Locating Finger" shown in Fig. 114. In *space I*, inclined surface *A* is perpendicular to the top plane of projection and is shown in the top view as the line 5-3. It is shown as a foreshortened surface in the front view at 7-8-11-10 and in the side view at 12-13-16-15.

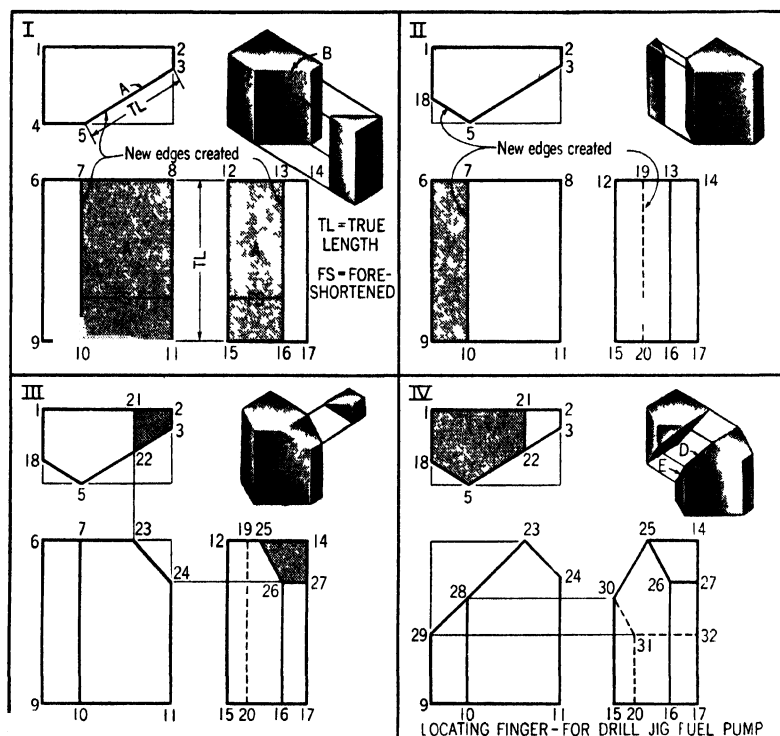


FIG. 114 MACHINING A "LOCATING FINGER" — INCLINED SURFACES

Each successive cut in Fig. 114 is perpendicular to a plane of projection and produces an inclined surface, and each step should be carefully studied. In *space IV* the final cut produces an inclined surface perpendicular to the front plane of projection showing as the line 29-23 in the front view, as a foreshortened surface in the top view at 1-21-22-5-18, and as a foreshortened surface in the side view at 25-14-32-31-30.

Inclined edges

An inclined edge is a line of intersection of two plane surfaces which is parallel to one plane of projection and not parallel to the others. In Fig. 114 *I*, inclined edge *B* is parallel to the top plane of projection, and its

true length is shown in the top view at 5-3. It is shown foreshortened in the front view at 7-8 and in the side view at 12-13. Each succeeding cut produces additional inclined lines, each of which you should analyze similarly.

An oblique edge is a line of intersection of two plane surfaces which is oblique to all planes of projection. Such edges may be produced by the intersection of inclined planes as in Fig. 114, but none appear until two inclined planes intersect as in *space III*. Here edge *C* is oblique to all planes of projection and is shown foreshortened

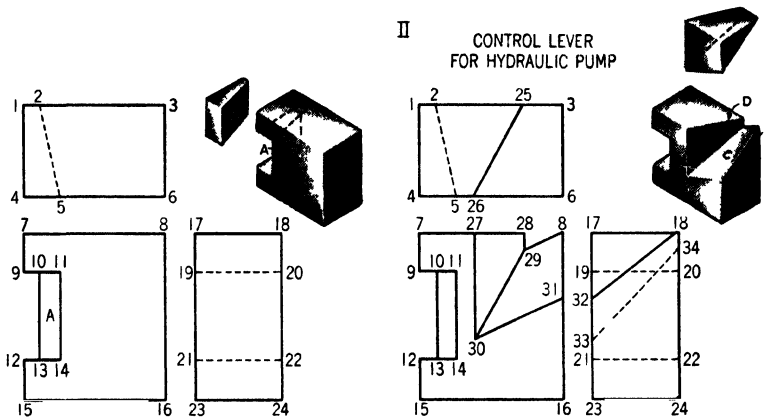


FIG. 115. MACHINING A "CONTROL LEVER" — INCLINED AND OBLIQUE SURFACES

in the top view at 22-3, in the front view at 23-24, and in the side view at 25-26. In *space IV* two additional oblique lines *D* and *E* are produced, and you should identify each line in the three views.

An oblique surface is a plane which is oblique to all planes of projection. It will show as a foreshortened surface (never as a line) in all views. In Fig. 115 an inclined surface and two normal surfaces are produced by the first cut in *space I*, but in *space II* an inclined surface *B* and an oblique surface *C* are produced by the V-shaped cut. The oblique surface *C* shows as a foreshortened surface in all views: in the top view at 25-3-6-26, in the front view at 29-8-31-30, and in the side view at 32-18-34-33. Edges *D* and *E* are inclined edges, and *F* is an oblique edge not showing in true length in any view.

Cylindrical surfaces are very common, because of the ease with which they can be produced on the lathe, the drill press, and other

machines using the principle of rotation either of the “work” or of the cutting tool. These surfaces may be classified as *normal*, *inclined*, or *oblique* cylindrical surfaces depending upon the position of the center line of the cylindrical surface with respect to the planes of projection. Normal cylindrical surfaces are illustrated in Fig. 116. In *space I* of this figure the removal of the two upper

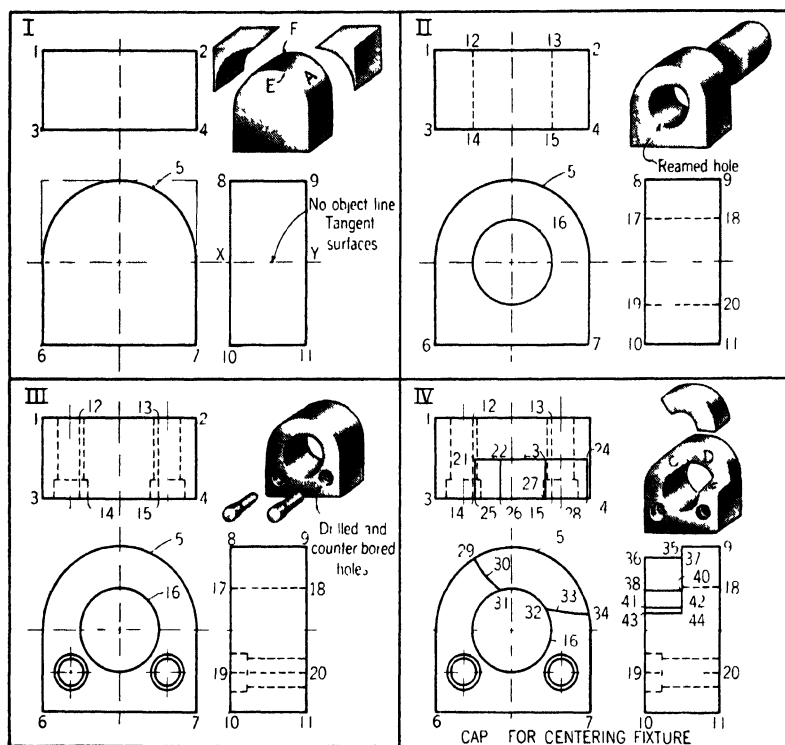


FIG. 116. MACHINING A “CAP” — CYLINDRICAL SURFACES

corners produces the cylindrical surface *A* which appears in the top view as the surface 1–2–4–3, in the front view as arc 5, and in the side view as the surface 8–9–Y–X.

Holes

In *space II* a reamed hole shows in the front view as circle 16, in the top view as surface 12–13–15–14, and in the side view as surface 17–18–20–19. In *space III* two drilled and counterbored holes are added, producing four cylindrical surfaces and two normal surfaces. The normal surfaces are the bottoms of the counterbores. In *space IV* a cylindrical cut is added, producing two cylindrical surfaces which show in the front view as arcs 30

and 33, in the top view as surfaces 21-22-26-25 and 23-24-28-27, and in the side view as surfaces 36-37-40-38 and 41-42-44-43. In addition, normal surface *D* is produced, showing in the front view as surface 29-34-32-31, in the top view as line 21-24, and in the side view as line 35-44.

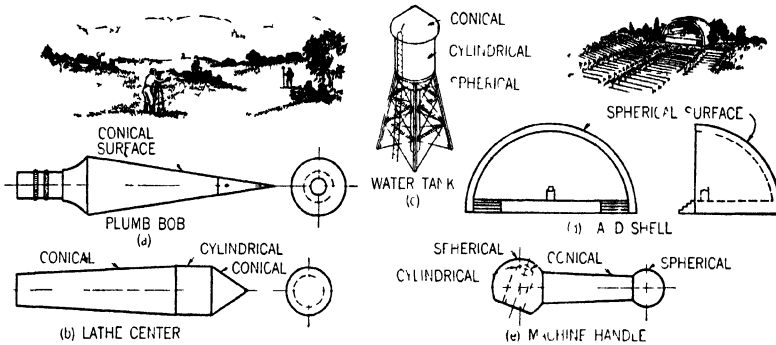


FIG. 117. CURVED SURFACES IN ENGINEERING.

Circles and circular arcs are also produced by cylindrical cuts. *Other curved surfaces* In *space I* of Fig. 116 arcs *E* and *F* (see pictorial), are shown in the top view at 1-2 and 3-4 respectively, in the front view at 5, and in the side view at 8-X and 9-Y respectively. Other common geometric shapes encountered in engineering include the cone and the sphere illustrated in Fig. 117.

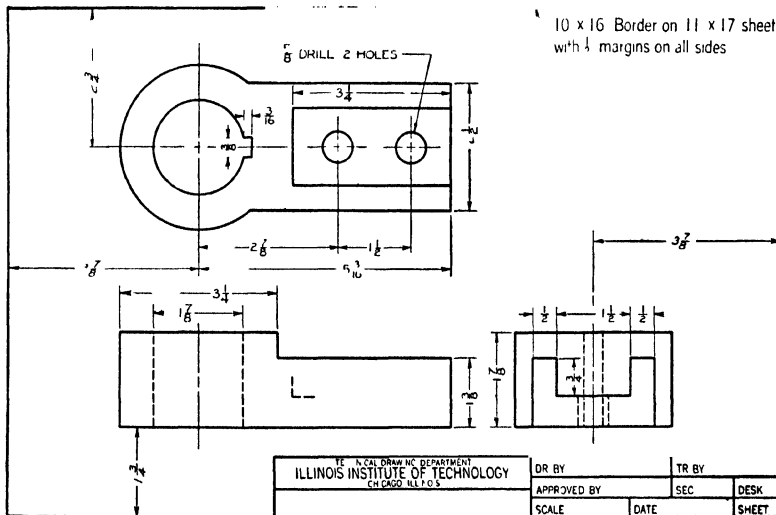


FIG. 118. CONTROL LEVER

Complete the detail drawing, omitting spacing dimensions only.

Problems

In Fig. 118 and Fig. 119 are shown two layouts to be completed. In Fig. 118 add any missing lines needed to complete the views. Omit spacing dimensions (inclined numerals) and the note in the upper right hand corner, but include the dimensions on the views. In Fig. 119 the views are almost completely blocked in with construction lines. Complete this drawing, omitting the pictorial drawing and the spacing dimensions. Be sure to construct all tangent points accurately. (See Fig. 83.) Do not dimension this drawing until you have studied the section, on dimensioning, page 214.

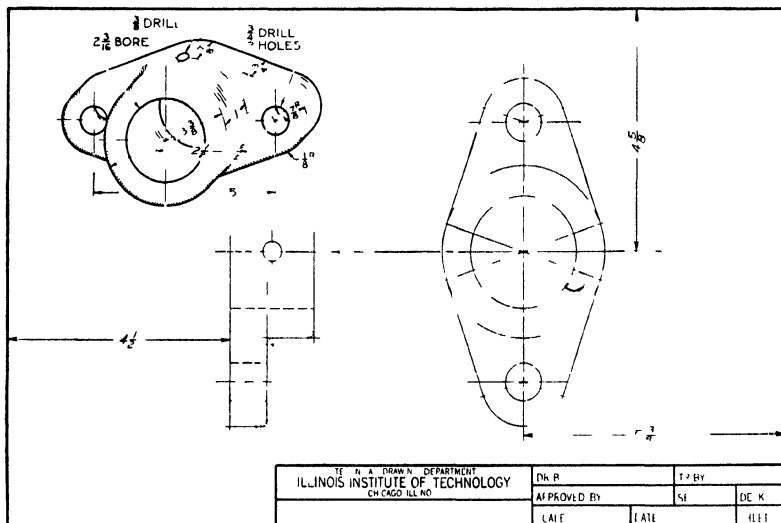


FIG. 119. BEARING.

Complete the detail drawing, omitting the pictorial and the spacing dimensions.

Machine problems

A wide selection of practical machine problems is shown in Figs. 120 to 137 inclusive. For each problem it is best first to make a freehand sketch of the necessary views on cross-section paper. Let each square on the paper represent some dimension so that your sketch will be to scale. Next, obtain your instructor's approval of your choice of views; then make a mechanical drawing, using the same sheet layout as shown in Fig. 118. Do not add dimensions until you have studied dimensioning, page 214.

SECTIONING

Natural sections

Sectioning an object is a procedure with which everyone has had experience, although he may not have realized it. We have all viewed with horror in the carnival side show the sawing of a woman in half. We have bitten into an apple and have thus made a section through a wormhole. We have wondered when purchasing a watermelon how it looks inside, and we then satisfied our curiosity about the interior structure of this luscious fruit by making a *cross section* through the melon with a long-bladed knife.

the front view the cutting plane appears as a line, called a *cutting-plane line*. The arrows indicate the direction of sight for the sectional view. This type of section is a *full section*, so called because the cutting plane passes entirely through the object.

In drawing a full section you will see that the sectioned areas *Section lining* are those portions that have actually been in contact with the cutting plane, and these are crosshatched with very sharp parallel section lines spaced by eye to look well. The lines are spaced from about $\frac{1}{16}$ " apart to about $\frac{1}{8}$ " apart depending upon the size of the drawing, and are usually drawn at 45° with the horizontal.

However, in a sectional view we also show the visible lines *Visible lines* behind the cutting plane, as in Fig. 140. The disconnected sections (Fig. 140 a) must be connected by adding the lines representing the vertical ribs, the rear rims of the holes, and other visible lines

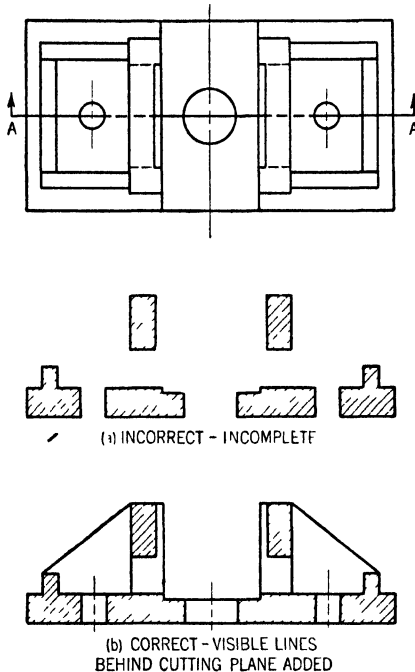


FIG. 140. VISIBLE LINES BEHIND THE CUTTING PLANE

back of the cutting plane as shown at *b*. Note that all section lines run in the same direction on any section or series of sections of a single object.

Sometimes, wishing to retain a portion of the exterior in the

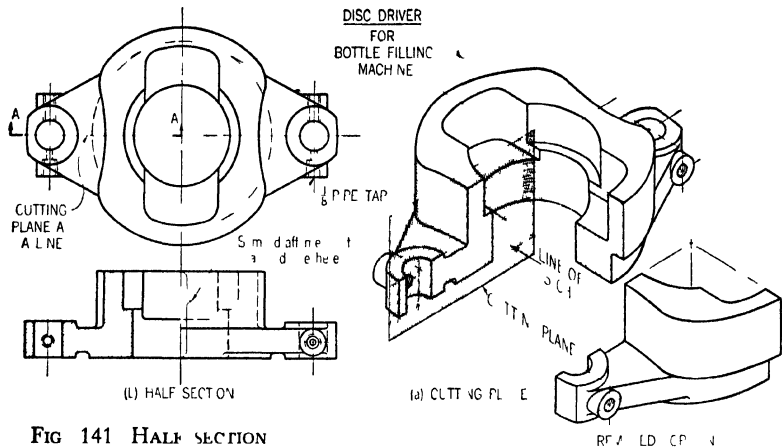


FIG 141 HALF SECTION

Half sections

sectional view, we section only a part of the object. A *half section*, so called because the cutting plane passes only halfway through the object, is shown in Fig 141. Another method of sectioning only

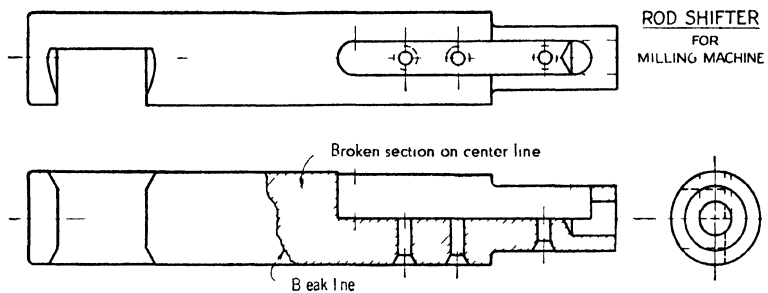


FIG 142 BROKEN SECTION

part of a view is to introduce a *break line* to separate the sectioned area from the rest of the view, this produces a *broken section* (Fig 142)

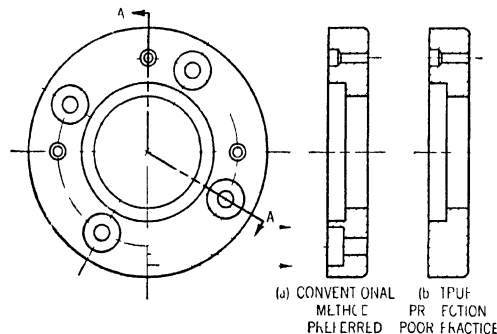
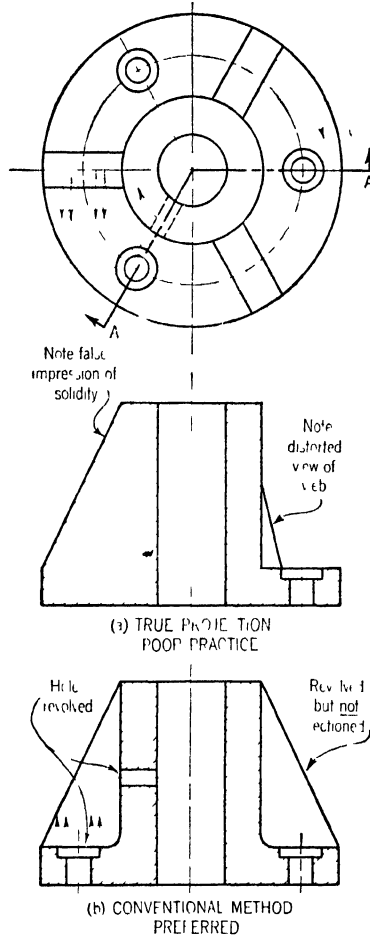


FIG 143 CONVENTIONAL VIOLATION OF PROJECTION

In practice it has become customary to violate the rules of projection in order to obtain a clearer drawing. For example, in Fig. 143 *b* the true sectional view does not give an accurate idea of the symmetry of the arrangement of holes about the center.

Violations of projections



Webs and similar thin parts not sectioned

Revolving features into the section

FIG. 144. WEBS IN SECTION

In order to show one of the large counterbored holes in the section, the hole in the front view is conceived to be revolved into the vertical center line so that it may appear in the section, as shown in Fig. 143 *a*. The cutting plane line makes this clear.

Another common violation of projection relates to webs in section. In Fig. 144 *a* the web on the left, shown in section, gives a false impression of mass. Furthermore, the rear web on the right

Webs in section

is seen obliquely, resulting in a very awkward and misleading representation. The preferred section is shown in Fig. 144 *b*, in

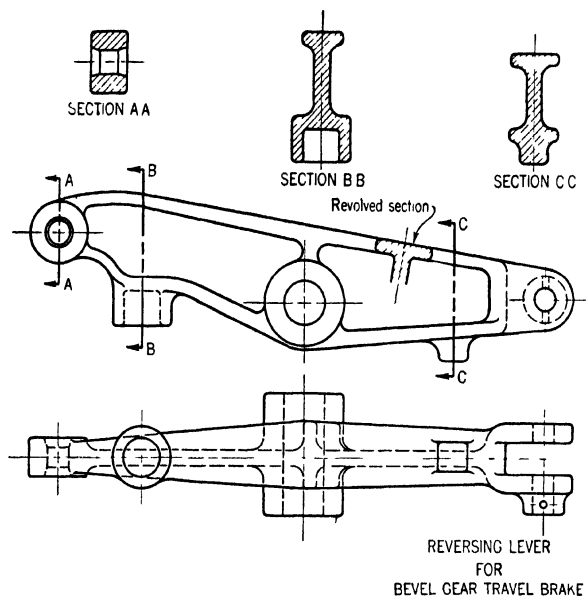


FIG. 145. DETAIL SECTIONS

which the web on the right has been revolved, two holes on the left have been revolved, and neither of the webs is crosshatched.

Representing construction materials

| | | | | | |
|--|---|--|--|--|---------------------------------------|
| | Cast iron. | | Sound or heat insulation. Cork, hair-felt, wool, asbestos, magnesia, packing, etc. | | Marble, slate, glass, porcelain, etc. |
| | Steel. | | Flexible material. Fabric, felt, rubber, etc. | | Earth. |
| | Bronze, brass, copper and compositions. | | Fire brick and refractory material. | | Rock. |
| | White metal, zinc, lead, babbitt and alloys. | | Electric windings, electro magnets, resistance, etc. | | Sand. |
| | Aluminum and aluminum alloys. | | Concrete. | | Water and other liquids. |
| | Electric insulation, Vulcanite, fibre, mica, Bakelite, etc. Show solid for narrow sections. | | Brick or stone masonry. | | Across grain } Wood. With grain } |
| | | | | | |

FIG. 146. AMERICAN STANDARD SECTION LINING

Sometimes it is not possible or convenient to show sections in the regular views. For example, see Fig. 145 where the sections A-A, B-B, and C-C could not be superimposed upon one another in a right side view. Instead, *detail sections* are placed at convenient spaces on the drawing and identified with their several cutting

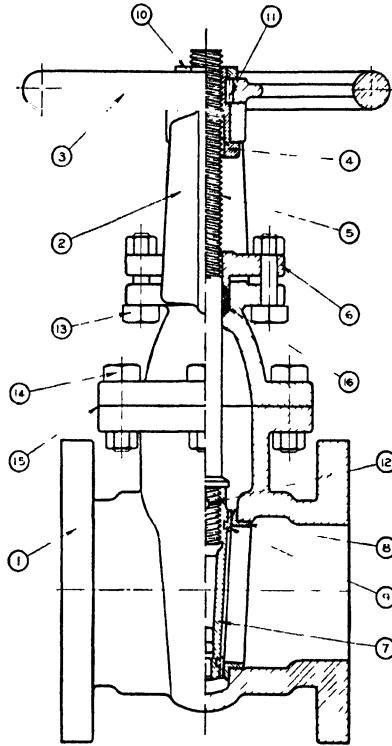


FIG. 147. SECTIONAL VIEW OF GATE VALVE ASSEMBLY — A HALF SECTION

planes by means of notes. Another method, little used in practice, is to revolve the section in place, as shown near the right-hand end of the lever in Fig. 145. It will be observed that none of the detail sections shown could have been *revolved sections* without covering important lines of the drawing.

Often on detail drawings (drawings of single pieces) the section lining shown on the preceding illustrations, which is the standard symbol for cast iron, is used regardless of the actual material. Material specifications are usually given in a more detailed manner by means of notes. However, if desired, the general type of material can be indicated by different symbols, as shown in Fig. 146.

*Assembly
sections*

A half section in an assembly drawing is shown in Fig. 147. Notice that section lines for different adjacent parts run in opposite directions, and that the different symbols for the several kinds of material are shown. Some violations of projection are

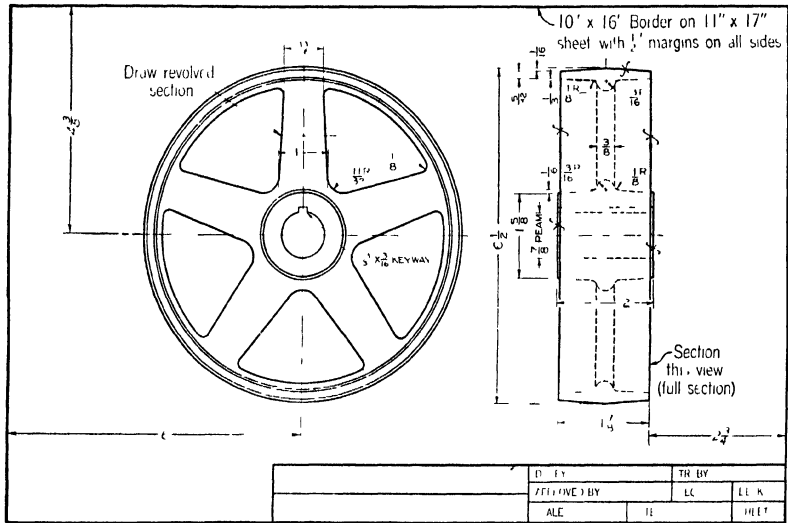


FIG. 148. PULLEY

Complete the drawing, omitting spacing dimensions and instructional notes.

illustrated with respect to the bolts, nuts, pins, threaded shaft, and similar forms which are not sectioned. These parts have no interior details worth showing and are therefore left in exterior view.

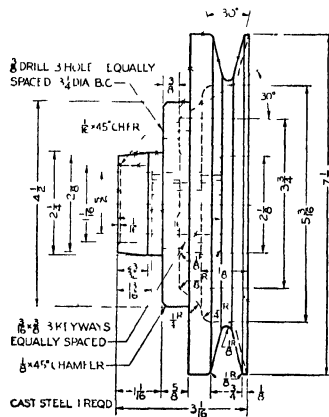


FIG. 149. SHEAVE

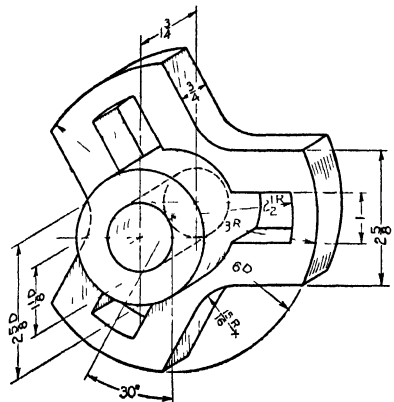


FIG. 150. ROTARY STOP

In Fig. 148 is given a practical problem in which the sheet layout of Fig. 118 is used again. Draw the views as indicated, but omit all spacing dimensions (inclined numerals) and additional notes. Add dimensions to the views only after you have studied the section on dimensioning. See p. 214.

The problems in Figs. 149–156 inclusive are selected to include a variety of sections. Choose the necessary views and sections needed and make a sketch of these on cross-section paper. Then proceed to make mechanical drawings on

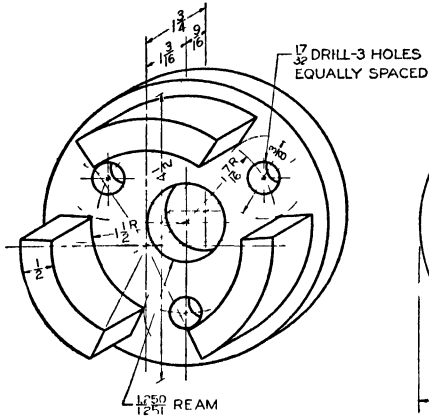


FIG. 151. CUP WASHER

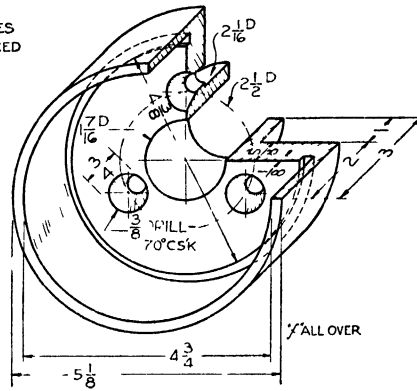


FIG. 152. SPECIAL BEARING

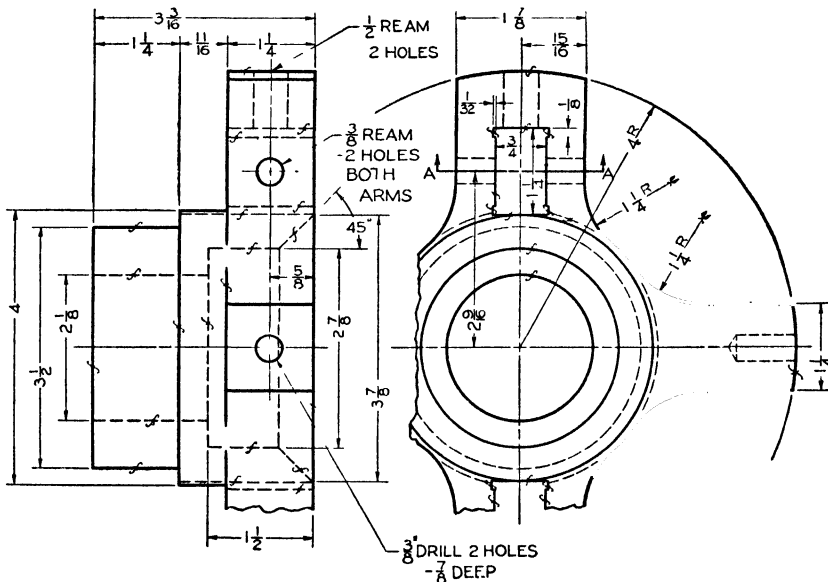


FIG. 153. WEB FOR LATHE CLUTCH

Given: Partial front and left side views. Required: Complete front view and right side view in full section, and draw detail section A-A.

detail paper or vellum, using the sheet layout of Fig. 148. Do not dimension the drawing until you have thoroughly studied dimensioning.

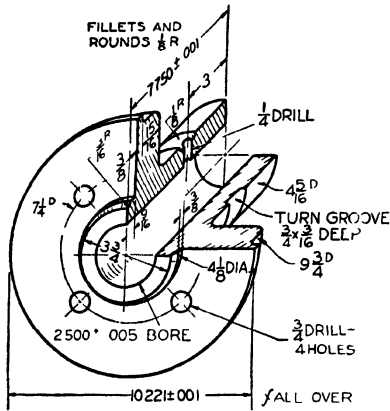


FIG. 154. FLANGED BLARING

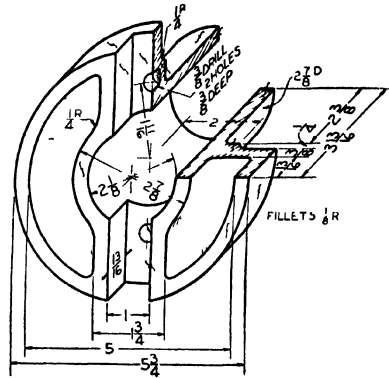


FIG. 155 STOCK GUIDE

AUXILIARY VIEWS AND REVOLUTION

Regular views

Up to this point we have considered the various *regular* views of an object of essentially rectangular shape. Usually two or three such views are sufficient to explain clearly the shape of the object, but occasionally this is not the case.

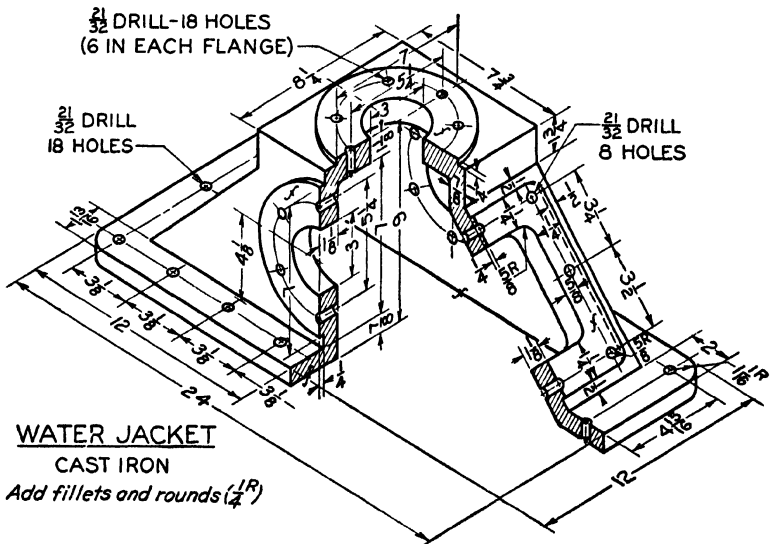


FIG. 156. WATER JACKET

For example, in Fig. 157 a clear description of the shape of the semicircular face is not shown in the regular views. In both the front and side views the circular arc appears as an ellipse because it is clear that the plane of the surface is not parallel to a plane of projection. In order to obtain a true view of such a surface it is necessary to view the object in a different direction, and this means that we must set up a different plane of projection — one which will be parallel to the inclined surface. A view thus obtained is called an *auxiliary view*.

If we imagine an auxiliary plane in our glass box (Fig. 158 *a*) which is parallel to the semicircular face of the object, such a plane will be perpendicular to the top plane of projection. Then if we revolve the auxiliary plane about a “hinge line” (line of intersection of auxiliary plane and top plane) until it coincides with the top plane, we will have two adjacent views of the object (*b*) obtained in a similar manner to the regular views. Observe that the true height of the object appears in the auxiliary view.

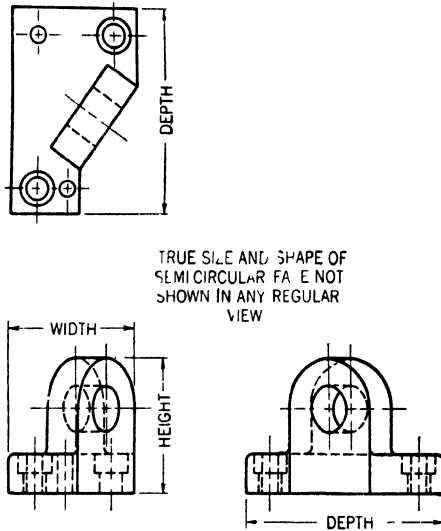
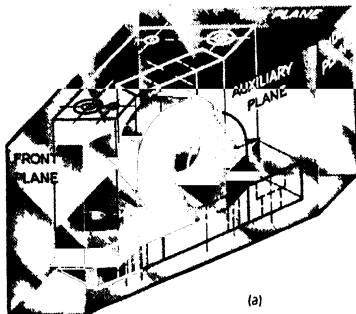
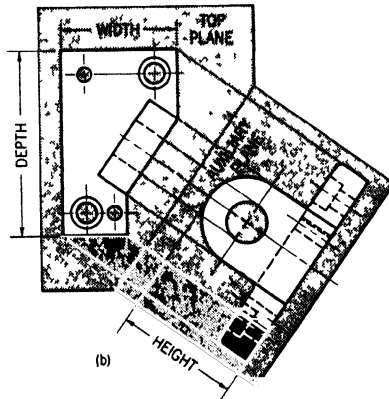


FIG. 157. INADEQUACY OF REGULAR VIEWS



(a)



(b)

FIG. 158. THE AUXILIARY PLANE

Projection

We can now add this auxiliary view to our drawing as shown in Fig. 159. The length L in the auxiliary view is obtained by direct projection from the top view as indicated by the arrows.

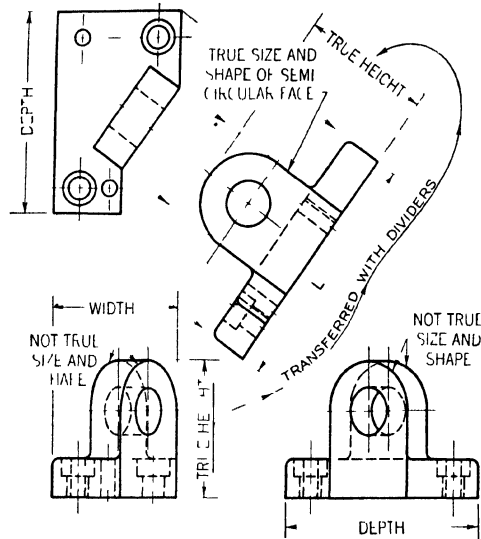


FIG. 159 AUXILIARY VIEWS ADDED

Any auxiliary view is projected from some other view, and one dimension of an auxiliary view can always be established thus by direct projection. The other dimension in the auxiliary view

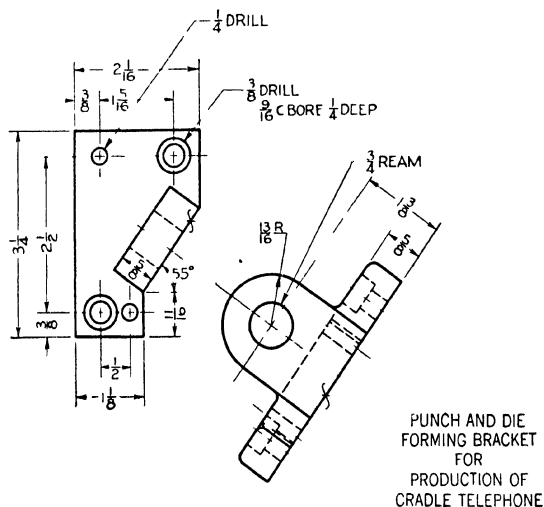


FIG. 160 MINIMUM REQUIRED VIEWS

*Saving time
and effort*

(height) was taken from the front view, but it might have been taken from any view which had been *projected from the top view*, the latter being the view we projected from to get our auxiliary view.

Your next question may well be: "If the front and side views are not useful in describing the object, why keep them?" In this you would be correct; these views now serve no purpose and should be omitted, as in Fig. 160. We now have a drawing which is complete because the three major dimensions *width*, *height*, and *depth* are shown, and because the shape of the object is completely described in these two views. That we have a complete drawing is proved by the fact that we can now place all dimensions on the views, and the object can then be built in the shop.

Most people seem to try to make something difficult out of *revolution*. As a matter of fact it is very simple and is another step

in our argument concerning auxiliary views. In the case of auxiliary views our given object has some feature which we do not see in its true size and shape through one of the regular planes of the glass box; so we merely set up an auxiliary plane parallel to the important feature and obtain a true view of that part. In revolution, instead of changing your position to suit the object, you revolve the object to suit your position. In either case you must look perpendicularly toward the surface in question. What difference does it make whether you move around to view this face perpendicularly, or you move the object around until you obtain the same view? In Fig. 161 the top view is simply redrawn in the "revolved" position so that the front view will show the true circular shape. Notice that the dimension (in this case — *height*) parallel to the imaginary axis of revolution is not changed.

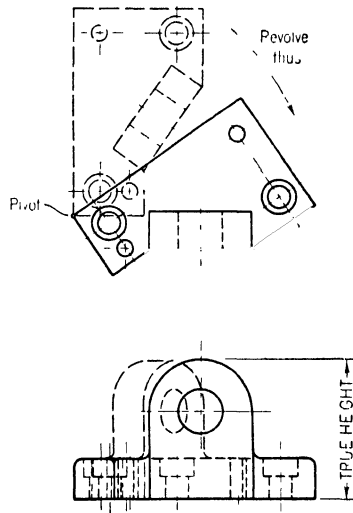


FIG. 161. REVOLUTION

Two typical auxiliary-view problems are given in Figs. 162 and 163. These can be drawn conveniently on a size 11" × 17" sheet as shown in Fig. 148. If you wish to dimension these problems, you should first study the section that follows on dimensioning.

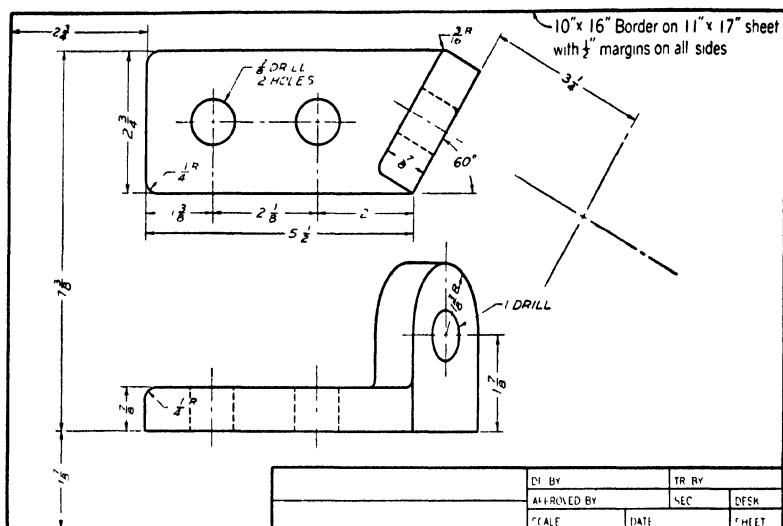


FIG. 162. ANGLE BRACKET

Complete the detail drawing, omitting spacing dimensions and note.

DIMENSIONING

Shape and size description

If a drawing is to be complete so that the object may be made from it exactly as intended by the designer, it must show not only a complete *shape description* but also a complete *size description*. It is the designing engineer's job to decide upon the shape of a new part, the material it is to be made of, and the kinds of fits involved

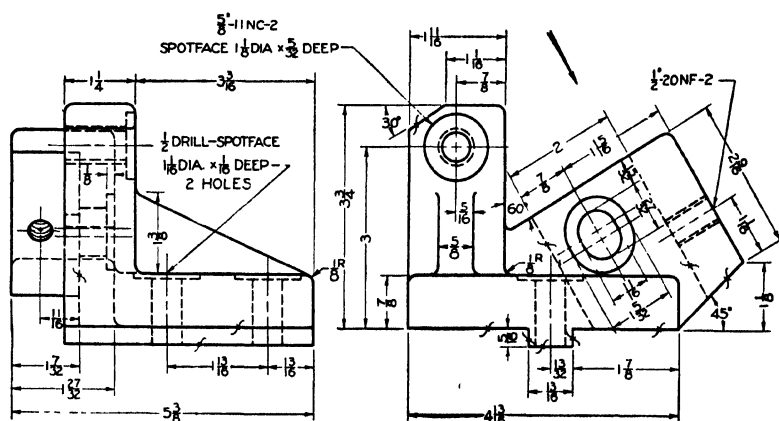


FIG. 163. SUPPORT BRACKET

Given: Front and left side views. **Required:** Front and right side views, and auxiliary view in direction of arrow.

with mating parts. The draftsman's job is to make the drawing so that it conveys this information unmistakably to the shop. The workman is not expected to exercise judgment in engineering matters, but only to carry out instructions on the drawings.

The shops are equipped with precision machine tools and measuring devices, and the draftsman can specify extreme degrees of accuracy merely by lettering the desired numbers on the drawing. Since this affects both cost and function, it is obvious that a great

Accuracy

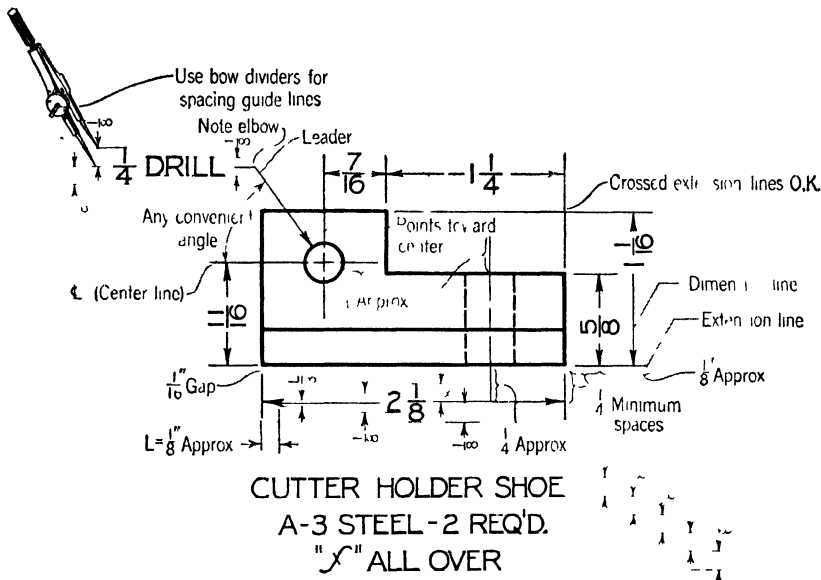


FIG. 164. APPLICATION OF DIMENSIONS

deal of thought must be given by the draftsman to dimensioning a drawing. Further, the dimensions must be such as to conform to the equipment in the shop and to the logical use of it by the workman. The draftsman, therefore, must be thoroughly familiar with the shop processes involved in the part to be dimensioned.

The terminology and the spacing of dimensions are illustrated in Fig. 164. Rigid uniformity and care in spacing should be exercised to insure a pleasing and legible drawing. It is well to measure spacings at first and to use guide lines for all lettering, as shown. Later, dimensions may be lettered without guide lines, but notes should never be lettered without them. For the average dimension sufficient space is available (Fig. 165), but frequently limited spaces necessitate adjustments, as shown.

Terminology and spacing

Arrowheads may be drawn with two strokes (Fig. 166 *a*) or filled in as shown at *b*, with preference for the latter.

Legibility of dimension figures requires special methods of spacing when dimensions are crowded

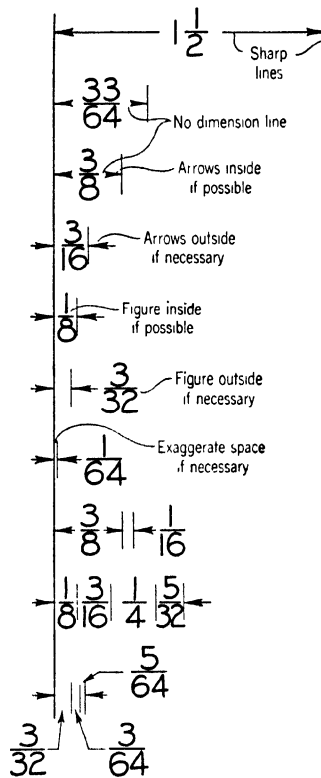


FIG. 165. DIMENSIONS IN LIMITED SPACES

Angles and radii

Angles are dimensioned in degrees, lettered horizontally, except for large arcs (Fig. 167). Radii are always designated by the letter R, as shown in Fig. 168, and small arcs require adjustments to assure clear, uncrowded numerals, as shown.

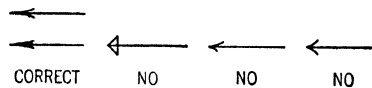
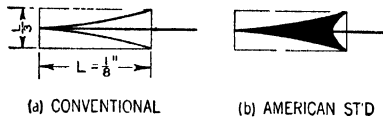


FIG. 166. ARROWHEADS

A surface on a casting or forging is said to be *rough* or *finished*, depending upon whether or not it has been machined (see Fig. 169).

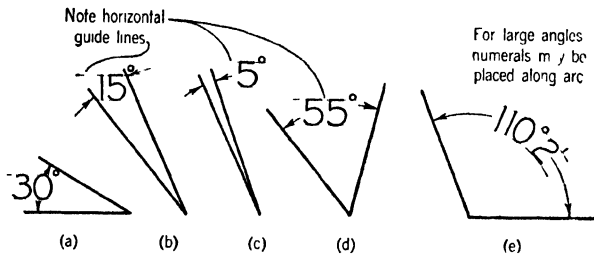


FIG. 167. ANGLES

If two rough surfaces intersect, the edge should be rounded, producing for an exterior edge a *round* and for an interior edge a

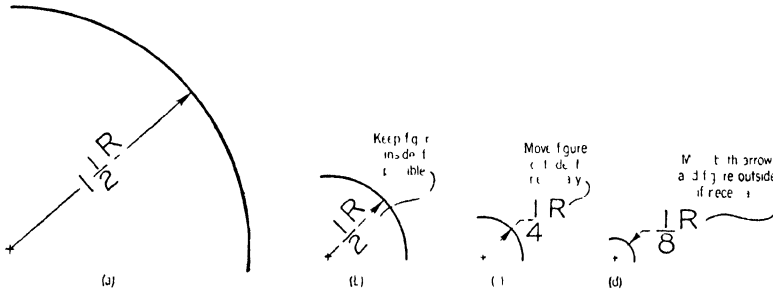


FIG. 168 RADII

fillet. These are intended by the designer to add strength to the edge or corner and to facilitate casting or forging. If either or both of two intersecting surfaces are finished, the edge will be sharp *Fillets and rounds*

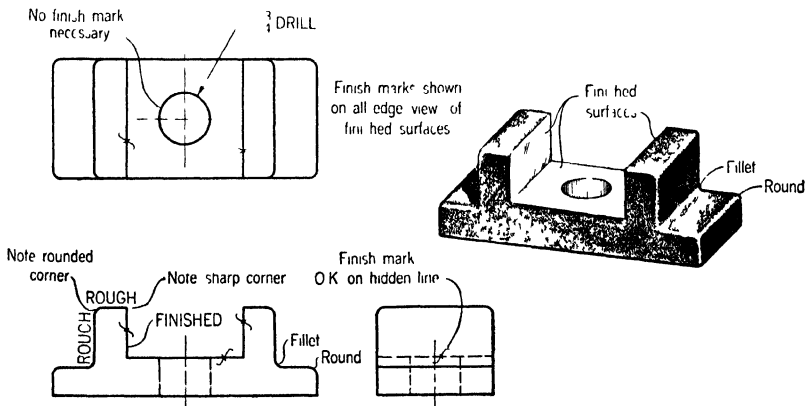


FIG. 169. FINISH MARKS

unless otherwise designated; if both are rough, the edge will be rounded or filleted.

Finish mark

The italic *f*, placed only on the edge views of finished surfaces (Fig. 169), is still the most nearly standard finish mark used in industry, despite all efforts to bring about an agreement for some form approaching the capital *V*.

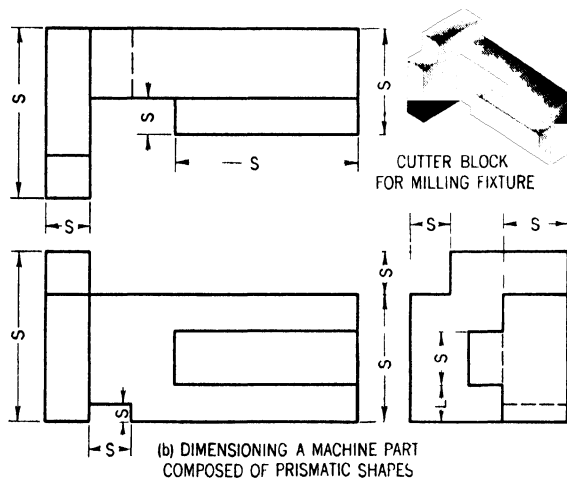
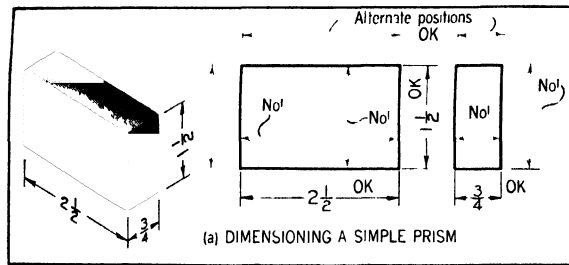


FIG. 170. DIMENSIONING PRISMS

Geometric shapes

Any machine or engineering structure, when broken down into its elemental forms, is found to be composed largely of geometric shapes, such as the prism, cylinder, pyramid, cone, and sphere. In dimensioning a drawing we must first indicate the sizes of the elements with *size dimensions*; second, we must locate these elements with respect to each other with *location dimensions*.

Size dimensions

Three dimensions, for *width*, *height*, and *depth*, are necessary in dimensioning a prism (Fig. 170 a). An application to a machine part made up of prismatic shapes is shown at b. In general a dimension should be placed outside or between the views. Dimen-

Dimension lines cutting across views are objectionable. Dimension numerals are lettered on guide lines parallel to the dimension lines and placed so that they are read either from the bottom or the right-hand side of the sheet.

Reading dimensions

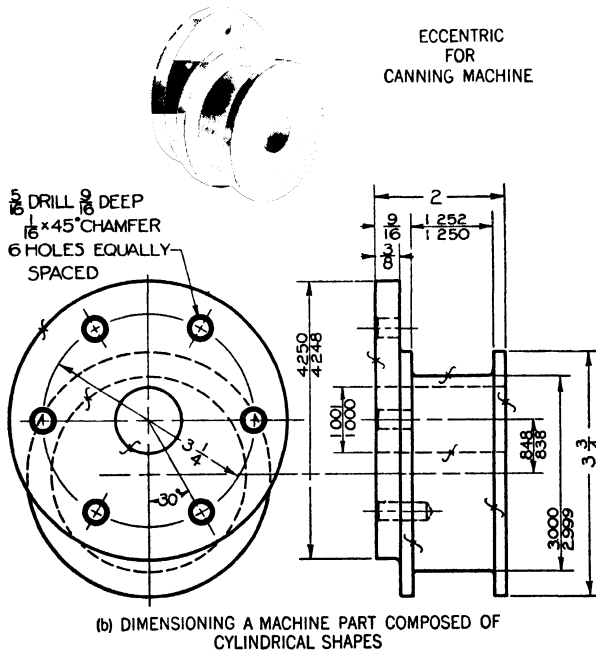
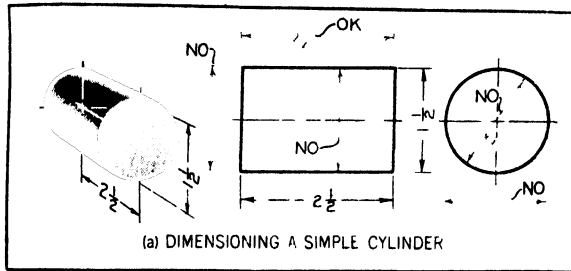


FIG. 171. DIMENSIONING CYLINDERS

The size-dimensioning of a cylinder and its application on a machine part are shown in Fig. 171. Cylindrical shapes are generally dimensioned with both the length and diameter in the rectangular view, as shown. Machined holes are usually designated by notes which specify the diameter and the operation required. Several of the most common types are illustrated in Fig. 172.

Machined holes

A cone is dimensioned by giving its altitude and diameter of base in the triangular view. A sphere is dimensioned by giving its diameter between two views.

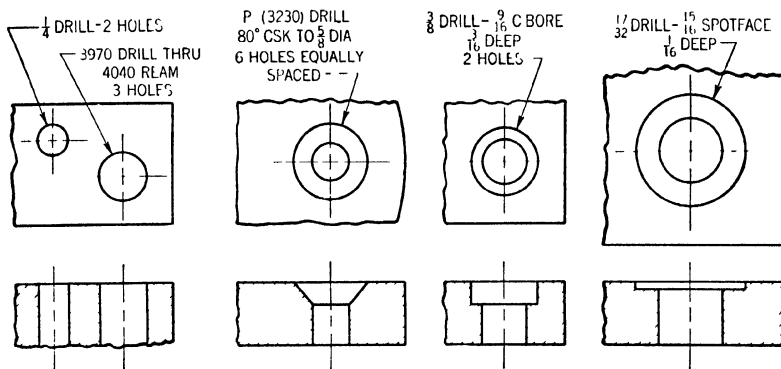
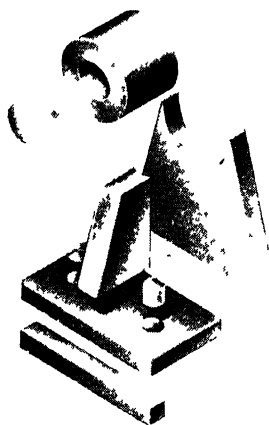


FIG. 172. SHOP NOTES FOR MACHINED HOLES

Location dimensions

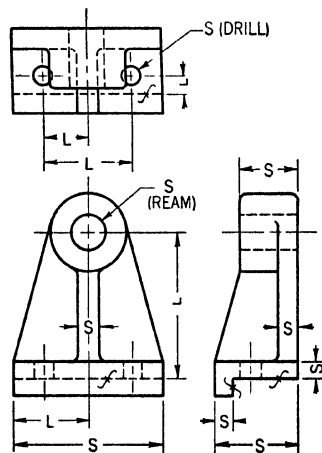
A practical application of location and size dimensions on a drawing is shown in Fig. 173. Notice that all cylinders are located from their center lines and with respect to finished surfaces. Observe also that dimensions are attached to the view which most clearly identifies the form dimensioned; this is called *contour dimensioning*. Location dimensions for holes are accordingly given on the view where the holes show as circles. Dimensions should never be repeated on a drawing.



(a) THE GEOMETRICAL ELEMENTS



(b) BEARING BRACKET



(c) SIZE & LOCATION DIMENSIONS

FIG. 173. SIZE (S) AND LOCATION (L) DIMENSIONS

A machine part cannot be intelligently dimensioned without consideration of its relation to mating parts. In Fig. 174 the dimension *A* must appear on both the drawings of the *bracket* and of the *frame*, and is therefore an important *mating dimension*.

In Fig. 175, showing a redesign of this part, dimension *A* is not used as it is not necessary to control closely the distance between cap screws; but dimensions *F* are now important mating dimensions and must appear in both drawings. Other dimensions *E*, *D*, *B*, and *C* are not mating dimensions, since they do not directly affect the matching of parts (Figs. 174 and 175).

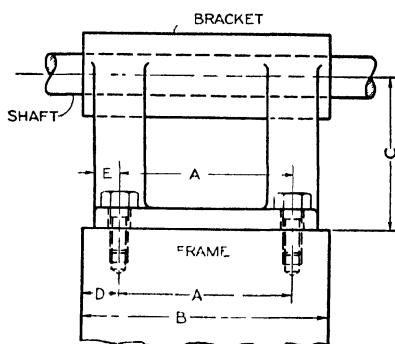


FIG. 174. BRACKET ASSEMBLY — SINGLE BRACKET

Where accurate fits are required, the old practice was to give the basic or nominal sizes of both mating parts in common whole numbers and fractions, and to specify in a note the types of *fits* desired, such as “running fit” or “drive fit.” The workman in the shop was relied upon to make the parts so that they would fit properly. Today the workman is no longer expected to decide upon fits of mating parts. Mass production has introduced *interchangeable manufacturing* by which mating parts may be made in different

Fits and interchangeability

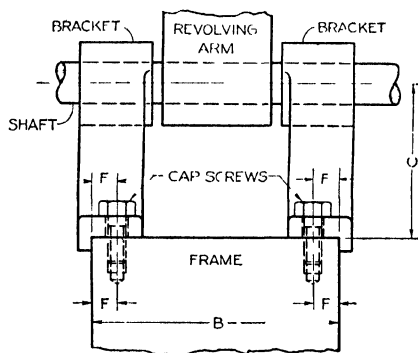


FIG. 175. BRACKET ASSEMBLY — DOUBLE BRACKET

factories and possibly assembled in a third factory. Thus it is necessary on drawings of closely fitting parts to specify sizes in such a manner that machine operators in widely separated shops can produce parts which are interchangeable.

No formal problems in dimensioning are included here because you can apply the principles directly to the problems given at the end of the sections on shape description, page 198, and working drawings, page 243.

Problems

PICTORIAL DRAWING

The four kinds of projection are illustrated in Fig. 176, and all except the regular multiview projection (a) are pictorial types since they show several sides of the object at once.

Axonometric projection

In axonometric projection you consider yourself (as the observer) to be an infinite distance away from the object so that the visual rays from your eye to the object are theoretically parallel to each other and perpendicular to the plane of projection (b). The object may be tilted at any position with respect to the plane of projection, and any of these projections would be *axonometric projections*, provided the visual rays are kept perpendicular to the plane of projection.

Isometric projection

If the object happens to be placed so that lines OA , OB , and OC make equal angles with the plane of projection (Fig. 176 b), the

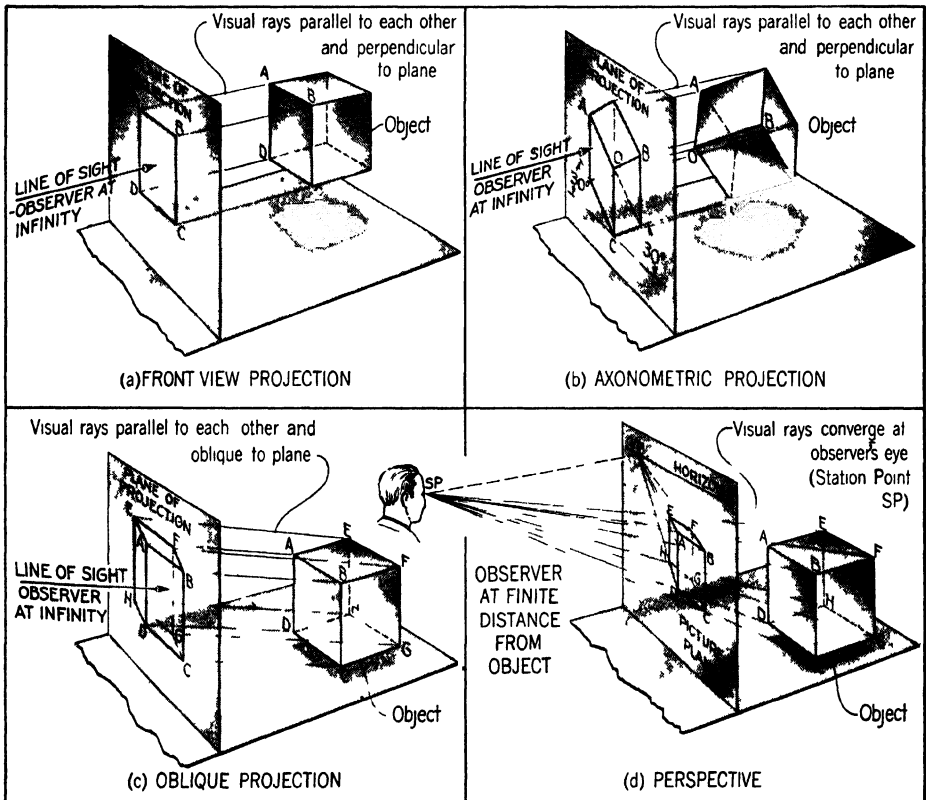


FIG 176 THE FOUR TYPES OF PROJECTION

result is an *isometric projection*; each of the lines is projected automatically to about 80 per cent of its actual length. An isometric projection may be obtained as in Fig. 177 *b* by revolving the object and then taking an auxiliary view. To accomplish this, the cube is revolved from its ordinary position in the top view (dotted)

Revolving the object

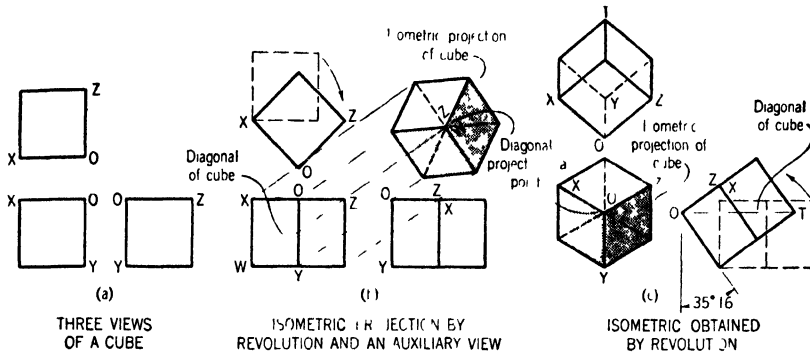


FIG. 177. ISOMETRIC PROJECTION

to the position shown. In this position the diagonal ZW through the cube would appear true length in the front view. An auxiliary view is then taken in which the observer looks in a direction parallel to the diagonal so that the diagonal is seen as a point in the auxiliary view. The cube then appears in true isometric in this auxiliary view. We have obtained an isometric projection by a combination of moving the object and moving the observer so that the two are properly related to produce an isometric projection.

Instead of making the auxiliary view shown in Fig. 177 *b*, you could further revolve the object as shown in the side view at *c*. Here in the new side view position, the diagonal OT is horizontal and is shown in true length. The front view must show the diagonal as a point, and the front view is therefore a true isometric projection.

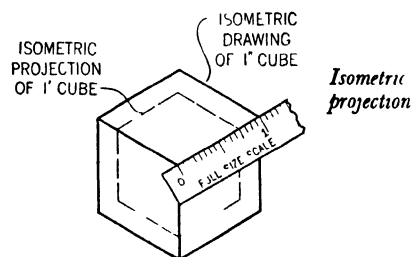


FIG. 178. COMPARISON OF ISOMETRIC DRAWING AND ISOMETRIC PROJECTION

Isometric projection

To draw an *isometric projection* directly on paper, a special scale would have to be prepared to reduce the lengths of all lines about 20 per cent. However, in practice, the draftsman simply ignores this reduction, drawing all lines full length; and the result is called an *isometric drawing*, to distinguish it from the "true" *isometric projection*.

Isometric drawing

jection, which is about 20 per cent smaller. This difference between the "true," or theoretical, *projection* and the full-size isometric *drawing* is clearly shown in Fig. 178.

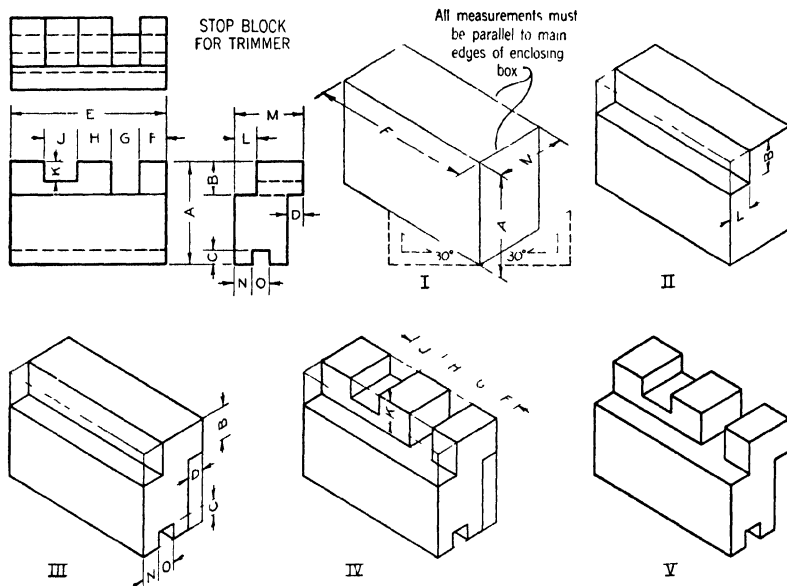


FIG. 179. ISOMETRIC DRAWING OF NORMAL SURFACES

The steps in constructing an isometric drawing of an object composed only of normal surfaces (see page 192) are illustrated in Fig. 179. Notice that all measurements are and must be made parallel to the main edges of the enclosing box. No measurement of a diagonal on any surface or through the object can be set off directly with the scale.

The method of constructing an isometric drawing of an object composed partly of inclined surfaces (and oblique edges) is shown

*Inclined
surfaces*

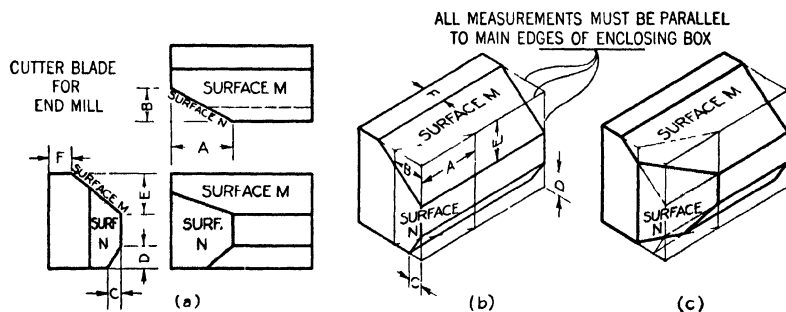


FIG. 180. INCLINED SURFACES IN ISOMETRIC

in Fig. 180. Notice that inclined surfaces are located by *offset measurements* along the main edges of the enclosing box. For example, dimensions *E* and *F* are set off to locate the inclined surface *M*, and dimensions *A* and *B* are used to fix the position of the surface *N*.

A circle in isometric is an ellipse, due to the fact that the observer *Circles in isometric* is not looking perpendicularly toward the plane containing the

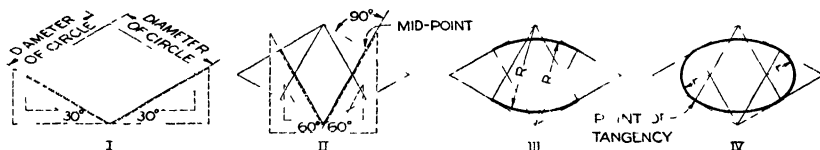


FIG. 181. STEPS IN DRAWING FOUR-CENTER OVAL

circle. The construction of the true projection of the circle is relatively difficult and time-consuming; therefore it is seldom used. The accepted approximate construction of the isometric ellipse is that of the *four-center oval* (Fig. 181). A typical example of the isometric ellipse is shown in Fig. 182. Notice that the centers for the large ellipses cannot be used for the smaller ellipses though the ellipses are concentric, and that the centers for the lower ellipses are obtained by projecting the upper centers down a distance *C* for each center.

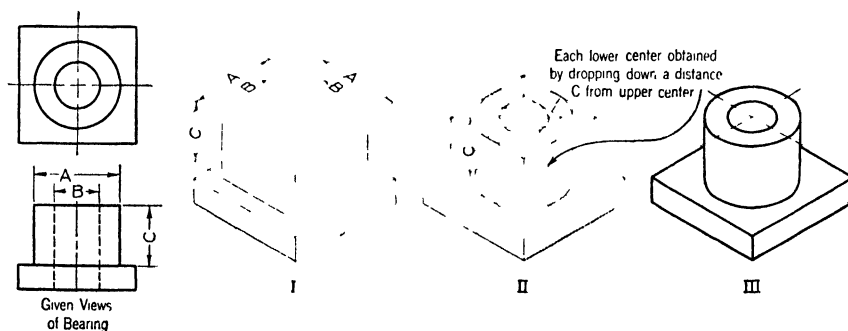


FIG. 182. ISOMETRIC DRAWING OF A BEARING

OBLIQUE PROJECTION

In oblique projection (Fig. 176 *c*) the observer is stationed at *Oblique projection* infinity, so that the visual rays are parallel to each other but oblique to the plane of projection. All surfaces of the object which are parallel to the plane of projection will be projected in their true

*Steps in
construction*

sizes and shapes on that plane. Thus the front surfaces $A-B-C-D$ will be projected to true size and shape. The chief advantage of oblique projection over isometric projection is that any object having circular arcs can easily be drawn if the object is placed so that these arcs are parallel to the plane of projection (Fig. 183). The steps in constructing an oblique drawing are illustrated in this figure. *The receding lines may be drawn at any angle, but 30 degrees or 45 degrees is usually used for convenience.*

Not only may you draw the receding lines at any angle with the horizontal but they may theoretically be drawn any length as well. Usually the receding distances are drawn full size, as it is more convenient to make the entire drawing to agree with the actual dimensions of the given object.

Note in Fig. 183 that the position of the object is such that all circles and circular arcs are in planes parallel to the front plane

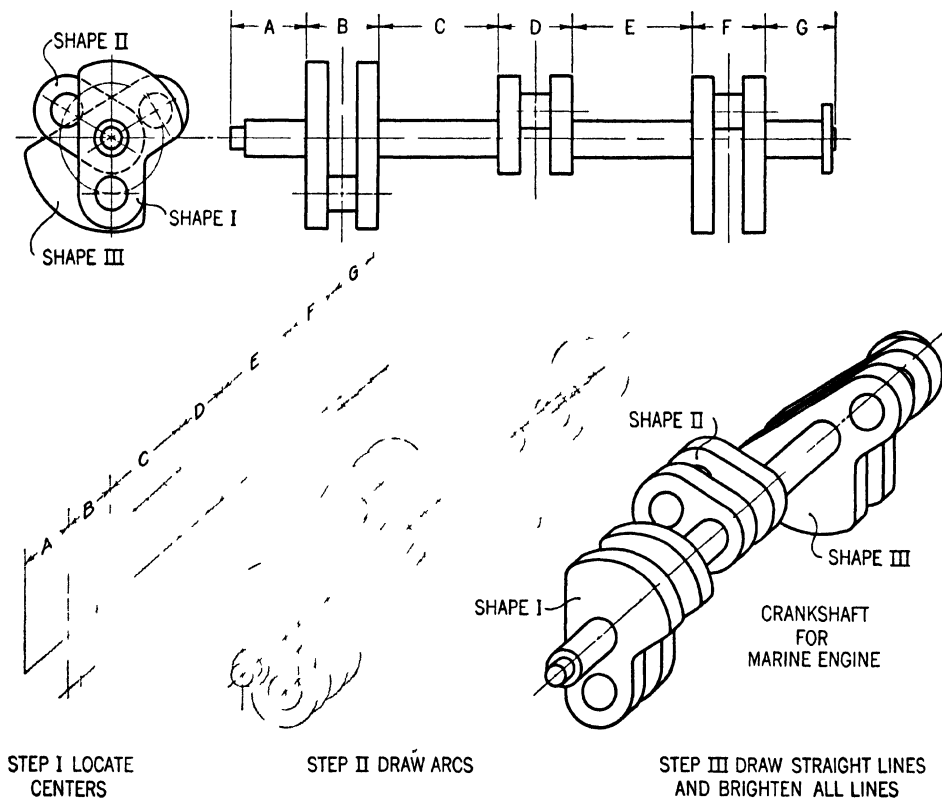


FIG. 183. OBLIQUE PROJECTION OF A CRANKSHAFT

of projection. This is purposely so in order that such curves could be drawn directly with the compass. If circular curves lie in receding planes of the object, they will appear as ellipses, and the construction would be made much more difficult and the result less pleasing. Notice that this drawing is built upon an important center line and that all receding measurements are set off along this line. *Circles in oblique*

PERSPECTIVE

In *perspective projection* the observer is relatively near the object being viewed, this position resulting in a convergence of the visual rays from all points on the object toward the observer's eye or *SP* (Station Point). See Fig. 176 *d*.

In Fig. 176 *d* the perspective is the picture on the *picture plane* (*PP*), formed by all the piercing points of the visual rays from the object to the eye, or station point (*SP*). The problem to be solved is how to *construct* the perspective from the ordinary top and side views of the object. In Fig. 184, we have the top and right side views of the object, of *PP*, and *SP*, and the perspective is shown in the center where the front view is usually placed. In the top and side views we have the edge views of *PP* and can see where the visual rays pierce *PP*. In the perspective, points *A-B-C*, and so on, are found by projecting down from these piercing points in the top view and across from the corresponding piercing points in the

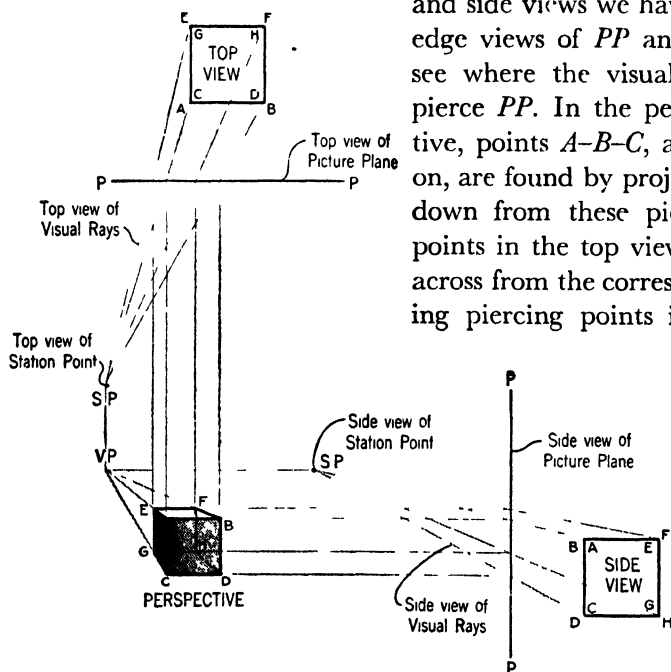


FIG. 184. PERSPECTIVE PROJECTION

side view — for example (top view) the piercing point of visual ray $SP-B$ is projected down to B in the perspective; and (side view) the piercing point of $SP-B$ is projected across to B in the perspective. All other points are projected in the same manner.

Notice that the receding lines $A-E$, $B-F$, $D-H$, and $C-G$ of the perspective meet at a vanishing point (VP), and that VP lines up with the top and side views of SP . Such a perspective, having only one vanishing point, is called a *one-point perspective*.

Two-point perspective

If, on the other hand, the top view is turned at an angle to PP (Fig. 185), there would be two vanishing points, VPL and VPR . No side view of PP , SP , and the visual rays is necessary or convenient, but the side view of the object is needed to obtain needed *height measurements*. Here, much depends upon the vanishing points, of which there are two in this case; hence, this is a *two-point perspective*.

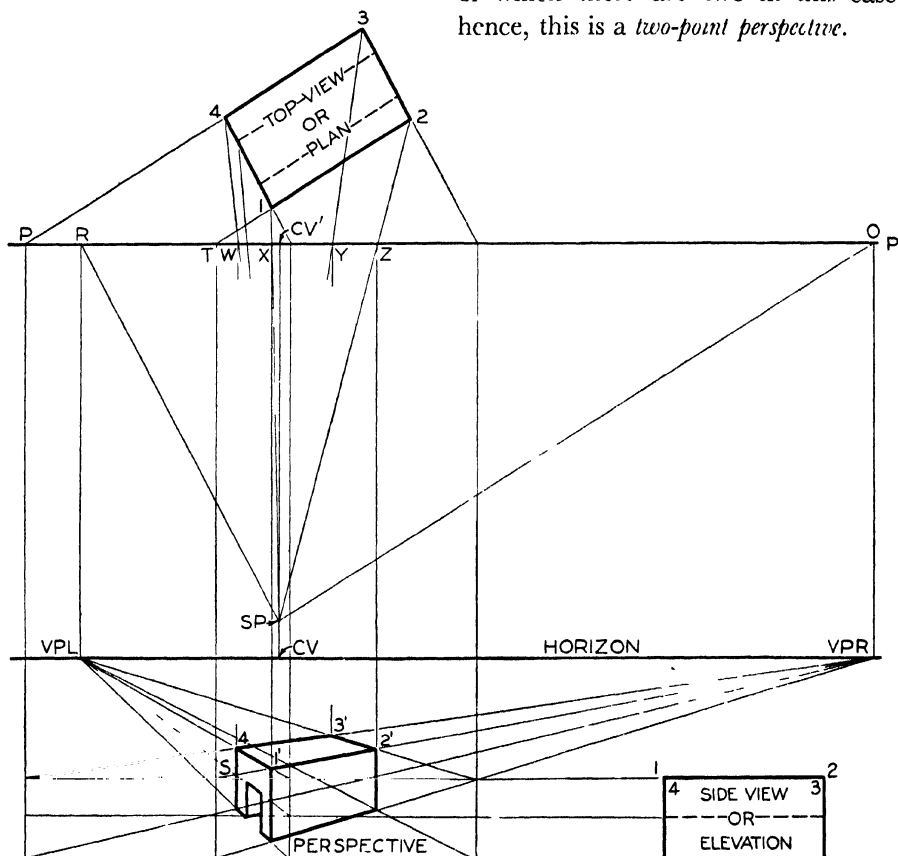


FIG. 185. PERSPECTIVE OF A SINGLE OBJECT

The top portion of the drawing in Fig. 185 (above the *horizon*) is a regular top view of the object, picture plane, and the observer as in Fig. 184. To draw the perspective you must simply learn how to draw the perspective of any line, and repeat this procedure until the perspective drawing is completed. *The setup*

In Fig. 185, to find the perspective of line 1-2 (top view), draw a line from *SP* parallel to 1-2, striking the picture plane *PP* at *O*. Then project downward to the horizon from *O* to obtain the vanishing point *VPR* of line 1-2 (and all lines of the object which are parallel to 1-2). Next (top view) extend the line 1-2 to the picture plane at *T* and project downward from *T* to *S*; then *S* is the *piercing point* of the line 1-2 in the picture plane, the height of *S* being projected across from 1-2 in the side view. *Steps in drawing perspective*

Then join *S* to *VPR* with a straight line. The visual rays (top view) from *SP* to the points 1 and 2 intersect the picture plane at *X* and *Y*. Project downward from these points to the line *S-VPR* to obtain the ends 1' and 2' of the perspective of the line 1-2. Follow this same procedure with all other lines of the object to complete the perspective. Always follow this procedure in finding the perspective of a given line:

1. Find the vanishing point *VP* of the line.
2. Find the piercing point of line in the picture plane *PP*.
3. Connect these two points with a straight line.
4. Find the end points of the perspective of line 1-2 by projecting downward from the piercing points of visual rays with the picture plane.

Principles

Remember the following:

1. Vanishing points of all horizontal lines are on the horizon.
2. Parallel lines have the same vanishing point.
3. Lines which are parallel to the picture plane have no vanishing points.

A perspective is geometrically the same as a photograph; and except for the fact that the retina of the eye is spherical, instead of flat like a sheet of drawing paper, it corresponds exactly to the impression obtained by the eye. By means of perspective the most complicated objects can be represented accurately and naturally. Architects use perspective extensively in the process of design to study the appearance of buildings before they are built. They also prepare perspectives to show clients how the building will look upon completion. As a matter of fact, perspectives are used in all kinds of illustration where natural appearance is desired.

*Production
illustrations*

Modern industry, particularly aircraft manufacturing, has found a new and valuable use for perspective. Because we find that many thousands of workers cannot "read" blueprints, especially the complex drawings of airplanes, perspectives have come into extensive use to supplement the blueprints by providing the extra visualization needed for a clear understanding. It was soon found that engineers, executives, and all other persons connected with the production of these intricate mechanisms came to regard these perspectives as a great help in clearly visualizing the separate parts and the assemblies thereof.

These drawings are known as "production illustrations," and are defined roughly as pictorial drawings used in the shop or the production line to show clearly how all the parts are fabricated and assembled. A pen-and-ink perspective drawing of a complete airplane showing the details of construction is shown in Fig. 186. Obviously a considerably more extensive study of perspective is necessary to produce a drawing of this character than can be provided in these pages.

A more natural gradation in rendering light and shade on a perspective is afforded through the use of the *air-brush*. An excellent example of this type of drawing is shown in Fig. 187. By means of this ingenious tool the paint is sprayed on the drawing while certain parts are "masked," or covered, to protect them from the spray.

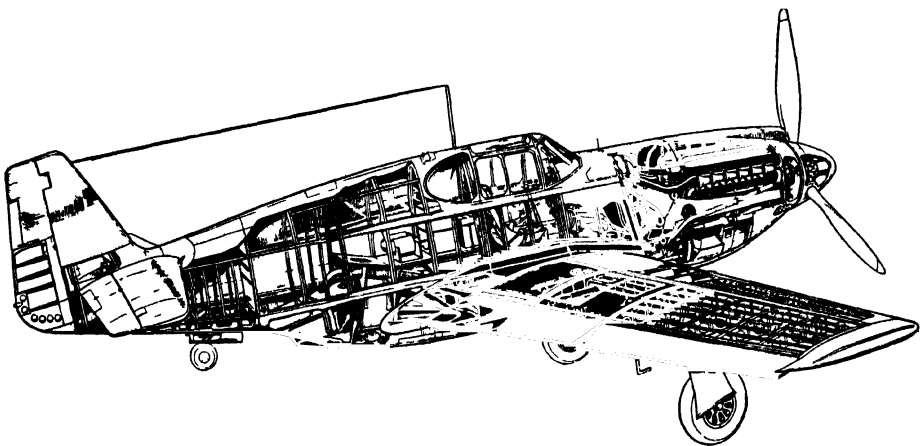
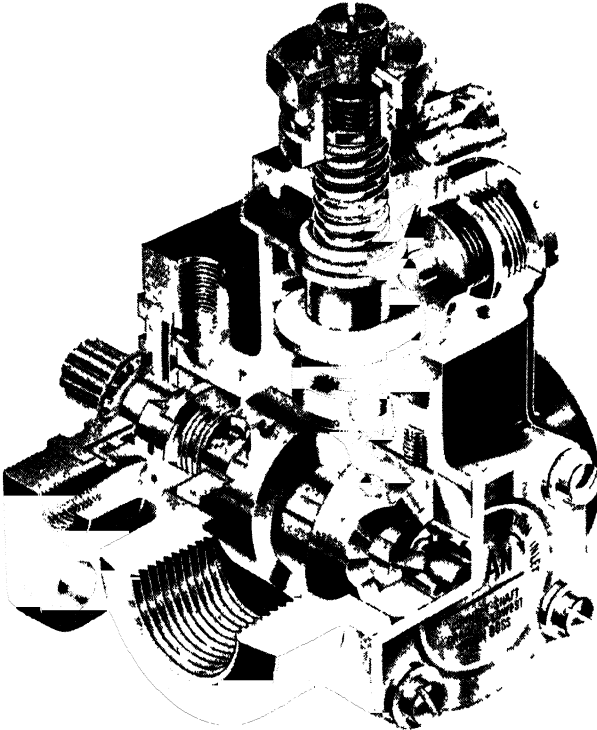


FIG. 186. PRODUCTION ILLUSTRATION OF THE P-51 MUSTANG FIGHTER
An example of complex technical drawing — a true perspective.

It is not expected that you can, as a result of the brief treatment of perspective *Problems* afforded here, produce pictorial drawings of complicated objects. However, a large selection of elementary problems is provided in Fig. 188 and you are urged to do as many of these as you can before attempting anything more difficult. The isometric and oblique problems can be drawn on the sheet layout of Fig. 162, page 214. For perspective drawings use larger paper sizes of your own choosing.



Courtesy Candler-Hill Corporation

FIG. 187. PICTORIAL DRAWING SHADED WITH THE AIR-BRUSH -- FIFAN AIRCRAFT FUEL PUMP

TECHNICAL SKETCHING

The value of freehand technical sketching to the engineer cannot be overstated. It enables him to convey technical ideas to others quickly and unmistakably. He can employ all of his knowledge of graphic expression and can record his ideas easily and completely. "A picture is worth a thousand words," according to an old Chinese proverb. *Importance of sketching*

Most original mechanical ideas find their first expression through a freehand sketch. Executives resort to it daily to explain their ideas to subordinates. Engineers usually prepare their designs and

Straight lines

In general, before drawing a straight freehand line, decide upon your starting and stopping points; then swing your pencil back and forth between these points until you feel that you are "set" to make the line. First make a very light sketch line between the points; then, more slowly, go over the line and make it the proper weight. Much can be learned by observation of good sketches, and

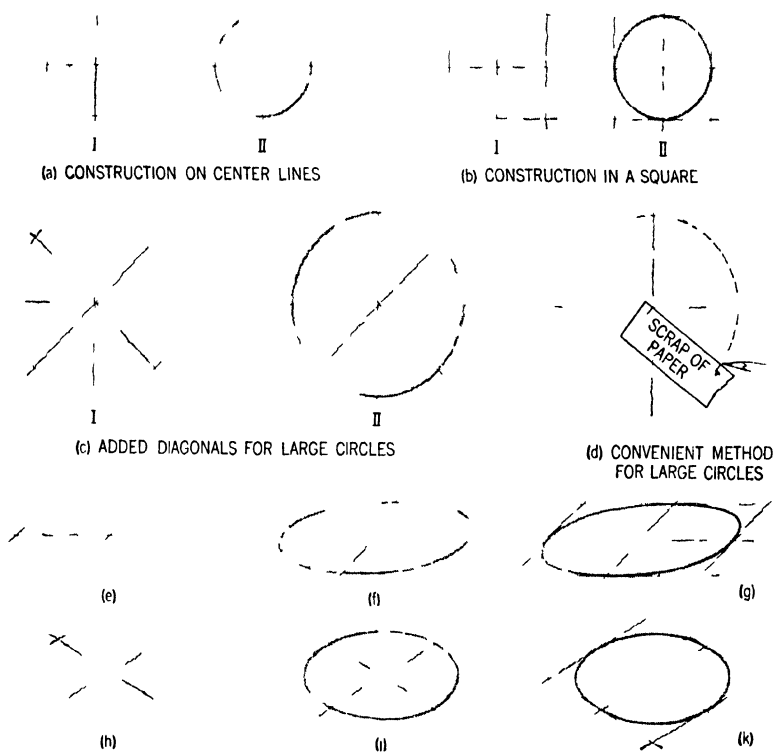


FIG 190 SKETCHING CIRCLES AND ELLIPSES

by practice. The draftsman who perseveres will eventually evolve a satisfactory technique of his own.

Some of the various methods used in constructing circles and ellipses are shown in Fig. 190. In general, construction lines should be so light that they will not need to be erased.

Sketches such as Fig. 191 are frequently made showing the views of objects carrying complete dimensions and shop notes. First, block in the main geometric elements in the views chosen; second, sketch in the details; and third, heavy-in the required lines, and add dimensions and notes. The final drawing should

Views of objects

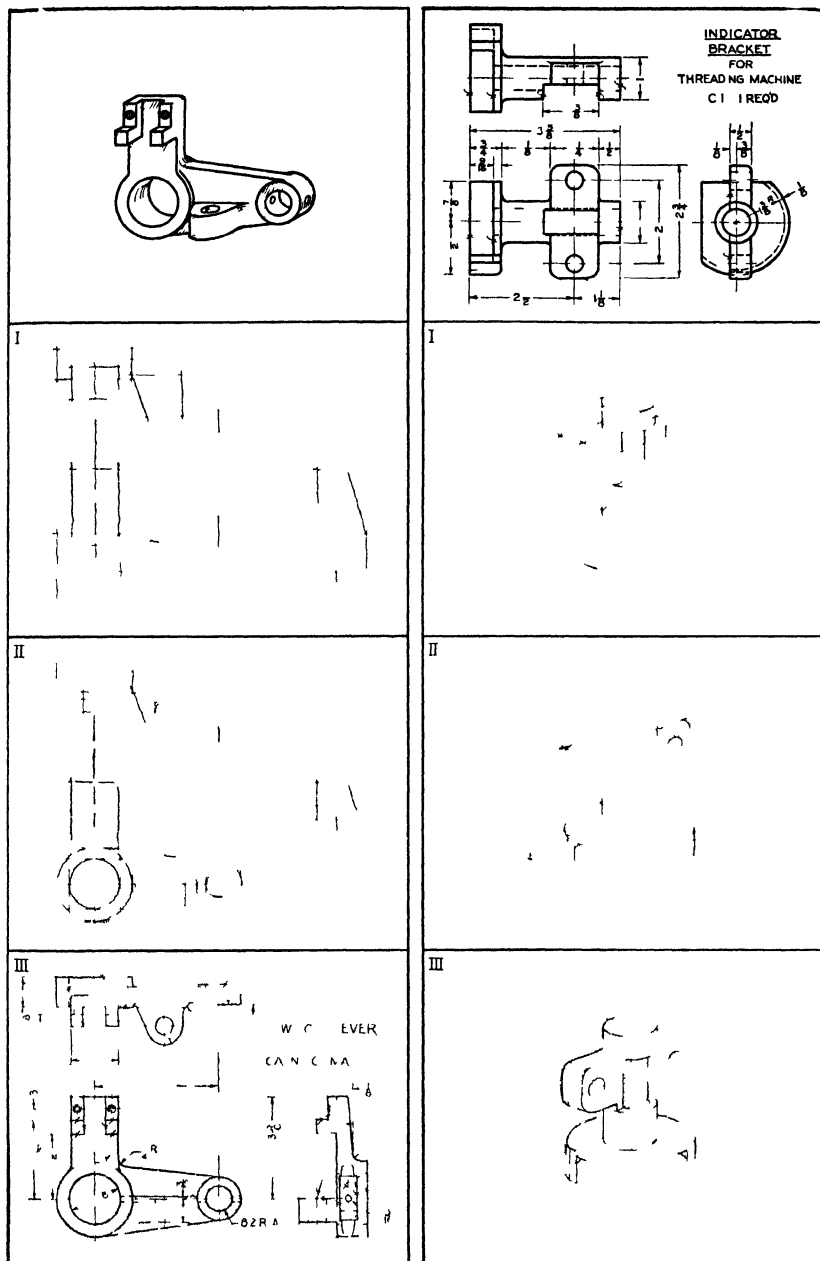


FIG 191 SKETCHING A WORKING
DRAWING

FIG 192 SKETCHING A PICTORIAL
DRAWING

Pictorial
sketches

have the contrast in lines of a mechanical drawing, but the character of the lines should be strictly freehand. In other words, do not try to make every line "perfect," but be very sure that your general proportions and estimates of distances and areas are well considered.

Pictorial sketches may be made to approximate isometric, oblique, or perspective projection. (See Fig. 192.) Considerable liberty can be taken in a pictorial sketch if the draftsman has a natural aptitude for sketching, but most persons should rely entirely upon the fixed methods of projection used in mechanical drawing. The pictorial drawing, unlike the working drawing, can be shaded if desired. Shading, however, is an art in itself and beginners are cautioned to keep shading to a minimum. Many a sketch is ruined by "overdoing" this part of the drawing. The best rule to follow is to use restrained shading, and to use it only when the shape is not clear without it.

Problems

Any of the problems given heretofore for mechanical drawing will make equally suitable problems for technical sketching, and you are therefore referred to pages 198, 209, 213 for problem material. Ordinary buff detail drawing paper of the same kind used for mechanical drawings will be found excellent for these sketches.

MACHINE FASTENERS

Some machine elements occur so often in engineering that the draftsman must be familiar with accepted methods of representation of such parts on drawings. These are chiefly permanent fasteners such as rivets, or movable fasteners such as bolts, screws, and keys.

Threads

Threads are of course the most common fastener. On the outside of a member they are called *external threads* and on the inside,

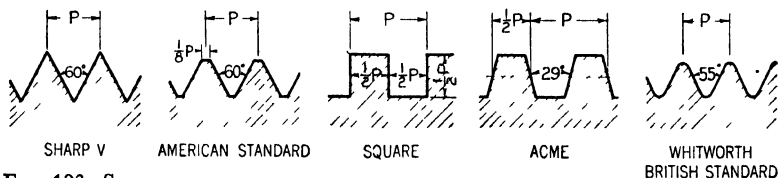


FIG. 193. SCREW THREAD FORMS

internal threads. The diameter is always understood to be the major diameter, or outside diameter. A number of thread forms, or profiles, are in use today, the most common of which are shown in Fig. 193. The American Standard is the general all-purpose thread, but either the *square thread* or the *Acme thread* is widely used to transmit power.

Screw threads are helicoidal in shape but the curved lines are never drawn because of the considerable labor and time involved.

Threads under approximately 1-in. diameter on the drawing are drawn by the conventional method (Figs. 194 and 195). Here the visible thread is represented by alternating long thin lines and short thick lines at right angles to the shaft. The long lines are spaced by eye to approximate the spacing between the ridges.

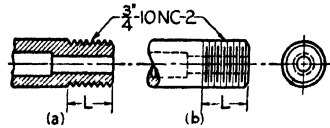


FIG. 194. EXTERNAL THREAD SYMBOLS — REGULAR

Complete tables covering the various threads giving the number of threads per inch and other data for various diameters are given in the handbooks. A few of the most important tables are included here.

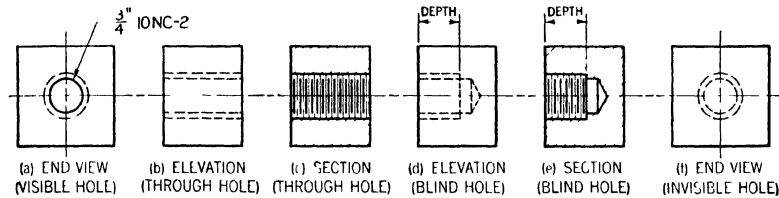


FIG. 195. INTERNAL THREAD SYMBOLS — REGULAR

AMERICAN STANDARD SCREW THREADS

| Sizes | Threads per Inch | | | Sizes | Threads per Inch | | |
|----------------|--------------------------|------------------------|-------------------------------|----------------|--------------------------|------------------------|-------------------------------|
| | NC National Coarse | NF National Fine | EF S.A.E. Extra Fine | | NC National Coarse | NF National Fine | EF S.A.E. Extra Fine |
| 0 | | 80 | | $\frac{9}{16}$ | 12 | 18 | 24 |
| 1 | 64 | 72 | | $\frac{5}{8}$ | 11 | 18 | 24 |
| 2 | 56 | 64 | | $\frac{3}{4}$ | 10 | 16 | 20 |
| 3 | 48 | 56 | | $\frac{7}{8}$ | 9 | 14 | 20 |
| 4 | 40 | 48 | | 1 | 8 | 14 | 20 |
| 5 | 40 | 44 | | $1\frac{1}{8}$ | 7 | 12 | 18 |
| 6 | 32 | 40 | | $1\frac{1}{4}$ | 7 | 12 | 18 |
| 8 | 32 | 36 | | $1\frac{3}{8}$ | 6 | 12 | 18 |
| 10 | 24 | 32 | | $1\frac{1}{2}$ | 6 | 12 | 18 |
| 12 | 24 | 28 | | $1\frac{3}{4}$ | 5 | | 16 |
| $1\frac{1}{4}$ | 20 | 28 | 36 | 2 | $4\frac{1}{2}$ | | 16 |
| $1\frac{5}{8}$ | 18 | 24 | 32 | $2\frac{1}{4}$ | $4\frac{1}{2}$ | | 16 |
| $1\frac{3}{8}$ | 16 | 24 | 32 | $2\frac{1}{2}$ | 4 | | 16 |
| $1\frac{7}{8}$ | 14 | 20 | 28 | $2\frac{3}{4}$ | 4 | | 16 |
| $1\frac{1}{2}$ | 13 | 20 | 28 | 3 | 4 | | 16 |
| | | | | Over 3 | | | 16 |

Thread spacing

Uses and fits

The *form* of the American Standard thread is shown in Fig. 193, and in addition several *series* have been standardized, the most important of which are shown in the preceding table. National Coarse threads are for general use, National Fine threads are for special uses requiring a larger number of threads per inch, and the S.A.E.

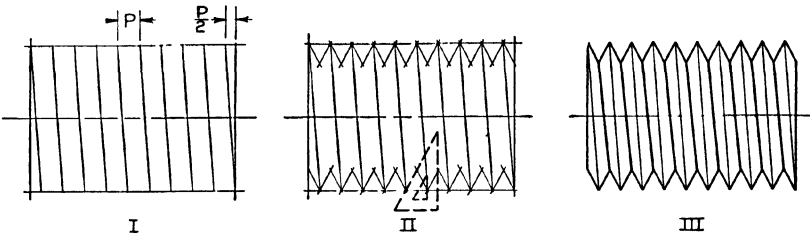


FIG. 196. STEPS IN DRAWING SEMICONVENTIONAL SHARP-V, OR AMERICAN STANDARD THREADS

Extra Fine threads are intended for airplane construction and other uses where extremely fine threads are needed.

Tables of fits for American Standard threads are also available, specifying the tightness of fit between internal and external mating threads. In Figs. 194 and 195 the first figure in the thread note designates the major diameter, followed by the number of threads per inch, the symbol for the series, and finally the class of fit.

Pitch of a thread

Threads of over 1-in. diameter on the drawing are drawn by the *semiconventional* method (Figs. 196 and 197), which more closely resembles the actual thread. In all cases the pitch, $P = \frac{1''}{\text{no. threads per inch}}$, and the number of threads per inch may be found in the tables given here.

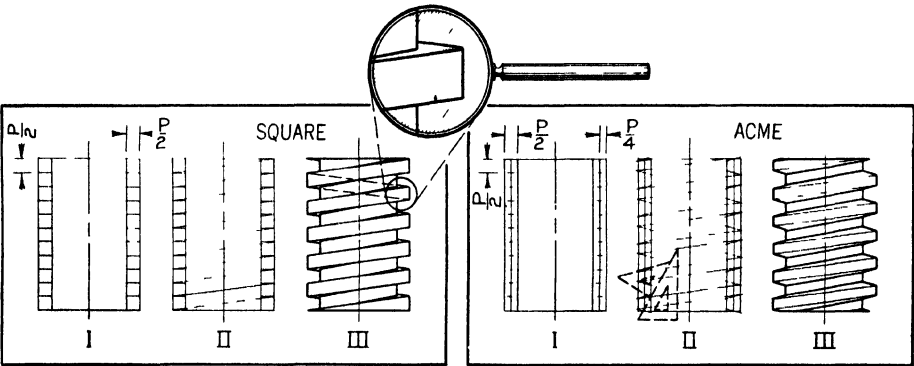


FIG. 197. DRAWING SEMICONVENTIONAL SQUARE AND ACME THREADS

SQUARE AND ACME * THREADS

| Size | Threads per Inch | | Size | Threads per Inch | | Size | Threads per Inch | | Thread spacing |
|----------------|------------------|------|----------------|------------------|------|----------------|------------------|------|----------------|
| | Square | Acme | | Square | Acme | | Square | Acme | |
| $\frac{1}{4}$ | | 16 | $\frac{7}{8}$ | 5 | 6 | 2 | $2\frac{1}{2}$ | 4 | |
| $\frac{3}{8}$ | 12 | | 1 | 5 | 5 | $2\frac{1}{4}$ | 2 | 3 | |
| $\frac{5}{8}$ | | 14 | $1\frac{1}{8}$ | 4 | 5 | $2\frac{1}{2}$ | 2 | 3 | |
| $1\frac{1}{8}$ | | 12 | $1\frac{1}{4}$ | 4 | 5 | $2\frac{3}{4}$ | 2 | 3 | |
| $1\frac{3}{8}$ | | | $1\frac{1}{2}$ | 3 | | 3 | $1\frac{1}{2}$ | 2 | |
| $1\frac{5}{8}$ | 10 | 12 | $1\frac{3}{8}$ | | 4 | $3\frac{1}{4}$ | $1\frac{1}{4}$ | | |
| $1\frac{7}{8}$ | 10 | 10 | $1\frac{5}{8}$ | | 4 | $3\frac{1}{2}$ | $1\frac{1}{3}$ | | |
| 2 | 8 | | $1\frac{1}{2}$ | | 4 | $3\frac{3}{4}$ | $1\frac{1}{3}$ | | |
| $2\frac{1}{8}$ | 8 | 8 | $1\frac{3}{4}$ | $2\frac{1}{2}$ | 4 | 4 | $1\frac{1}{3}$ | 2 | |
| $2\frac{1}{4}$ | 6 | 6 | | | | $4\frac{1}{4}$ | $1\frac{1}{3}$ | | |
| | | | | | | $4\frac{1}{2}$ | 1 | | |
| | | | | | | 5 | | 2 | |

* Acme threads are American Standard (ASA B1.3-1941)

Threads in section are shown in Fig. 198. Note that the stud and nut are not sectioned, as there is no need for showing internal construction. Note also that at the bottom end of the stud the back side of the threaded hole and a portion of the tap drill hole are shown. The angle of the point of the drill is 60° with the center line.

A thread is understood to be R.H. (right hand) unless specified L.H. (left hand) at the end of the thread note. A right-hand thread advances when the screw is turned clockwise as illustrated by Fig. 199 *b*; a left-hand thread advances when the screw is turned counterclockwise (Fig. 199 *a*).

The shop man remembers this by the position of his thumb around a screw thread, as shown.

Regular American Standard bolts are illustrated in Fig. 200, *Bolts* showing convenient draftsman's proportions based on diameter *D*. These bolts have National Coarse threads.

A number of other miscellaneous machine fasteners are shown in Fig. 201. These are generally shown only in assembly drawings and their dimensions are obtained from handbooks.

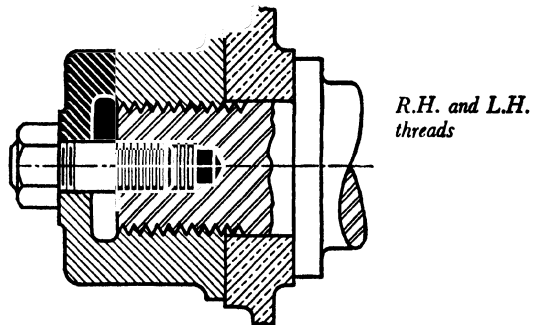


FIG. 198. THREADS IN SECTION

WORKING DRAWINGS

Originality in design

When you look at some complicated structure, such as an airplane, a full-color printing press, or a late-model milling machine, don't be discouraged. Not one but hundreds of persons have con-

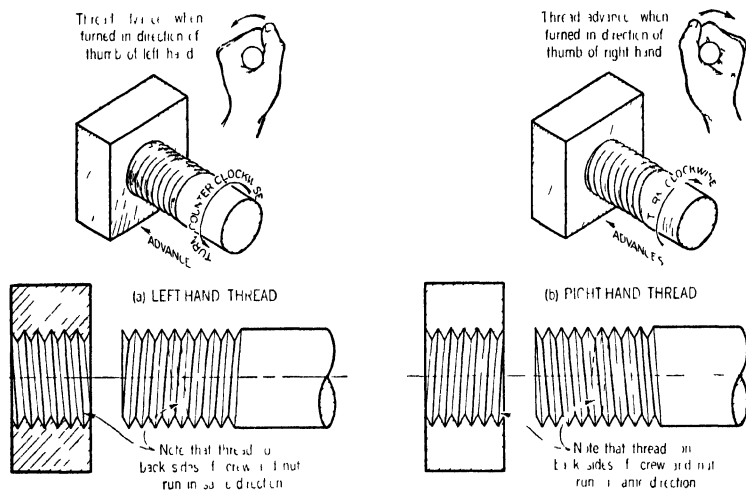


FIG. 199. RIGHT-HAND AND LEFT-HAND THREADS

tributed their brains and originality to these modern wonders, and you may be sure that the first design was relatively crude by our standards, revolutionary as it might have been at the time.

Leonardo's inventions

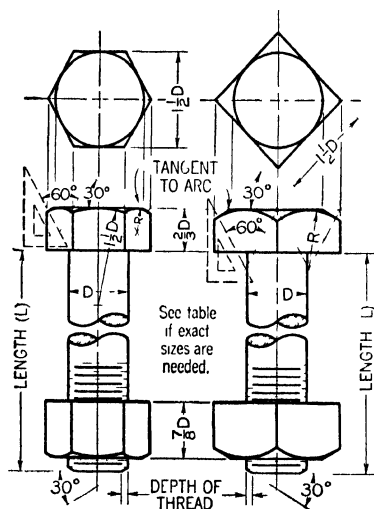


FIG. 200. BOLT PROPORTIONS

Leonardo da Vinci, perhaps the greatest scientific and artistic genius of all time, is credited as being the first to tackle seriously the problem of human flight. One of his many strange sketches, with left-handed mirror-written notes, is shown in Fig. 214.

Since Leonardo, thousands of other brilliant and daring minds, notably the Wright brothers, have added their ideas. Each designer has contributed small improvements through the years to produce the intricate mechanisms of today (such as Fig. 186), and

often bitter tragedy taught men to abandon one idea and try another. The point is that engineering design is cumulative, and each designer hopes only to improve a little upon existing mechanisms.

DRAFTING PROCEDURE

Usually a designing engineer conceives the basic idea, but often the ideas come from draftsmen, shopworkers, or salesmen. In any case this fortunate individual visualizes the mechanism in three-dimensional space and not in terms of a word description. It is

1. *The idea sketch*

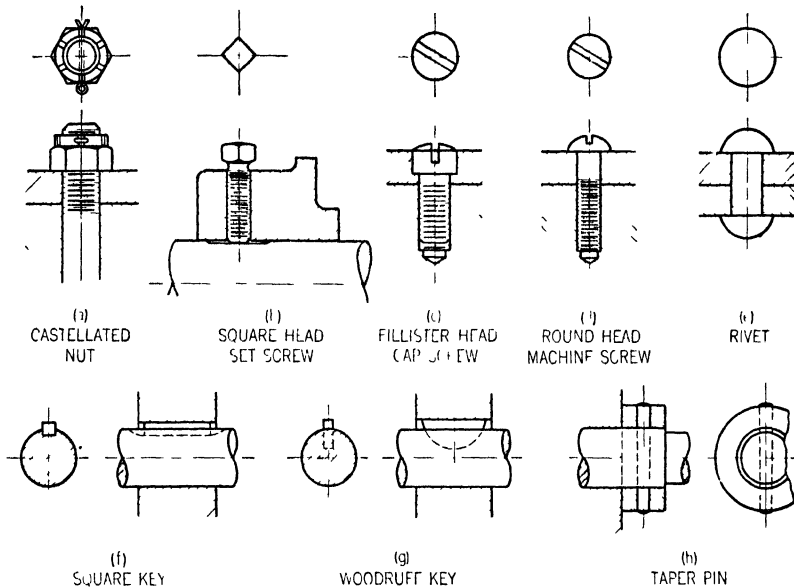


FIG. 201. MISCELLANEOUS FASTENERS

only natural that at this moment he would seize a pencil and any scrap of paper to record the new idea at once in the form of a freehand technical sketch (page 235). Such a sketch is shown in Fig. 202, of an "Arm Adjusting Mechanism" for a milling machine.* See Fig. 203 (handle in upper left portion of milling machine).

Then the originator, or another designer-draftsman, uses the idea sketch as a basis for the *layout drawing* of Fig. 204, drawn in pencil with instruments to exact scale but without dimensions. This is a *design assembly drawing*, showing the part in its natural

2. *The layout drawing*

* Figures 202 to 207 are reproduced through courtesy of the Brown & Sharpe Mfg. Co.

position and in relation to all adjacent parts. The layout is usually full size to enable the designer to visualize more clearly the actual sizes of the parts. When working with drawings to half-size or less, a designer has a tendency to design large, heavy, or clumsy parts, while in working double size or more he is apt to get an exaggerated idea of strength and rigidity.

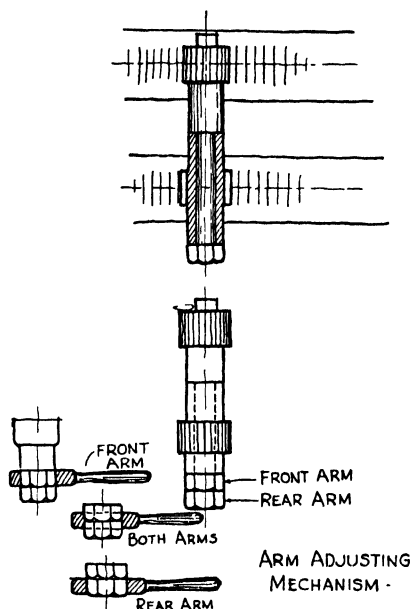


FIG. 202. IDEA SKETCH FOR IMPROVEMENT
ON MILLING MACHINE

The designer-draftsman makes use of all the sources of information available to him, including physics, mathematics, chemistry, and mechanics. He adopts the conclusions and recommendations obtained from experimental tests and laboratory studies and he uses the sound experience that he and his company have obtained from success and failure of past machines, following carefully the performance and maintenance records of machines in use. Much design information is compiled in handbooks and designers use these so frequently that they usually own their own copies.

Some mechanisms are relatively simple in form and operation and little thought is required to design the shapes and arrangement. They may be, however, subject to great loads, so that careful computation of strength is necessary to design the parts to withstand the stresses and strains. A good example of this type of

design problem would be a simple hoist. Other mechanisms, such as typewriters and adding machines, are not subject to large stresses, and the chief problem is one of arrangement and shape of parts for effective operation and low production cost.

When the designer's layout has finally been approved, it is turned over to the detailer to make *detail working drawings* (Fig. 205).

3. *The detail drawing*



FIG. 203. "ARM ADJUSTING MECHANISM" ON THE MILLING MACHINE

These are separate drawings of the individual parts, completely dimensioned and including all shop notes and other data required in the manufacture of the parts. The detailer takes his dimensions directly from the layout with scale and dividers.

The shop also needs an assembly drawing to direct the workers in assembling the parts properly. Since the original layout is an assembly, it usually can be traced, omitting unnecessary adjacent parts, and used as the assembly drawing (Fig. 206).

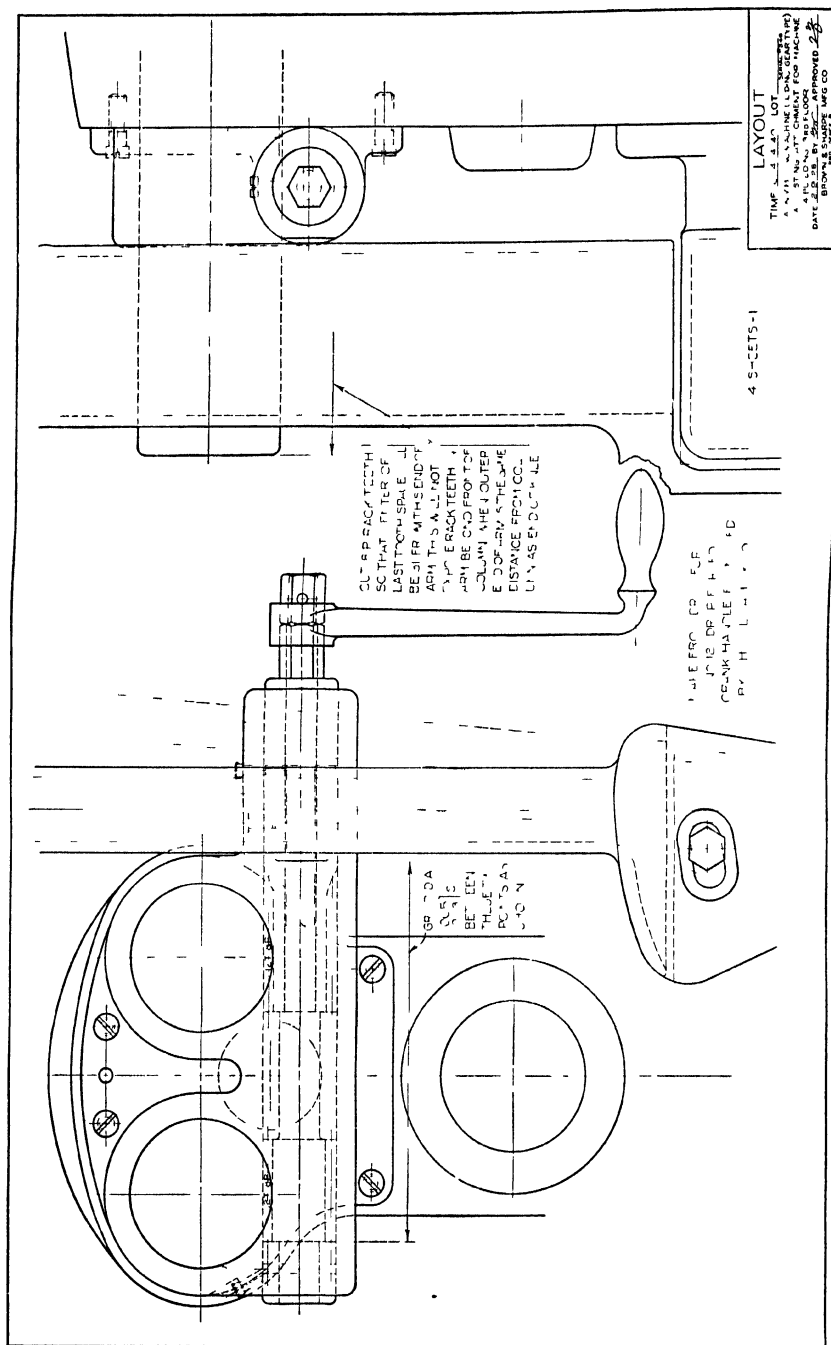
4. *The assembly drawing*

Finally, the patent drawing is made and sent to the United States Patent Office (Fig. 207). Patent drawings are shaded, lettered in script, and otherwise follow rigid Patent Office specifications.

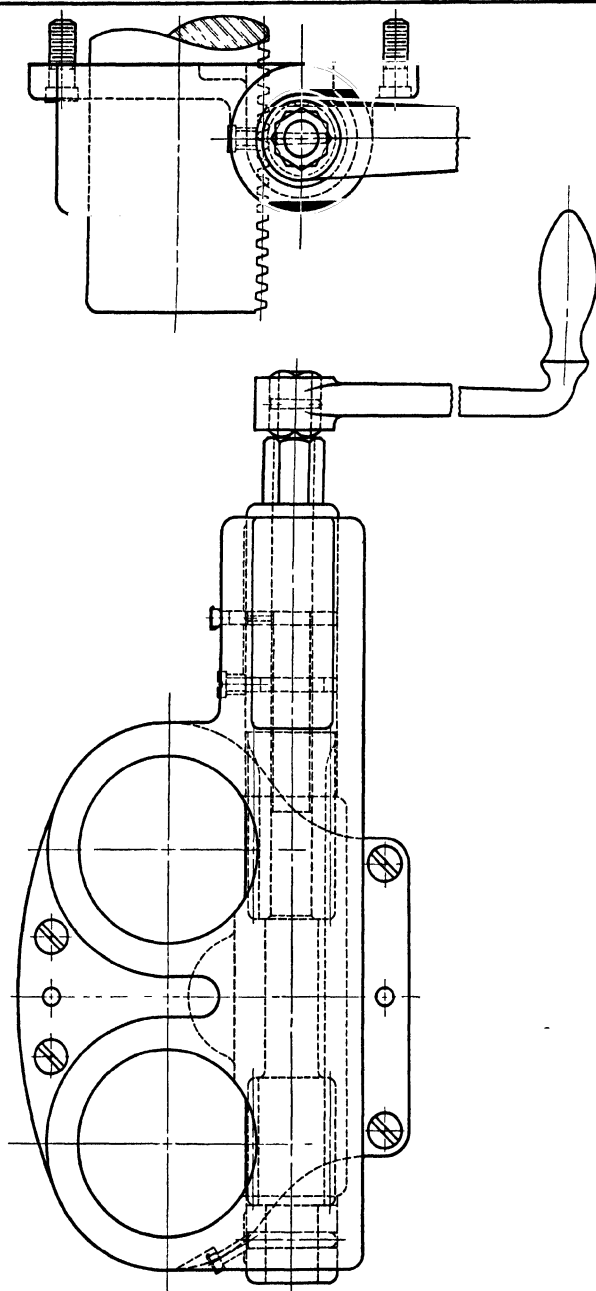
5. *The patent drawing*

Three detail drawing problems are given in Figs. 208–210. These will test your ability to read a drawing and to visualize fairly complicated details. Use a 17" × 22" sheet, design your own title block, and follow the instructions given with each problem. Dimension the problems fully and letter all notes. Place the dimensions where they should be in the new drawing, remembering that in many cases they could not be placed properly on the limited views given. This drawing can be drawn directly on vellum in pencil, if desired, or the pencil drawing on detail paper can be traced on cloth in ink. (Continued on p. 248.)

Problems



2A-7190



| | | | |
|----------------------------|------|-----|--|
| TIME | BT2 | LOT | |
| MACH. USED ON | FACE | | |
| FINISHED ON | | | |
| ARM ADJUSTING MECHANISM | | | |
| NAME ASSEMBLER Drawing | | | |
| DATE 1-1-39 10:11 P.M. 7 H | | | |

| | |
|-------------|----------------------------|
| MATERIAL | |
| STOCK SIZE | |
| ON TIME NO | Q 2 4 5 4 - 40 |
| DESIGNED BY | Desmet & Schmitt, Mfg. Co. |
| CHANGES | REVISIONS |
| 1.23 | |

FIG. 206. ASSEMBLY OF "ARM ADJUSTING MECHANISM"

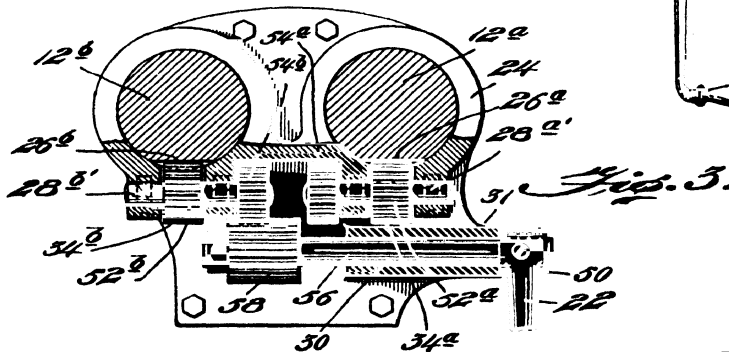
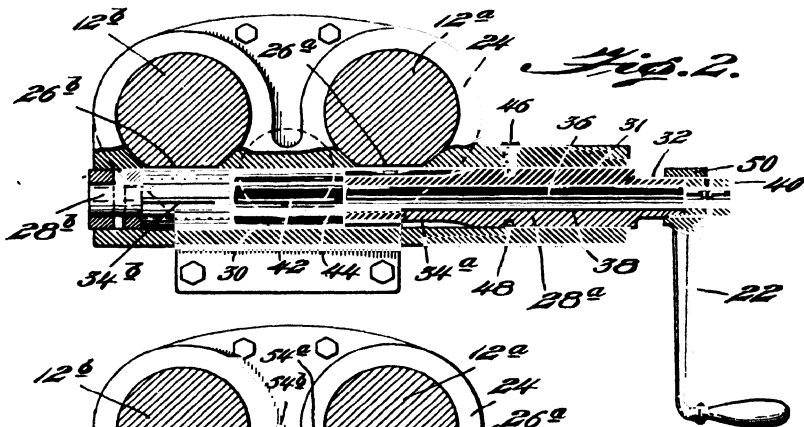
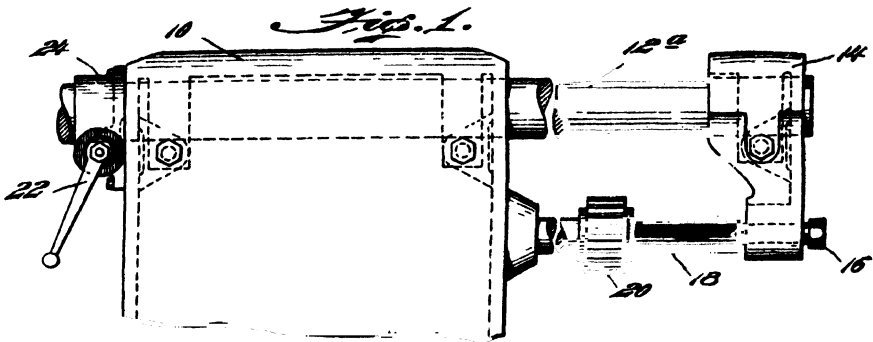
June 30, 1931.

B. P. GRAVES

1,812,649

ADJUSTING MEANS FOR OVERHANGING ARMS

Filed Sept 11, 1928



Inventor
Benjamin P. Graves,
By Thomas A. Jewell,
Attorney

FIG. 207. PATENT OFFICE DRAWING FOR "ARM ADJUSTMENT MECHANISM"

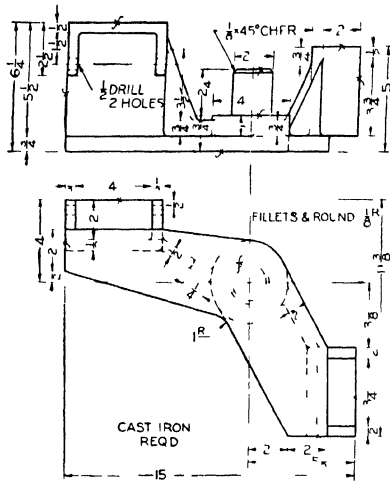


FIG. 208 SKIP DRUM BRACKET FOR A CONCRETE PAVER

Given: Front and bottom views

Required: To draw front, top and right side views.

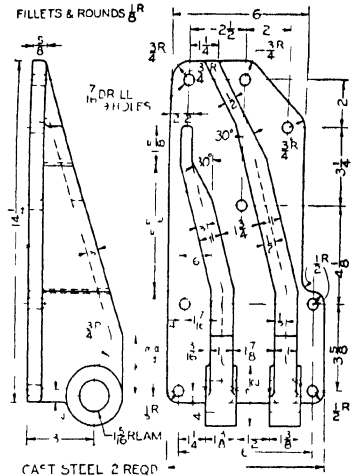


FIG. 209. BRACKET FOR A DIESEL ENGINE

Given: Front and left side views

Required: Revolve present front view 90° clockwise, and let it become a top view. Then add front and right side views.

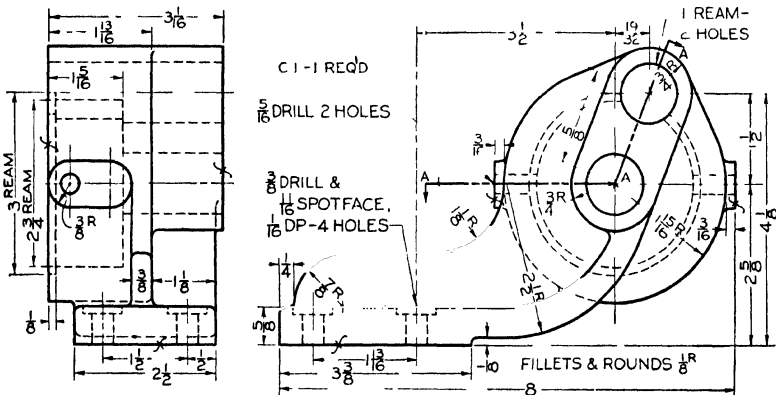


FIG. 210 PUMP BRACKET FOR A THREAD MILLING MACHINE

Given: Front and L. S. views. Required: Draw front view, R. S. view, and top view sectioned on A. A.

*Problems
continued*

All of the details, together with a pictorial assembly drawing of a "Fixture for Centering Connecting Rod," are given in Figs 211-213. Draw complete detail and assembly working drawings. Use 17" × 22" sheets, placing several details per sheet. Include a complete parts list on the assembly drawing.

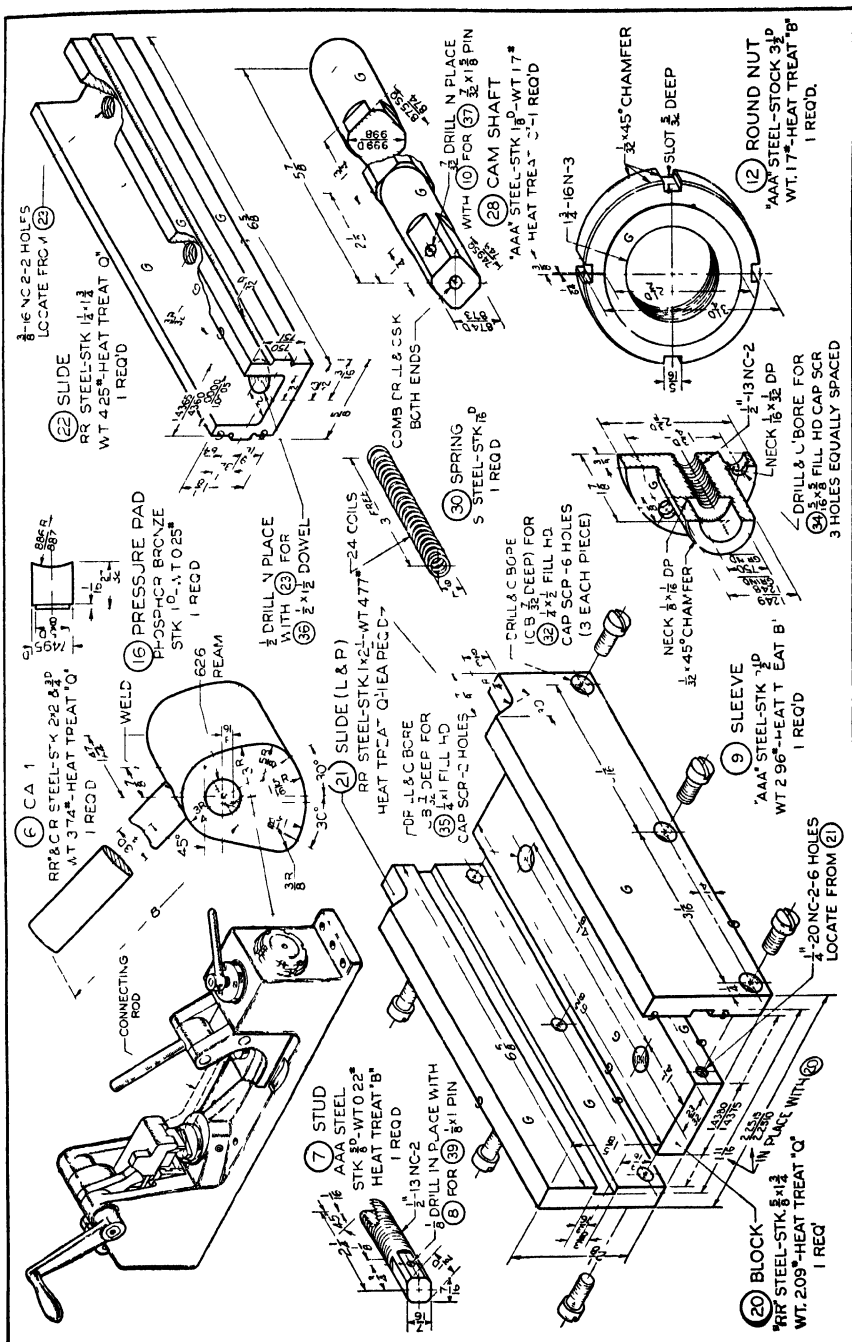


Fig. 211. Fixture for centering connecting rod

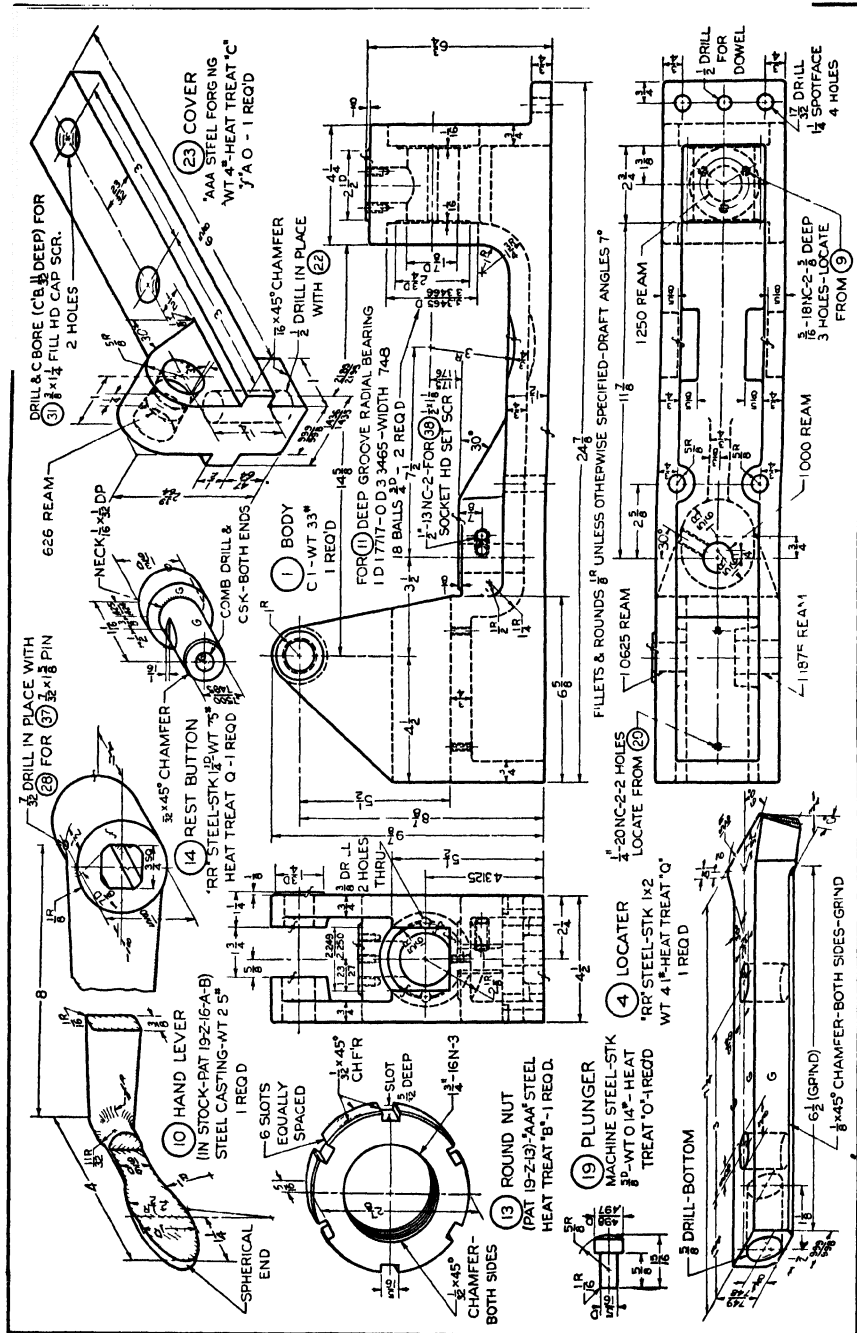


FIG. 212. FIXTURE FOR CENTERING CONNECTING ROD

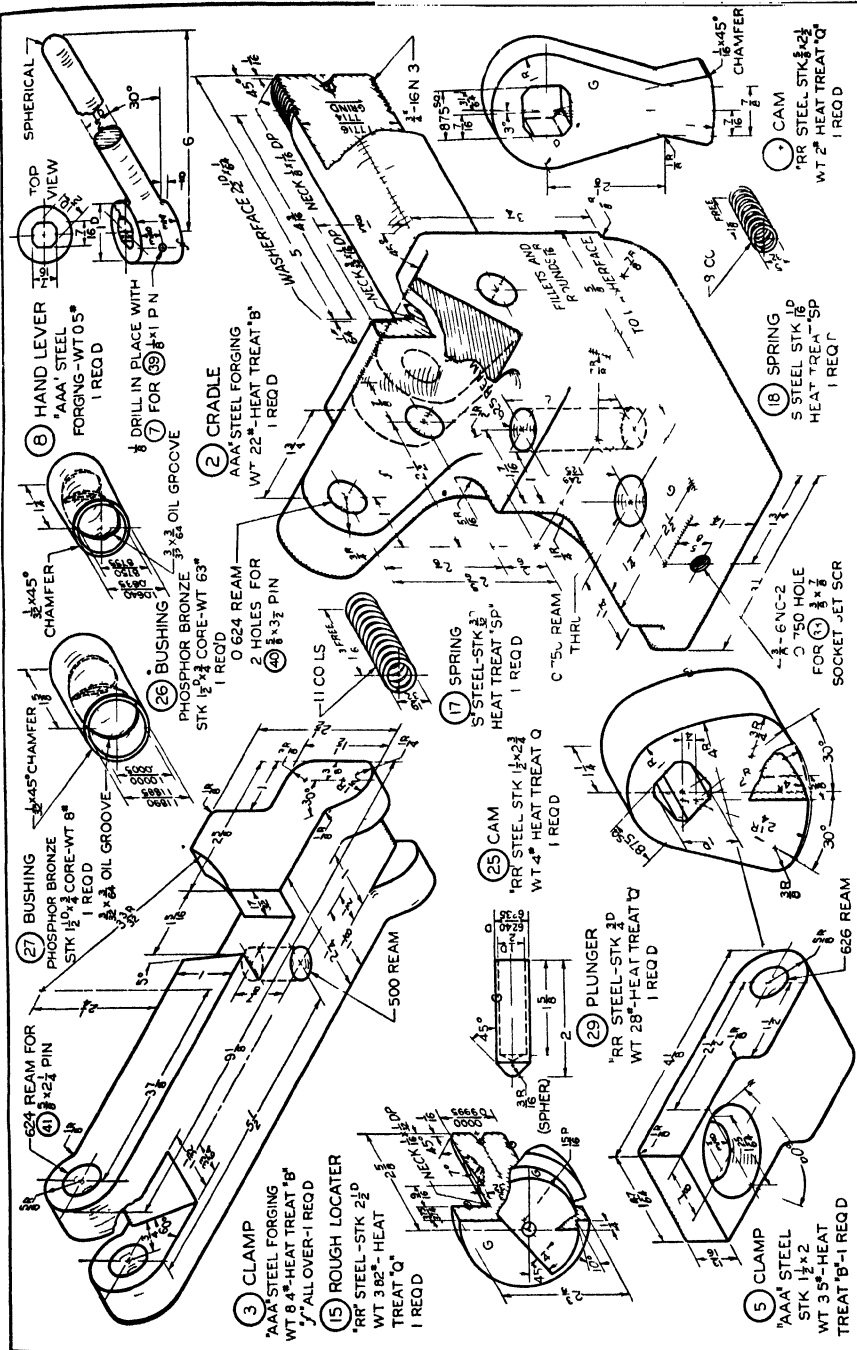
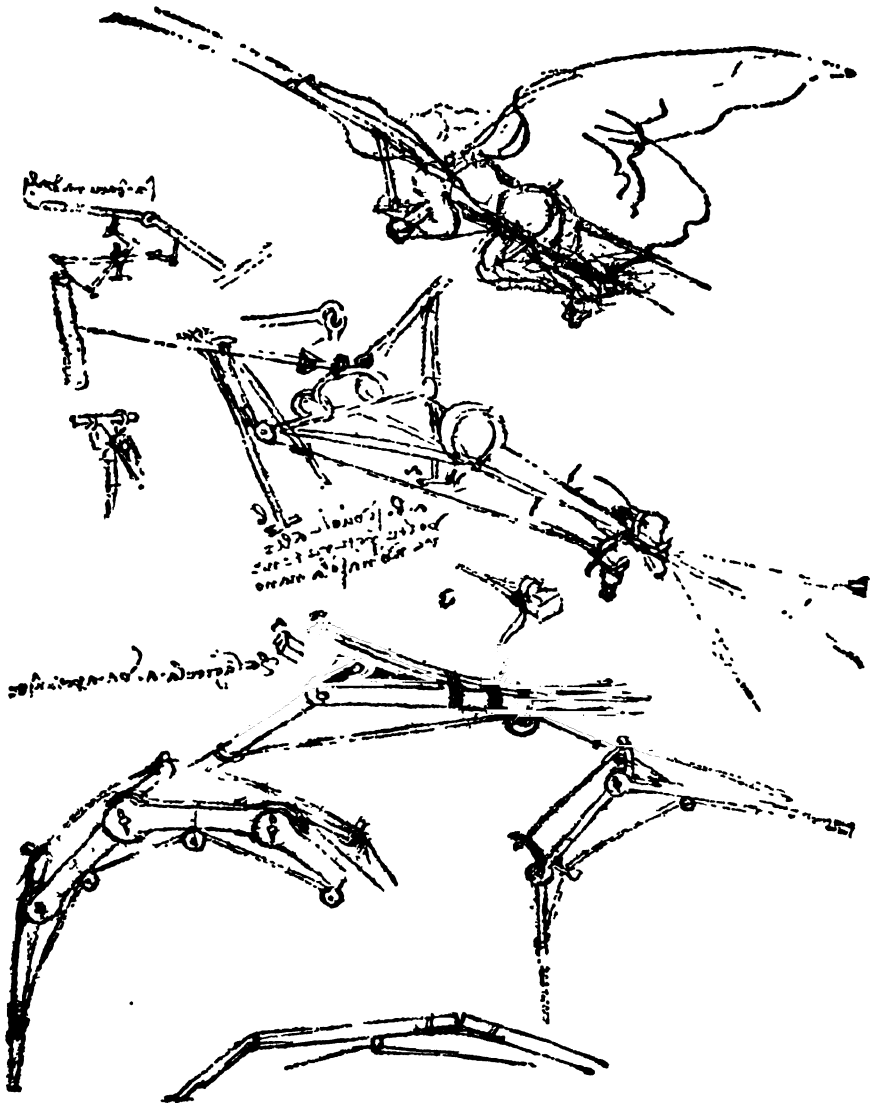


FIG. 213. FIXTURE FOR CENTERING CONNECTING ROD



Courtesy Random House, Inc

FIG. 214. LEONARDO DA VINCI'S IDEA SKETCH OF AN AIRPLANE

In conclusion, from Leonardo da Vinci's sketches (Fig. 214), to the complicated drawings used in modern industry, it is evident that technical ideas have always been expressed graphically. Hence, your objective should be to produce drawings which are neat and simple, yet clear and complete.

Mathematics

A UNIVERSAL TOOL OF ENGINEERING

Basic to all fields of engineering is the prediction of how some thing designed on paper will work. To make such predictions an engineer must know various laws of nature under which physical systems operate. These laws are usually expressed in mathematical form because they are relations between the measures of quantities, such as the mass and acceleration of a projectile and the force acting on it. From such laws the engineer can determine how to increase the efficiency of a given device. He can also diagnose failures in equipment and suggest improvements. When the engineer uses numbers, and hence mathematics, his work becomes more exact, and his field more scientific. *Mathematical laws*

Certain outstanding inventions like the radio were discovered through mathematics, whereas others such as the dynamo were found experimentally, and subsequently improved by mathematics. If, therefore, you wish to build or maintain some engineering device such as a bridge, electrical generator, airplane, or oil-cracking plant, or if you wish to add to the comforts of man through original scientific inventions, you will need mathematics.

Mathematics had its origin in the need of our ancestors to manipulate numbers rapidly. Prehistoric man probably used the whole numbers 1, 2, 3, and so on, and excavations show that even before 2000 B.C. the Babylonians divided one of these whole numbers by another. Mathematicians speak of whole numbers as *positive integers*.

The result of such division is called a *quotient*. The quotient of 4 by 3 is represented by $\frac{4}{3}$. In a quotient the first, or top, number is called the *numerator* and the second, or bottom, is the *denominator*. In the fraction $\frac{4}{3}$ the numerator is 4 and the denominator is 3.

If the denominators of two fractions are identical, they are said to have a *common denominator*. In this case the fractions may be added to yield a fraction whose numerator is the sum of the numerators of the given fractions and whose denominator is the *Addition of fractions*

same as the common denominator of the given fractions. For example,

$$\frac{4}{3} + \frac{7}{3} = \frac{11}{3}.$$

This rule for the addition of fractions probably arose from a consideration of parts that may be added up to make a whole, as in the following summations:

$$\frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1, \quad \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1, \quad \frac{1}{3} + \frac{2}{3} = \frac{3}{3} = 1.$$

*Symbols
representing
numbers*

The rule for the addition of two fractions may be stated very briefly by the use of symbols to represent numbers, whose values may be anything we please to make them. If a , b , and c represent numbers, the relation

$$(1) \quad \frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

says that the sum of the fractions a/b and c/b is the sum of the numerators over the common denominator, or $(a+c)/b$.

If we let $a = 4$, $b = 3$, and $c = 7$, equation 1 reduces to the sum of the fractions $\frac{4}{3}$ and $\frac{7}{3}$ above, whereas if $a = 1$, $b = 3$, and $c = 2$ we have the last of the above sums. The formula 1 thus represents an unlimited number of equations, because an unlimited number of equations can be obtained from it by assigning different values to a , b , and c .

The result of multiplying numbers together is called the *product* of these numbers. The given numbers are termed the *factors* of the product. Thus the product of the numbers 2, 3, and 7 is $2 \times 3 \times 7$, or 42. The numbers 2, 3, and 7 are factors of 42, that is, of the product $2 \times 3 \times 7$.

*Notation for
product*

To avoid confusion between the symbol \times meaning "times" and the symbol x standing for a number, and to simplify notations, it will be convenient to omit the "times" sign when no confusion will result. For example, $2a$ means 2 times a , and $12abc$ represents 12 times a times b times c . If, however, you wish to write 3 times 4, you will have to use a times sign because 34 stands for thirty-four.

*Reducing
fractions*

Since $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{3}{6}$ have the same value, it is obvious that fractions with different numerators and denominators may be equal to one another. Because it is easier to work with small numbers, the fraction $\frac{1}{2}$ is to be preferred to $\frac{2}{4}$ or $\frac{3}{6}$. In the fraction $\frac{2}{4}$ the numerator and denominator have the common factor 2; and, similarly, for $\frac{3}{6}$ the common factor is 3. Removal of the common

factor by dividing it into numerator and denominator yields $\frac{1}{2}$ in each instance. The process of removing common factors of the numerator and denominator of a fraction is said to be *reducing the fraction*.

The process of splitting a number into a product of factors is called *factoring the number*. It is customary in factoring a whole number to use only whole numbers as factors. Thus 462 can be factored as shown:

$$462 = 2 \times 3 \times 7 \times 11.$$

Likewise, the number 75 is a product of the factors 3, 5, and 5.

You can readily see that any whole number can be split up uniquely into a product of whole numbers, each of whose only factors are that number and 1.

To reduce a fraction you should follow these steps:

*Rules for
reducing
fractions*

1. *Factor the numerator and denominator completely so that further factoring with whole numbers other than 1 is not possible.*

2. *Remove common factors from numerator and denominator.*

To reduce the fraction $\frac{462}{308}$, we factor numerator and denominator to obtain

$$\frac{462}{308} = \frac{\cancel{2} \times 3 \times \cancel{7} \times \cancel{11}}{2 \times \cancel{7} \times \cancel{7} \times \cancel{11}} = \frac{3}{2},$$

removing common factors as shown. By using algebraic terms we may write

$$(2) \quad \frac{ab}{cb} = \frac{a}{c},$$

where a , b , and c stand for whole numbers. Thus you can multiply or divide the numerator and denominator of a fraction by the same positive integer (whole number) without changing the value of the fraction. The right side of formula 2 is read " a over c ."

The rule given in equation 2 may be used to add fractions with different denominators. If we wish to add $\frac{4}{10}$ and $\frac{6}{100}$, we first multiply numerator and denominator of the first fraction $\frac{4}{10}$ by 10 to get $\frac{40}{100}$ because we know (from equation 1) that the denominators of both fractions to be added must be alike. Addition now yields

*Lowest
common
denominator*

$$\frac{40}{100} + \frac{6}{100} = \frac{40 + 6}{100} = \frac{46}{100}.$$

To add $\frac{1}{2}$ and $\frac{1}{3}$ we perform the steps indicated:

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{3 + 2}{6} = \frac{5}{6}.$$

Naturally, in changing fractions to a common denominator we should choose the smallest possible denominator, since $\frac{1}{5}$ is a simpler result than $\frac{2}{15}$, for example. This smallest possible denominator is called the *lowest common denominator*. For the fractions $\frac{1}{2}$ and $\frac{1}{3}$, the lowest common denominator is 6. Of course, we can add $\frac{1}{2}$ and $\frac{1}{3}$ by writing

$$\frac{1}{2} + \frac{1}{3} = \frac{6}{12} + \frac{4}{12} = \frac{6+4}{12} = \frac{10}{12} = \frac{5}{6},$$

but this involves a final unnecessary operation. The use of the lowest common denominator 6 is simpler. For $\frac{3}{10}$, $\frac{5}{7}$, and $\frac{1}{4}$, the lowest common denominator is $5 \times 7 \times 4$, namely, 140. That 140 is the lowest common denominator of these fractions can be seen from writing the denominators in the factored forms $10 = 2 \times 5$; $7 = 7$; $4 = 2 \times 2$ respectively. Hence, the lowest common denominator is $5 \times 7 \times 2 \times 2$, or 140.

Decimals

The numbers 0, 1, 2, \dots 9 are called *digits*. Just remember that there can be only ten digits, the same as the number of fingers. To facilitate operations with the quotients of whole numbers, the Babylonians invented the *decimal system*. This system is the process of writing such numbers as 5.2 instead of $5\frac{1}{5}$. In decimals, the position and value of the digit designates a given multiple of one of the following numbers: one, ten, one hundred, and so forth; or one tenth, one hundredth, one thousandth, and so on. The use of several digits in a number indicates that corresponding multiples are to be added together. For example, the decimal 23.784 stands for the sum

$$2 \times 10 + 3 \times 1 + 7 \times \frac{1}{10} + 8 \times \frac{1}{100} + 4 \times \frac{1}{1000}.$$

This sum is the quotient of two positive integers, namely,

$$\frac{23,784}{1000}.$$

Decimal numbers

A number written in decimal notation is referred to as a *decimal number*. The ordinary decimal numbers are quotients of whole numbers. We use the word *ordinary* to distinguish the common decimal number from unending decimals such as $0.333 \dots$ which is understood to be $\frac{1}{3}$. Although this fact will not be proved here, each decimal number, ending, or formed from some point on by the eternal repetition of a block of digits as in 1.23,756,756,756, \dots ,

can be written as an ordinary fraction. Accordingly, many decimal numbers are not new numbers, but rather equivalents of ordinary fractions. The representation 23.784 is easier to write than the long sum given above, and has the advantage over

$$\frac{23,784}{1000}$$

of omitting the denominator.

An ordinary fraction can be changed to its *decimal equivalent* by dividing the numerator of the fraction by its denominator. For example, the quotient of 7 by 8 is 0.875 as shown.

$$\begin{array}{r} 0.875 \\ 8 \overline{) 7.0} \\ \underline{64} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

Hence 0.875 is the decimal equivalent of the fraction $\frac{7}{8}$.

The rule for the subtraction of one fraction from another is *Subtraction of fractions* similar to rule 1 for addition, namely,

$$(3) \quad \frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}.$$

Thus, we obtain

$$\frac{12}{5} - \frac{4}{5} = \frac{12 - 4}{5} = \frac{8}{5}.$$

Fractions with denominators not necessarily the same can be combined according to the following laws:

$$(4) \quad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}; \quad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

Cross multiplication

The denominator bd will often not be the lowest common denominator of the given fractions. Such an example is the following:

$$\frac{7}{10} + \frac{4}{15} = \frac{7 \times 15 + 10 \times 4}{10 \times 15} = \frac{105 + 40}{150} = \frac{145}{150} = \frac{29}{30}.$$

Use of the lowest common denominator yields

$$\frac{7}{10} + \frac{4}{15} = \frac{7}{2 \times 5} + \frac{4}{3 \times 5} = \frac{7 \times 3}{2 \times 5 \times 3} + \frac{4 \times 2}{3 \times 5 \times 2} = \frac{21 + 8}{30} = \frac{29}{30}.$$

Combining operations

Operations on several fractions can be performed at once as shown in the example:

$$\frac{5}{8} - \frac{1}{12} + \frac{3}{10} = \frac{75}{120} - \frac{10}{120} + \frac{36}{120} = \frac{75 - 10 + 36}{120} = \frac{101}{120}.$$

In this example the denominators can be factored into

$$2 \times 2 \times 3, \quad 3 \times 2 \times 2, \quad \text{and} \quad 2 \times 5,$$

whence the lowest common denominator is $2 \times 2 \times 2 \times 3 \times 5$, which is 120. Each fraction is changed to one with this denominator by multiplying above and below by the proper factor. The numerators are then combined according to the operations indicated. A method analogous to equation 4 can be used for working the same example:

$$\begin{aligned} \frac{5}{8} - \frac{1}{12} + \frac{3}{10} &= \frac{5 \times 12 \times 10 - 1 \times 8 \times 10 + 3 \times 8 \times 12}{8 \times 12 \times 10} \\ &= \frac{600 - 80 + 288}{960} = \frac{808}{960} = \frac{101}{120}. \end{aligned}$$

Multiplication of fractions

From such examples as

$$2 \times \frac{3}{4} = \frac{6}{4}, \quad \frac{3}{4} = \frac{3}{12} = \frac{1}{4},$$

we are led to the rule that the product of two fractions is *the fraction whose numerator is the product of the numerators of the given fractions and whose denominator is the product of the denominators of these fractions*. In brief, using numbers, a , b , c , and d , we may write

$$(5) \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

As an example we obtain the product of the two fractions

$$\frac{3}{4} \times \frac{7}{8} = \frac{21}{32}.$$

Also, since 11 is $1\frac{1}{1}$, we have

$$11 \times \frac{7}{8} = \frac{11}{1} \times \frac{7}{8} = \frac{11 \times 7}{1 \times 8} = \frac{77}{8}.$$

Division of fractions

Since division by 8 is the same as multiplication by $\frac{1}{8}$, the fraction $\frac{3}{4}$ divided by 8 is

$$\frac{\frac{3}{4}}{8} = \frac{3}{4} \times \frac{1}{8} = \frac{3 \times 1}{4 \times 8} = \frac{3}{32}.$$

Division of one fraction by another can always be reduced to multiplication by use of the following relations

$$(6) \quad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}}{\frac{c}{d}} \times \frac{\frac{d}{d}}{\frac{c}{c}} = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}} = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{cd}{dc}} = \frac{a}{b} \times \frac{d}{c}.$$

When we change from a fraction a/b to its *reciprocal* b/a , we say that we *invert* a/b . The formulas in equation 6 imply that to divide one fraction by another, we invert the second fraction and multiply the result by the first. In particular,

$$\frac{\frac{3}{4}}{\frac{7}{8}} = \frac{3}{4} \times \frac{8}{7} = \frac{24}{28} = \frac{6}{7}.$$

Fractions with denominator 100 occur so frequently that it is *Per cent* desirable to write such fractions in a shorter way than usual. We use the symbol % to mean "divided by 100." Thus $\frac{4}{100}$ is 4%, and $\frac{5}{200}$ is $2\frac{1}{2}\%$. The expression 4% is read 4 per cent, an expression derived from the latin "per centum."

The concept of per cent was introduced into mathematics to enable us to compare one number with another more easily. It is often difficult to decide which of two numbers is the larger when these numbers are fractions with different denominators. When the denominators are the same, the comparison is easy, because the larger numerator goes with the larger number. It may be difficult at a glance for you to pick the larger of the numbers $\frac{7}{11}$ and $\frac{8}{12}$. When, however, we write $\frac{7}{11}$ and $\frac{8}{12}$ respectively as

$$\frac{84}{132} \quad \text{and} \quad \frac{88}{132},$$

we can see at once that the second fraction $\frac{8}{12}$ is the larger. The number $\frac{7}{11}$ is about 63.6% and $\frac{8}{12}$ is about 66.7%, from which you can verify that $\frac{7}{11}$ is smaller than $\frac{8}{12}$. Because of the common use of the decimal system it is particularly convenient to employ the denominator 100 as the basis for comparing these numbers.

To write an ordinary fraction in terms of "per cent," simply divide the denominator into the numerator; then move the decimal point two units to the right in the answer. To change $\frac{7}{11}$ to a decimal we divide as indicated, *How to use per cent*

$$\begin{array}{r} 0.636 \\ 11 \overline{) 7.0} \\ \underline{66} \\ 40 \\ \underline{33} \\ 70 \\ \underline{66} \\ 4 \end{array}$$

Moving the decimal in 0.636 two places to the right we have 63.6. Thus $\frac{7}{11}$ is 63.6% approximately. The number 1 is 100%, the

number 2 is 200%, and so on. By 5% of 10 we mean $\frac{5}{100}$ ths of 10, or $\frac{1}{2}$. Thus we can operate with fractions written as per cents just as with ordinary fractions. For example, 10% of 50% is 5%, or

$$\frac{10}{100} \times \frac{50}{100} = \frac{500}{10,000} = \frac{5}{100}.$$

Ratio

A fraction may be thought of as a way of comparing the sizes of two numbers. This comparison is emphasized in mathematics by writing the given fraction, say a/b , as $a : b$, read as the *ratio of a to b*. Hence 3 : 4 is the ratio of 3 to 4, that is, the fraction $\frac{3}{4}$.

The splitting of 28 into the unequal parts 12 and 16 is a divi-

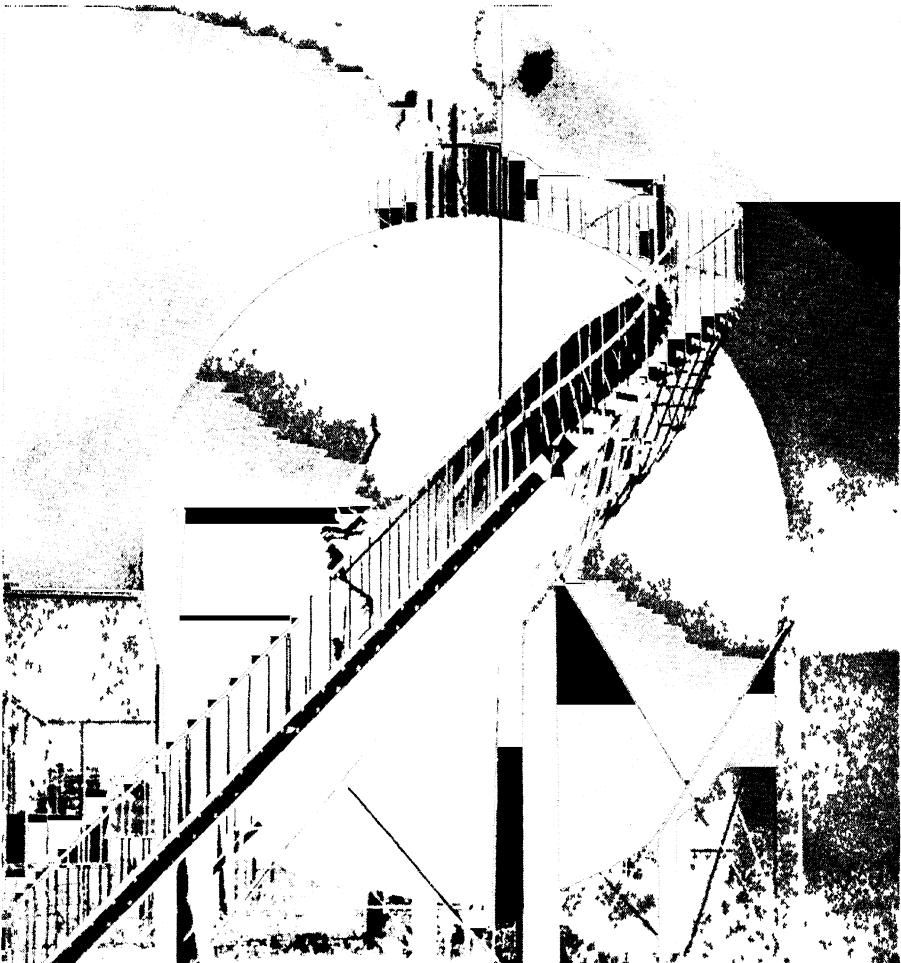


FIG. 215. GEOMETRICAL SHAPES ARE FOUND IN INDUSTRY

sion of the number 28 into two parts whose ratio is 3 : 4. If one gear weighs 2 lb. and another $5\frac{1}{2}$ lb., the ratio of their weights is $2 : 5\frac{1}{2}$, or $4 : 11$.

The equality of two ratios is a *proportion*. It follows that

Proportion

$$\frac{2}{3} = \frac{4}{6}$$

is a proportion. This equality is read "2 is to 3 as 4 is to 6." If the values which x and y can take on in a given problem are such that the ratio $x : y$ is constant, y is said to be *directly proportional* to x .

Since the circumference c of a circle divided by the diameter d is a constant π , the ratio $c : d$ is π , and d is directly proportional to c . Also, c is directly proportional to d .

If y is directly proportional to x , we also say that y *varies directly* as x . If y varies directly as $1/x$, we say that y *varies inversely* as x . *Variation*

We will let p denote the pressure of a gas, and v the volume, keeping the temperature of the gas fixed. The gas then satisfies Boyle's law (discussed on pp. 67-68), that is, for a constant k whose value depends only on the kind and amount of gas used,

$$(7) \quad p = \frac{k}{v}, \quad \text{or} \quad pv = k.$$

The constant k is called the *constant of proportionality*. For example, if the pressure of a given weight of gas is known to be 20 lb. per square inch when the volume is reduced to 1000 cu. in., substitution of 20 for p and 1000 for v gives *Constant of proportionality*

$$k = 20 \times 1000 = 20,000 \text{ (units being inch pounds).}$$

Knowing k , we can now find the pressure of the gas for any volume. Such a problem arises when we change the volume of gas in a cylinder by moving a piston in the cylinder. With $v = 2000$, the formula

$$p = \frac{20,000}{v} = \frac{20,000}{2000} = 10$$

implies that the pressure p is 10 lb. per square inch, whereas with a volume v of 500 cu. in., the pressure p is increased to 40 lb. per square inch.

Boyle's law is a special case of a general gas law,

General gas law

$$(8) \quad pv = RT,$$

where T is the *absolute temperature*. (See p. 69.) Here T varies directly as p and directly as v . On the other hand, p varies directly as T and inversely as v .

If T is 492° absolute temperature on the Fahrenheit scale when p is 14.7×144 lb. per square foot and v is 12.39 cu. ft., the value of R in the general gas law is

$$R = \frac{14.7 \times 144 \times 12.39}{492} = 53.3,$$

where R is in foot-pounds per degree absolute on the Fahrenheit scale. This value of R applies to air.

From the formula

$$pv = 53.3 T$$

for the given gas, all possible sets of values of p , v , and T can be found by substitution. If the pressure p is 30 lb. per square foot and the absolute temperature T is 200° , the volume v in cubic feet is

$$v = \frac{53.3 \times 200}{30} = \frac{10,660}{30} = 355\frac{1}{3}.$$

Unknowns

It is often desirable to solve a proportion for one of the terms. For example, for what value of the *unknown* x is

$$\frac{2}{x} = \frac{4}{7}?$$

Multiplying both sides by the lowest common denominator $7x$, and dividing by 4, we obtain the solution for x , as shown:

$$x = \frac{7 \times 2}{4} = \frac{7}{2}.$$

ZERO AND THE NEGATIVE NUMBERS

Zero

You are so accustomed to working with the number zero that it probably seems to be the easiest number to employ, and you might assume therefore that it was one of the first numbers used in arithmetic. On the contrary, the Hindus introduced a notation for it around 500 A.D. They treated zero like the other numbers, multiplying, adding, and subtracting with it as with the whole numbers. *The number zero is, by definition, an integer.*

Negative numbers

The Hindus found that relations such as

$$x + 1 = 0, \quad x + 2 = 0$$

cannot be solved for x by the use of those numbers described up to this point. Wishing to assign a meaning to such equations, and also desiring to think of such expressions as

$$\frac{1}{2} - \frac{3}{2}$$

as numbers, the Hindu mathematicians were led to devise negative quantities. The whole numbers used above are positive. The quotient of any two of these is also said to be positive. These are examples of positive numbers which will be defined later.

Given a positive number a , the *negative number* $-a$ is by definition the solution for x from the equation

$$(9) \quad x + a = 0.$$

When we say "solution," we mean that whenever we come across the sum $(-a) + a$, we can replace this sum by zero. We have put parentheses around the number $(-a)$ to separate this number from the rest of the text. It has no other significance. The sum $(-4) + 4$ is thus zero.

The negative integers are the numbers $-1, -2, -3, \dots$, *Negative integers* and so on; the whole numbers $1, 2, 3$, etc. used above are the positive integers. The negative and positive integers and the number zero comprise the class of numbers ordinarily called *integers*.

Negative numbers enable us to subtract a number from a smaller number conveniently. Thus you can write $8 - 12$ as -4 . If the temperature is 8°F. and drops 12°F. , the resulting temperature is $(8 - 12)^\circ \text{F.}$, which is -4°F. The Hindu mathematicians thought of positive numbers as *possessed* and negative numbers as *owed*. In this sense if we have \$200 on hand but owe \$400 our "assets" amount to $-\$200$.

The number zero has caused considerable difficulty because it *Meaning of zero* cannot always be employed like the other integers. It can be

shown that if we say that $\frac{0}{0}$ is 1 we soon arrive at a contradiction in our mathematics. To give some meaning to $\frac{0}{0}$, Isaac Newton (1642-1727), Gottfried Wilhelm Leibniz (1646-1716), and others introduced the branch of mathematics called the *calculus*.

We shall illustrate with numbers a method whereby we can show that *zero times any number is zero*. Let us consider 0×5 . We can write 0 as $1 - 1$. You can easily verify that the law which follows applies for all positive integers a, b , and c , and in fact holds for all numbers. *Multiplication by zero*

$$(10) \quad (a - b) \times c = ac - bc.$$

Therefore, we may write

$$(5 - 2) \times 3 = 5 \times 3 - 2 \times 3,$$

or, in words, the quantity $(5 - 2)$ times 3 is the same as 5 times 3 minus 2 times 3. Applying this law to the product of 0 and 5 by writing zero as $(1 - 1)$, we get

$$(1 - 1) \times 5 = 1 \times 5 - 1 \times 5 = 0.$$

Therefore, $0 \times 5 = 0$ and we conclude that the product of zero and any other number is also zero.

*Division by
zero*

We shall show why *division by zero is not allowed* in mathematics. That a number not zero cannot be divided by zero can be illustrated with the fraction $\frac{5}{0}$. We recall that c is the quotient of a number a by a number b if $a = bc$. It follows that if there is a number m such that

$$\frac{5}{0} = m,$$

then we may write

$$5 = 0 \times m.$$

Contradiction

But since $0 \times m = 0$ instead of 5, we have arrived at a contradiction. Thus we conclude that there is no number m equal to $\frac{5}{0}$.

Suppose now that there is a number m such that

$$\frac{0}{0} = m.$$

From the meaning of *quotient* this expression may be rewritten as

$$0 = 0 \times m.$$

The last relation is satisfied by *any number* m . It would be absurd to say that $\frac{0}{0}$ is any number, such as 10, -15 , and 100, all at the same time. Therefore, $\frac{0}{0}$ is not a number.

*Addition and
subtraction
with zero*

We have treated multiplication and division by zero. To complete the picture you will note that if you add zero to or subtract zero from a given number, you again obtain this same number.

Signs

From the rules $(-1) \times a = -a$, $(-1) \times (-1) = +1$, and $(-1) \times (+1) = -1$, the *product of two negative numbers is positive as is the product of two positive numbers*, whereas the *product of a positive number and a negative number is negative*. The same remarks hold for division since the division of a by b can be thought of as the product of a by the reciprocal $(1/b)$ of b . If no plus or minus sign is written before a number, the sign before the number is understood to be plus.

The sign conventions and procedures for handling fractions given above enable us to simplify fractions whose numerators and denominators may themselves contain fractions. Thus

$$\frac{\frac{2}{3} \times \frac{8}{9} - \frac{5}{6}}{\frac{4}{9} - \frac{1}{3} \times 2} = \frac{\frac{16}{27} - \frac{5}{6}}{\frac{4}{9} - \frac{2}{3}} = \frac{\frac{32}{54} - \frac{45}{54}}{\frac{8}{18} - \frac{20}{18}} = \frac{-\frac{13}{54}}{-\frac{12}{18}} = \frac{1}{12} \times \frac{9}{2} = \frac{3}{4 \times 2} = \frac{3}{8}.$$

Similarly,

$$\begin{aligned} 2 + \frac{1}{3 - \frac{4}{1 + 6}} &= 2 + \frac{1}{3 - \frac{4}{7}} = 2 + \frac{1}{\frac{21 - 4}{7}} = \frac{34 + 7}{\frac{17}{8}} \\ &= \frac{\frac{41}{8}}{\frac{1}{8}} = \frac{41}{17} \times \frac{8}{1} = \frac{328}{17}. \end{aligned}$$

Any number that can be written as the quotient of two integers (division by zero excluded) is called a *rational number*. The numbers

$$0, 2, -3, \frac{4}{3}, 1.741, 0.333 \dots,$$

are rational numbers. The number zero is rational because it can be written as $\frac{0}{1}$. Examples of numbers which are not rational will be given presently.

Algebra is primarily concerned with the addition, subtraction, multiplication, and division of numbers (division by zero excluded). Under all of these operations, rational numbers always yield rational numbers. The rational numbers are the ordinary fractions of arithmetic, negative integers being included in the numerators and denominators. Because rational numbers with the above operations go into rational numbers, the rational numbers are said to form a *system of numbers*. When we work with them and limit our mathematical operations to those already mentioned, we are in a sort of prison from which we cannot escape.

REAL NUMBERS

The quotients of positive integers met the needs of the Egyptians and Babylonians who made use of their knowledge of mathematics in the construction of the pyramids, the evolution of the calendar, and other advances in astronomy and engineering. Much later the Greeks, who were greatly interested in mathematics for its own sake, continued investigations into the properties of numbers and added much to the knowledge of them.

Through its interest in the Pythagorean theorem, one of the most significant accomplishments in mathematics, the school of

Pythagoras about 500 B.C. was led to devise "irrational numbers," which will be discussed later. The importance of the Pythagorean theorem is evident when we consider that it is used in marine and aeronautical navigation, land surveying, airplane design, bridge construction, and indirectly to carry out innumerable functions of modern life.

*Geometrical
concepts*

To state the Pythagorean theorem we shall need some concepts from plane geometry. If you remove from a straight line everything to one side of a fixed point P on the line, you obtain a *ray* as shown in Fig. 216. The point P is the initial point of the ray.



FIG. 216. RAY

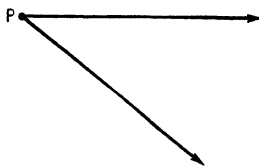


FIG. 217. ANGLE

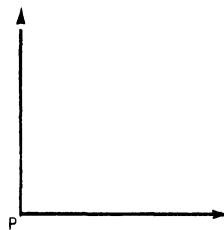


FIG. 218. RIGHT ANGLE

*Angles and
triangles*

Two rays with the same initial point form an *angle* as shown in Fig. 217. Two perpendicular rays form a *right angle* as in Fig. 218. Thus the hands of a clock at 12 : 15 form a right angle. A geometric figure bounded by three lines is said to be a *triangle* because the sides form three angles. Two triangles are shown in Fig. 219.

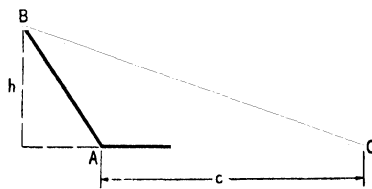
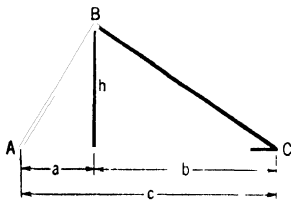


FIG. 219. TRIANGLES ABC WITH ALTITUDE h AND BASE c

The points A , B , and C of either triangle in Fig. 219, where the lines intersect, are called the *vertices* of the triangle. The line segments AC , CB , and BA are called the *sides* of the triangle. The notation AC used here designates the line segment from the point A to the point C . The entire triangle is designated as the triangle ABC . The angle "formed" by the sides AB and AC is called the

angle A . Sometimes this angle is termed the *angle CAB*. A triangle with a right angle is called a *right triangle*. The triangle of Fig. 220 is a right triangle with right angle C and sides designated by a , b , and c . The angles A , B , and C are opposite the sides a , b , and c respectively. This is a convention often used in treating geometric figures. We shall also let a denote the length of the side BC . The side c opposite the right angle C is termed the *hypotenuse* of the right triangle.

We know that $4^2 = 4 \times 4$, or in general terms a^2 (a "squared") means a times a . The *Pythagorean theorem* states that *the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides*. In terms of the right triangle of Fig. 220 this implies that

$$(11) \quad a^2 + b^2 = c^2.$$

The Pythagorean theorem is of such importance that proof of it will be given. Since this proof will be based on the use of areas, we shall need a few special geometrical concepts.

The height of a triangle is called the *altitude* of the triangle. The side AC of each triangle in Fig. 219 is the *base*, a term to be used also for the length of the side AC . The vertical line in the left triangle of Fig. 219 cuts this triangle into two right triangles. In Fig. 221 two triangles identical with the triangle of Fig. 220 are adjoined to build up a rectangle. The area of this rectangle is ab . The area of the triangle in Fig. 220 is therefore said to be $\frac{1}{2}(ab)$.

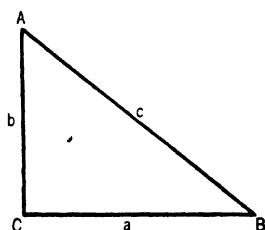


FIG. 220. RIGHT TRIANGLE

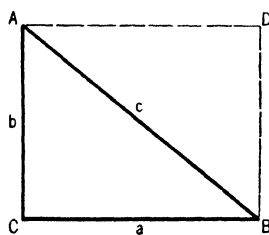


FIG. 221. RECTANGLE CUT INTO RIGHT TRIANGLES

The areas of the right triangles in the left triangle of Fig. 219 are $\frac{1}{2}(ah)$ and $\frac{1}{2}(bh)$ respectively. The area of the entire triangle is thus

$$\frac{1}{2}h(a + b).$$

A similar treatment holds for the other triangle of Fig. 219. In words, the *area of a triangle is one-half of the altitude times the base*.

If a triangle has a base which is 3 in. in length and an altitude of 4 in., the area of the triangle is $\frac{1}{2} \times 3 \times 4$, or 6 sq. in.

*Proof of
Pythagorean
theorem*

In Fig. 222 squares have been placed on the sides of a right triangle. Using the same type of notation for rectangles as for

triangles, we designate the square on the hypotenuse to be $ABDE$. This square has the area c^2 , whereas the squares $BCHK$ and $AFGC$ have the areas a^2 and b^2 respectively. To prove that $c^2 = a^2 + b^2$ we therefore wish to show that the area of the square $ABDE$ is the sum of the areas of the squares $BCHK$ and $AFGC$.

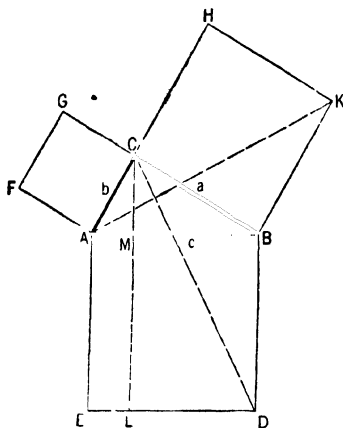


FIG. 222. RIGHT TRIANGLE BOUNDED BY SQUARES

The broken line CL is drawn parallel to the line AE . Broken lines join the points A and K , and the points C and D respectively. The side DB of the triangle DCB is identical in length with the side

AB of the triangle AKB because they are sides of the same square. Similarly, the side BC of the triangle DCB is identical with the side BK of the triangle AKB . The angles DBC and ABK are the same; that is, one can be superimposed on the other because each is equal to a right angle plus the angle ABC .

*Geometrical
relations*

It follows that the triangle DCB can be superimposed on the triangle AKB , for they are identical. The side BK of length a can be taken to be the base of the triangle AKB . The altitude of the triangle AKB is the distance CB , which is also a . The area of the triangle AKB is thus $\frac{1}{2} a^2$. The triangle DCB has the base BD of length c and altitude LD . Since this triangle has the same altitude and base as the rectangle $LMBD$, the area of the rectangle $LMBD$ is twice the area of the triangle DCB , and hence is a^2 .

We have thus proved that the area of the rectangle $LMBD$ is the same as the area of the square $BCHK$, namely a^2 . In the same way we can show that the area of the rectangle $AMLE$ is equal to the area of the square $AFGC$, that is, b^2 . Since the area of the square $ABDE$ is the sum of the areas of the rectangles $AMLE$ and $LMBD$, we have shown that $c^2 = a^2 + b^2$, as stated in equation 11, is true.

Q.E.D.

The sides of a right triangle distinct from the hypotenuse are called the *legs*. We recall that $\sqrt{29}$ means the *square root* of 29. The square root of 29, or $\sqrt{29}$, is a number whose square is 29. If the lengths of the legs of a right triangle are 2 and 5 respectively, the Pythagorean theorem states that the hypotenuse of the triangle has the length $\sqrt{2^2 + 5^2}$, that is $\sqrt{4 + 25}$, or $\sqrt{29}$. We shall show later how to approximate $\sqrt{29}$ by a decimal number.

It was early discovered that if the legs of a right triangle are both equal to one, the length $\sqrt{2}$ of the hypotenuse cannot be written as a rational number, that is, the quotient of one integer by another. We can prove this as follows. *Irrationality of $\sqrt{2}$*

We start by assuming that $p/q = \sqrt{2}$, where p and q are positive integers with no common factor other than 1. Squaring both sides of the above equation, and then multiplying both sides by q^2 , we obtain the equation

$$p^2 = 2 q^2.$$

If q contains a factor 2, the square of q contains this factor twice. Thus the square of a positive integer can contain only an even number of factors equal to 2. For example, if $q = 2 \times 2 \times 2 \times 3$, the square of q has six factors equal to 2. It follows that $2 q^2$ has an odd number of factors equal to 2, while p^2 has an even number, if any at all. We have thus arrived at a contradiction, for which reason we conclude that $\sqrt{2}$ cannot be written as the quotient of two positive integers, and in fact, of any two integers.

Since there are numbers, like $\sqrt{2}$, which are not rational, the number $\sqrt{2}$ and the rational numbers belong to a larger system which we shall describe but not define because an exact treatment requires some of the most advanced concepts of mathematics. René Descartes (1596–1650), philosopher and mathematician, wished to be able to distinguish between the points on a line just as we differentiate people by their names, or by associating numbers as we do with automobiles in a given state. Of course, you can count all the automobiles, and designate each one by a distinct positive integer. But not all the points on a line can be counted, and it is, therefore, impossible to assign a distinct integer to each point. *Points on lines*

Descartes marked off equal distances from a fixed point 0 on a line as shown in Fig. 223, and to each one of these marked points he assigned an integer, namely, the *directed distance* from the fixed *Numbering points*

point 0. He used the distance from 0 to the point marked 1 as the unit of measure. The direction from 0 was taken care of by the sign $+$ or $-$ of the integer, the plus sign referring to a point to the right of 0, minus referring to the left. Descartes then further

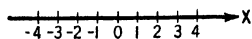


FIG. 223. COORDINATE AXIS

subdivided the line segments between the marked points, assigning to the point midway between 0 and 1 the value $\frac{1}{2}$, to the point midway between 0 and -1 the value $-\frac{1}{2}$, to the point one-third of the distance from 1 to 2 the value $\frac{2}{3}$, and so on. Thus to each rational number there is an assigned point on the line.

Real numbers

When the rational numbers are connected in this manner to points on the line, there are still points left over in the mathematical sense. There is certainly a point between 1 and 2 to which $\sqrt{2}$ may be assigned, and the number $\sqrt{3}$ also goes with a point in this range. We recall that the length of the circumference of a circle is the length of the diameter times π . This famous number π (3.14159 to five decimal places) must be assigned to a point between 3 and 4.

The numbers thus associated with the points on a line are called the *coordinates* of the points on the line, and the coordinates of the points on a line form what are called the *real numbers*. Real numbers which are not rational are termed *irrational*. The coordinates of points to the right of the *origin* 0 are the *positive numbers*, and coordinates to the left are the *negative numbers*.

Irrationality of π

Whether or not π is rational is a problem which baffled the best mathematicians for two thousand years. Solved only about sixty years ago by Ferdinand Lindemann, who worked with exact determinations of π obtained by means of the calculus, Lindemann showed that π cannot be expressed as a quotient of two integers, and is therefore irrational.

The real number system

In advanced mathematics it is proved that the addition, subtraction, multiplication, and division (division by zero excluded) of real numbers always yields real numbers, so that *the real numbers form a system*. This system satisfied scientists and engineers until the Middle Ages, when imaginary numbers were discovered. Imaginary numbers, associated with the $\sqrt{-1}$, will be discussed later.

New number systems

The process of discovering new systems of numbers is still going on. Such systems were originally and are still likely to be developed only by mathematicians interested mainly in the philosophy of

mathematical concepts. Later such fundamental studies become of material benefit to humanity as practical applications are found by scientists and engineers.

FUNDAMENTAL OPERATIONS OF ALGEBRA

The product $2 \times 2 \times 2 \times 2 \times 2 \times 2$ is written as 2^6 for the *Powers* sake of brevity. More generally, a product of n successive a 's is said to be the n th power of a , and is written as a^n . In this notation 1,000,000 is written as 10^6 .

The product of the n th power of a by the m th power of a is a product of $(n + m)$ factors, all equal to a . This gives the rule:

$$(12) \quad a^n \times a^m = a^{n+m}.$$

For example,

$$5^4 \times 5^3 = (5 \times 5 \times 5 \times 5) \times (5 \times 5 \times 5) = 5^7.$$

You have perhaps already become accustomed to the use of the *Terms* term *factor* for any number multiplied by another. There is also a name for a number added to another, namely, the word *term* itself. In the expression

$$2x^2 + 3ay + 5$$

the quantities $2x^2$, $3ay$, and 5 are terms; 2, x^2 , 3, a , and y are factors, not of the whole expression, but of these terms.

It is to be emphasized that when letters are used in expressions such as the above, they represent numbers from a system with which we are understood to be working, and not merely abstract symbols. Unless otherwise specified, in this chapter such letters will always represent numbers from the *real number system*. For example, the product ab is the product of a real number a by a real number b , and not the product of one letter by another. Algebra is not the study of letters, but rather the study of numbers. The use of letters merely enables us to treat all numbers from a given class at the same time.

The factors in the product

Products

$$(2 + 6 + 4)(5 + 7 + 1)$$

are 12 and 13 respectively, whence the product has the value 156. If we multiply each term of one factor by each term of the other and add the resulting terms, we have

$$\begin{aligned} 2 \times 5 + 2 \times 7 + 2 \times 1 + 6 \times 5 + 6 \times 7 + 6 \times 1 + 4 \times 5 + 4 \times 7 \\ + 4 \times 1 = 10 + 14 + 2 + 30 + 42 + 6 + 20 + 28 + 4, \end{aligned}$$

which also adds up to 156. This illustrates the principle which follows:

To multiply one expression by another, multiply each term in the first expression by each term in the second expression, and add the resulting products.

Thus,

$$\begin{aligned}(3 + 2x + a)(2x + 4) &= 6x + 12 + 4x^2 + 8x + 2ax + 4a \\ &= 12 + 4a + 14x + 2ax + 4x^2.\end{aligned}$$

This rule also applies with negative terms. For example,

$$\begin{aligned}(2 - 4)(3 + 7 - 1) &= 2 \times 3 + 2 \times 7 - 2 \times 1 - 4 \times 3 - 4 \times 7 + 4 \times 1 \\ &= 6 + 14 - 2 - 12 - 28 + 4 = 24 - 42 = -18.\end{aligned}$$

*Arrangement
of terms in
products*

If the terms in the factors of a product involve powers of the same number, say x , the multiplication can be conveniently performed by arranging the terms in descending powers of the number, as in the following illustration of the procedure for multiplying $4x^2 + 7x + 1$ by $x + 4$.

$$\begin{array}{r}4x^2 + 7x + 1 \\ x + 4 \\ \hline 4x^3 + 7x^2 + x \\ \underline{\quad\quad 16x^2 + 28x + 4 \quad\quad} \\ 4x^3 + 23x^2 + 29x + 4\end{array}$$

It will be observed that in the computation above, terms involving the same powers of x are arranged in the same vertical column for convenient addition. Multiplication of $x^2 - 2 - 3x^2 + x^4$ by $x^2 + 2x^3 + 1$ can be performed as follows.

$$\begin{array}{r}x^4 - 3x^2 + x - 2 \\ 2x^3 + x^2 + 1 \\ \hline 2x^7 \quad \quad - 6x^5 + 2x^4 - 4x^3 \\ \quad x^6 \quad \quad - 3x^4 + \quad x^3 - 2x^2 \\ \quad \quad \quad x^4 \quad \quad - 3x^2 + x - 2 \\ \hline 2x^7 + x^6 - 6x^5 \quad \quad - 3x^3 - 5x^2 + x - 2\end{array}$$

The product of the expressions in this case is

$$2x^7 + x^6 - 6x^5 - 3x^3 - 5x^2 + x - 2.$$

Procedure

In carrying out the multiplication we have first taken the product of $2x^3$ by each term in the first expression, obtaining thus the first line of terms under the long bar. The second line of terms

under this bar is the result of multiplying x^2 by each term of the first expression, whereas the third line of terms, just above the bottom bar, is the product of 1 by the first expression. The bottom row is derived by adding the like terms in each vertical column between the two bars.

On the left below we have carried out the ordinary long division of 32,342 by 183. *Long division*

| | |
|--|--|
| $ \begin{array}{r} 176 \\ 183 \overline{) 32342} \\ \underline{183} \\ 1404 \\ \underline{1281} \\ 1232 \\ \underline{1098} \\ 134 \end{array} $ | $ \begin{array}{r} 1 \times 10^2 + 7 \times 10 + 6 + 134/183 \\ 183 \overline{) 32342} \\ \underline{18300} \\ 14042 \\ \underline{12810} \\ 1232 \\ \underline{1098} \\ 134 \end{array} $ |
|--|--|

You will find that this procedure of long division is just a system of trial divisions with intermediate subtractions. On the right we have written the quotient in terms of powers of 10. First we find the largest multiple of 10^2 which when multiplied by 183 gives a number not greater than 32,342. This multiple of 10^2 is 1×10^2 because 100×183 is less than 32,342 while 200×183 is greater than 32,342. Subtracting 100×183 from 32,342 we find that we have 14,042 left over. We next find the largest multiple of 10 such that the product of this multiple by 183 is not greater than the remainder 14,042. The desired multiple of 10 is 7×10 because 70×183 is less than 14,042, while 80×183 is more than 14,042. Subtracting 70×183 (or 12,810) from 14,042 we have 1232 left over. Thus far we have 170 in the quotient. We must add another 6 to the quotient since 183 will go into 1232 six times, with a remainder 134. We add the *remainder* 134 divided by the *divisor* 183 to the *quotient*. To check this computation multiply the divisor 183 by the quotient $(176 + \frac{134}{183})$. *Remainder*

The standard procedure of dividing one algebraic expression by another is like the long division method above. The system of arranging terms according to descending powers of one number, as used in the multiplication of algebraic expressions above, is also helpful in dividing one algebraic expression by another. *Division of algebraic expressions*

We shall divide $1 + x^3 - 3x^2$ by $2x + x^2 + 3$. The computation performed below will be explained step by step.

Algebraic
division

$$\begin{array}{r}
 \text{(divisor)} \quad x^2 + 2x + 3 \overline{) x^5 - 3x^2 + 1} \quad \text{(quotient)} \\
 \text{(dividend)} \\
 \underline{x^5 + 2x^4 + 3x^3} \\
 -2x^4 - 3x^3 \\
 \underline{-2x^4 - 4x^3 - 6x^2} \\
 x^3 + 3x^2 \\
 \underline{x^3 + 2x^2 + 3x} \\
 x^2 - 3x \\
 \underline{x^2 + 2x + 3} \\
 -5x - 2 \quad \text{(remainder)}
 \end{array}$$

The quotient of $1 + x^5 - 3x^2$ by $2x + x^2 + 3$ is $x^3 - 2x^2 + x + 1$ with the remainder $(-5x - 2)$. In the computation above, the dividend $x^5 - 3x^2 + 1$ is written immediately under the bar, and the divisor $x^2 + 2x + 3$ is written to the left of the dividend.

First term of
quotient

The first term x^2 of the divisor is divided into the first term x^5 of the dividend yielding the first term of the quotient x^3 written above the bar. The quotient term x^3 is now multiplied by the divisor and the result written under the dividend, a bar being placed beneath. The quantity just above the bar is subtracted from the dividend giving us the *temporary remainder* as indicated below and $-3x^2 + 1$, which is not repeated.

$$\begin{array}{r}
 \text{(divisor)} \quad x^2 + 2x + 3 \overline{) x^5} \quad \text{(first term of quotient)} \\
 \text{(dividend)} \\
 \underline{x^5 + 2x^4 + 3x^3} \\
 -2x^4 - 3x^3 \quad \text{(temporary remainder)}
 \end{array}$$

It is to be emphasized that appropriate *spaces* be left between the terms of the dividend to take care of *missing terms*.

Procedure

The first term x^2 of the divisor is next divided into $-2x^4$, the first term of the temporary remainder, and the result becomes the second term of the quotient. This quotient term $-2x^2$ is multiplied by the divisor and the result placed underneath the temporary remainder $-2x^4 - 3x^3$. Subtraction (top of page) yields $x^3 + 3x^2 + 1$, of which 1 is omitted from the new temporary remainder, and the procedure of division by the first term x^2 of the divisor is repeated. This process is carried on until the remainder $-5x - 2$ is obtained. Since x^2 does not “divide into” $-5x$, the division ends.

Checking a
division

To check the above division, simply multiply the *quotient* $x^3 - 2x^2 + x + 1$ by the *divisor* $x^2 + 2x + 3$ and add the *remainder* $-5x - 2$. The result should be the *dividend* $x^5 - 3x^2 + 1$.

In dealing with quotients of numbers we showed that fractions *Factoring* may be simplified by removing factors possessed in common by numerator and denominator. It is therefore desirable to have rules for finding the factors of a given algebraic expression. Among these the following are particularly helpful since they occur frequently in engineering calculations:

- (13) $ab + ac = a(b + c)$
 (14) $a^2 + 2ab + b^2 = (a + b)^2$
 (15) $a^2 - 2ab + b^2 = (a - b)^2$
 (16) $a^2 - b^2 = (a - b)(a + b)$

Rule 13 has already been used above. We can verify rule 14 by *Proof* performing the multiplication

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ \quad ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

The others may be established in the same way. In Fig. 224 the area of the whole square is $(a + b)^2$. Adding the areas of the regions into which this square is cut up, you will obtain another verification of rule 14,

$$(a + b)^2 = a^2 + 2ab + b^2.$$

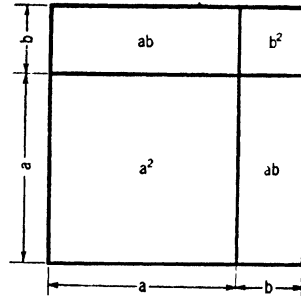


FIG. 224. FIGURE TO ILLUSTRATE

$$(a + b)^2 = a^2 + 2ab + b^2$$

Some illustrations of the above rules follow.

- $x^2 + 5x = x(x + 5)$ (Rule 13)
 $xy + y^2 - 2y = y(x + y - 2)$ (Rule 13)
 $x^2 + 6x + 9 = (x + 3)^2$ (Rule 14)
 $4a^2 - 20ab + 25b^2 = (2a - 5b)^2$ (Rule 15)
 $4x^2 - 9 = (2x - 3)(2x + 3)$ (Rule 16)

Examples of factoring

The following examples involve more than two factors.

$$\begin{aligned} x^3 + 3x^2 - 4x - 12 &= x(x^2 - 4) + 3(x^2 - 4) = (x + 3)(x^2 - 4) \\ &= (x + 3)(x - 2)(x + 2) \\ \frac{x^2 - 4}{2x + 4} &= \frac{(x - 2)\cancel{(x + 2)}}{2\cancel{(x + 2)}} = \frac{x - 2}{2} \end{aligned}$$

SQUARE ROOTS

The problem of finding the hypotenuse of a right triangle is *Accuracy* only one of many places where the square root of a number is

used. With the slide rule (to be described later), square roots can be obtained to three places at most. We shall describe the standard process for the extraction of the square root of a number to as many places as desired. As in the case of division, this process is one of successive subtractions of properly chosen quantities.

General square Rule 14 above gives the square of a sum of two terms a and b . This rule can be generalized to apply to the square of the sum of any number of terms, as follows.

The square of a sum is the sum of the squares of the terms plus the sum of twice all possible products which can be formed by multiplying distinct terms.

For example,

$$\begin{aligned}(2 + 3 + 5)^2 &= 2^2 + 3^2 + 5^2 + 2(2 \times 3) + 2(2 \times 5) + 2(3 \times 5) \\ &= 4 + 9 + 25 + 12 + 20 + 30 = 100.\end{aligned}$$

Similarly, $(a - 3)^2$ is $a^2 + 9 - 6a$. (See rule 15.)

Extraction of square root

We shall illustrate the process of extracting the square root of a number, and then show why it is correct. You will recall that a positive integer which is the square of another integer is termed a *perfect square*. For instance, 1, 4, 9, and 16 are perfect squares.

To obtain the square root of a number such as 52,346.008 you should perform the steps below.

1. Starting at the decimal point mark off successive pairs of digits in each direction by means of primes. You get

$$5'23'46.00'8.$$

First digit

2. Find the largest perfect square which is less than or equal to the number to the left of the first prime. This perfect square is 4 in our example.

3. Write this perfect square, 4, beneath the number, and again as a divisor to the left below. So far you have

$$\begin{array}{r} \sqrt{5'23'46.00'8} \\ 4 \\ 4 \overline{) } \end{array}$$

4. Place the square root 2 of the perfect square above the given number. You have

$$\begin{array}{r} 2 \\ \sqrt{5'23'46.00'8} \\ 4 \\ 4 \overline{) } \end{array}$$

5. Perform the indicated subtraction and bring down the next two digits. This gives

$$\begin{array}{r} 2 \\ \overline{) 5'23'46.00'8} \\ 4 \\ \hline 4 \overline{) 123} \end{array}$$

6. Choose a digit x to follow the digit 4 in the divisor so that $(40 + x)$ times x is the largest integer which is still less than or equal to the number beneath the last bar. For the case at hand the number beneath the bar is 123. When x is 2 the product $(40 + x)x$ or 42 by 2 is 84, which is less than 123, whereas the product of 43 by 3 is 129, which is too great. The desired value of x is thus 2.

7. Place the digit 2 after the digit 4 as a new divisor at the left and after the first 2 at the top, and write 2×42 or 84 beneath 123. You then have

$$\begin{array}{r} 22 \\ \overline{) 5'23'46.00'8} \\ 4 \\ \hline 42 \overline{) 123} \\ \underline{84} \end{array}$$

8. Perform the indicated subtraction. The result is 39. Bring down the next two digits, double the partial root 22 and write the divisor 44 to the left. Hence, you now have

$$\begin{array}{r} 22 \\ \overline{) 5'23'46.00'8} \\ 4 \quad . \\ \hline 42 \overline{) 123} \\ \underline{84} \\ 44 \overline{) 3946} \end{array}$$

9. Choose a digit x so that $(440 + x)$ times x is no larger than the number 3946 beneath the bottom bar, but $[440 + (x + 1)]$ times $(x + 1)$ is larger than 3946. In this case x is 8, for 448 times 8 is 3584, while 449 times 9 is 4041.

10. Place 3584 beneath 3946, and adjoin the digit 8 to the right of both 44 and 22. Having arrived at the decimal point in the given number, place a decimal to the right of 228. You have completed these steps:

$$\begin{array}{r}
 228. \\
 \overline{) 5'23'46.00'8} \\
 4 \\
 \hline
 42 \overline{) 1\ 23} \\
 84 \\
 \hline
 448 \overline{) 39\ 46} \\
 35\ 84
 \end{array}$$

*Meaning
of result*

Actually, the square of 228 is the largest perfect square that does not exceed 52,346. The next two digits should now be brought down, and the process repeated. The computation carried out to two decimal places is

$$\begin{array}{r}
 228.79 \\
 \overline{) 5'23'46.00'8} \\
 4 \\
 \hline
 42 \overline{) 1\ 23} \\
 84 \\
 \hline
 448 \overline{) 39\ 46} \\
 35\ 84 \\
 \hline
 4567 \overline{) 3\ 62\ 00} \\
 3\ 19\ 69 \\
 \hline
 45749 \overline{) 42\ 31\ 80} \\
 41\ 17\ 41 \\
 \hline
 1\ 14\ 39
 \end{array}$$

We have stopped at the remainder 11,439. We could have brought down the next two digits, which are zeros, and continued, but such precision would rarely be needed.

*Explanation of
square root
process*

In step 2 above, the largest multiple of 100 whose square is not greater than the given number 52,346 is found. This partial root, which we will call a , is 200. In step 5 we subtract a^2 from the original number which gives us the equivalent of this simple calculation:

$$52,346.008 - 200^2 = 12,346.008.$$

In step 6 we find the largest multiple of 10, which we shall call b , so that $(a + b)^2$ with $a = 200$ does not exceed 52,346.008. This value of b is 20. You will recall that $(a + b)^2 = a^2 + 2ab + b^2$.

*Successive
subtraction*

The number 420 (of which the zero is not written) in step 7 is the term $(2a + b)$, whereas 8400 is $(2a + b)b$, that is, the sum $2ab + b^2$. When we arrive at step 9 we have removed $a^2 + 2ab + b^2$ by subtraction from the given number, subtracting first a^2 and then $2ab + b^2$, with $a = 200$ and $b = 20$. The process of the extraction of the square root is thus a method of finding a, b, c, \dots

which are successive multiples of powers of 10, and the multiples of the numbers 1, $\frac{1}{10}$, \dots so that

$$52,346.008 = (a + b + c + \dots)^2.$$

A second example of this procedure for obtaining the square root follows.

$$\begin{array}{r} 0.067 \\ \overline{0.00'45'37} \\ 6 \overline{)45} \\ \underline{36} \\ 127 \overline{)9 \ 37} \\ \underline{8 \ 89} \\ 48 \end{array}$$

ROUNDING OFF NUMBERS

The computations and answers to engineering problems are usually in terms of rational numbers. Instead of using $\sqrt{2}$, it is customary to employ 1.414, which is an approximation to $\sqrt{2}$. In place of π the number 3.1416, or even $\frac{22}{7}$, is often employed. The representation 3.1416 is a "rounded off" form of the actual decimal representation of π . The latter representation requires an unlimited number of decimal places, and therefore cannot even be written as a decimal number. Although it is necessary to approximate irrational numbers by rational numbers, it is customary also to approximate rational numbers by simpler rational numbers with fewer decimal places. Thus in place of 5.73634219 an engineer would probably use 5.736. He might even be able to get along with 5.7 if the number represented cubic yards of earth or tons of coal. This process of dropping decimals in a number is called *rounding off* the number. *Use of decimal numbers*

The American Standards Association recommends the following method of rounding off numbers: *Method of rounding off numbers*

(1) If a digit larger than 5 is dropped, the preceding digit is increased by 1. Hence 3788 and 3.146 are rounded off to 3790 and 3.15 respectively when the last digits are dropped. We cannot drop the zero in 3790 because we need it to locate the position of the decimal point. We shall, however, speak of going from a digit not zero to zero as dropping the given digit.

(2) If a digit smaller than 5 is dropped, the preceding digits are left unchanged. Thus 3782 and 15.3 can be rounded off to 3780 and 15 respectively.

The digit 5

(3) *If the last digit in a number is 5*, the number can be rounded off to two equally close numbers. Thus 96.45 is midway between the numbers 96.4 and 96.5. If we always dropped the 5, we would invariably decrease numbers ending in 5 and adversely effect the accuracy of summations. A much more desirable method is that of making the preceding digit an even number.

Thus in dropping the digit 5 the numbers 34500, 37.05, 347500, and 0.0135 can be rounded off to 34000, 37.0, 348000, and 0.014 respectively. In dropping the 5 from the number 0.00995 we notice that the digit 9 preceding 5 becomes 10; but this increases the preceding 9 to 10 giving 0.01 as the rounded or approximate value.

Significant digits

You should notice that we did not round off the number 37.05 to 37 because the 0 in 37.0 is *significant* in that it tells us the digit to the right of the decimal place cannot be 1, for example, which fact cannot be inferred from the number 37 alone. In fact, 37 would be the number obtained by rounding off 36.6, 36.7, 36.8, 36.9, 37.1, 37.2, 37.3, or 37.4. In working with numbers you should be sure that each digit is significant and needed.

If x is an approximation to a number between $0.95x$ and $1.05x$, the number x is said to be accurate to 5 per cent. Again, when 20 is an approximation to a number between 15 and 25, the number 20 is said to be accurate to 25 per cent.

Number of digits to be kept

The distance an automobile travels in 3 hours is measured and found to be 200 miles. The quotient of the distance 200 by the time 3 gives the speed of the automobile in miles per hour. It would be absurd to perform the division and say that the automobile averaged 66.666667 m.p.h. for the 3 hours. There are errors in measuring both the distance and time, so that the first two or three digits are probably all that should be kept, the result in the case of three digits being 66.7 m.p.h.

Simple rules for the number of digits to be dropped in computations cannot be given in such a way as to apply to all situations. From experience you will be able to know how many "significant digits" to retain in a particular engineering problem. That rules can be developed to cover individual computations will be illustrated with a couple of examples.

Suppose that we are interested in the product AB of numbers A and B where there are inaccuracies in the values of A and B . These numbers may, for example, be the lengths of the sides of a

rectangle, and AB is the area. We assume that A and B are known to be accurate to x per cent and y per cent respectively. The number A is then an approximation of a number between

$$A - \frac{x}{100} A \quad \text{and} \quad A + \frac{x}{100} A.$$

Similarly B is an approximation of a number between

$$B - \frac{y}{100} B \quad \text{and} \quad B + \frac{y}{100} B.$$

The product AB becomes then an approximation of a number between

$$(17) \quad \left(A - \frac{x}{100} A\right)\left(B - \frac{y}{100} B\right) \quad \text{and} \quad \left(A + \frac{x}{100} A\right)\left(B + \frac{y}{100} B\right).$$

The first expression in (17) is obtained by replacing A and B by the minimum values $A - \frac{x}{100} A$ and $B - \frac{y}{100} B$ respectively. The other expression is derived from the maximum values.

We subtract AB from the expressions in (17) and obtain the following pair of limits after simplification:

$$(18) \quad -\left(\frac{x}{100} + \frac{y}{100} - \frac{xy}{10,000}\right)AB \quad \text{and} \quad \left(\frac{x}{100} + \frac{y}{100} + \frac{xy}{10,000}\right)AB.$$

The error in AB due to the errors in A and B lies between the expressions of (18).

Since $xy/10,000$ is usually small compared to the terms $x/100$ and $y/100$, we are ordinarily justified in dropping the term $xy/10,000$ from the expressions in (18). The expressions in (18) then become

$$(19) \quad -(x + y)\% \text{ of } AB \quad \text{and} \quad (x + y)\% \text{ of } AB.$$

In this case we have: *If the factors A and B are accurate to x per cent and y per cent respectively, the product AB is accurate to $(x + y)$ per cent.*

For example, let us suppose that a rectangular lot is paced off and found to be about 100 ft. by 200 ft. We write $A = 100$, and $B = 200$. Let us suppose further that A and B are paced by different persons known to be accurate in their pacing to 4 per cent and 5 per cent respectively.

By the above rule the product AB is accurate to $(4 + 5)$ per cent, that is, 9 per cent. Since the area AB in square feet is $AB = 100 \times 200 = 20,000$, the number AB may be in error by

$$\frac{9}{100} \times 20,000 = 1,800.$$

The answer 20,000 sq. ft. may thus be in error by as much as 1800 sq. ft. The true area then lies between 18,200 and 21,800 sq. ft.

Check

We shall carry through the computation of this example directly. The true width and length of the rectangle may be as much as 104 and 210 ft. respectively. In this case the area is 104×210 , or 21,840 sq. ft. Subtracting 20,000 from the last figure, we find that the error in AB is 1840. If on the other hand the true width and length are 96 and 190 respectively, the correct area of the rectangle is 18,240 sq. ft. The product AB is now in error by 1760. You can see that 1800 is a good approximation to 1840 and 1760, the approximation being due to our use of the limits (19) rather than the exact expressions (18).

Error in quotient

By methods similar to those used above, we can show that if A and B are accurate to x per cent and y per cent respectively, the quotient A/B is accurate to $(x + y)$ per cent. To get this result we have neglected terms such as $xy/10,000$, which are ordinarily small compared to the terms retained. Hence, for practical purposes we may say in the case of either a quotient or a product that the possible percentage error in the answer is the sum of the possible percentages of error in the terms.

Error in speed

As an example, let us suppose that the distance an airplane traveled in 2 hours at constant speed was measured and found to be 300 miles. We shall find the possible error in the speed obtained as 150 m.p.h. by dividing 300 by 2 (distance by time), upon the basis that the distance is accurate to 2 miles, and the time to 1 min.

The distance is now accurate to $\frac{2}{300}$, or about 0.67%, whereas the time is accurate to $\frac{1}{120}$, or approximately 0.83%. The speed 150 m.p.h. is thus accurate to 0.67% + 0.83%, or 1.5%. Taking 1.5% of 150 we have 2.25. The error in the speed may thus be as much as 2.25 m.p.h., so that the true speed lies between 147.75 and 152.25 m.p.h.

Several significant digits

It is sometimes necessary in engineering calculations to work with several significant digits. This is especially true when we must employ a small difference of large values. Such an example arises when one considers an ordinary compressed spring in the shape of a coil. For this spring Hooke's law

(20)

$$F = ks$$

applies, where for a change F in the force compressing the spring, the length of the spring changes by a distance s . The constant k

here is simply called the *spring constant*. If in a given state of compression the compressive force of the spring is measured and found to be 210.8 lb., and a decrease of this force to 210.5 lb. causes an increase of $\frac{1}{100}$ in. in the length of the spring, the constant k is $(210.8 - 210.5) \div 0.01 = 30$ (in pounds per inch), so that an inch change in the length of the spring corresponds to a 30-lb. change in the compressive force. *It would be clearly absurd here to round off the values of the compressive force to three figures before computing the value of k .*

1. What kinds of numbers are the following:

Problems

$$\sqrt{3}, \quad 10^2, \quad 13.076, \quad \frac{7}{5}?$$

2. Perform the indicated operations:

$$(a) \frac{2}{3} + \frac{3}{7}; (b) \frac{5}{8} - \frac{3}{16}; (c) \frac{4}{5} \times \frac{25}{16}; (d) \frac{\frac{2}{3}}{\frac{1}{15}}.$$

$$\text{Ans. (a) } \frac{23}{21}; (b) \frac{7}{16}; (c) \frac{5}{4}; (d) 4.$$

3. Give the values of:

$$(a) 0 \times 2; (b) \frac{3}{0}; (c) \frac{0}{2}; (d) 0 + 8; (e) 0 - 5; (f) \frac{1}{4 - 4}.$$

$$\text{Ans. (a) } 0; (b) \text{ no meaning}; (c) 0; (d) 8; (e) -5; (f) \text{ no meaning}.$$

4. Write as quotients of two integers:

Fractions

$$(a) 0.046; (b) -4; (c) 7; (d) 1\frac{1}{2}; (e) 0.666 \dots; (f) 14\%.$$

$$\text{Ans. (a) } \frac{23}{500}; (b) \frac{-4}{1}; (c) \frac{7}{1}; (d) \frac{3}{2}; (e) \frac{2}{3}; (f) \frac{14}{100}.$$

5. Simplify these expressions:

$$(a) \frac{\frac{1}{2} - \frac{3}{4}}{\frac{8}{3} + \frac{7}{6}}; (b) 1 + \frac{1}{1 + \frac{1}{1+2}}; (c) \frac{1 - \frac{3}{4} \times \frac{12}{5}}{\frac{1}{5}}; (d) \frac{1\frac{9}{4}}{\frac{1}{4}}.$$

$$\text{Ans. (a) } -\frac{3}{20}; (b) \frac{7}{4}; (c) -4; (d) \frac{4}{3}.$$

6. Perform the indicated operations:

$$(a) \frac{-4}{-12}; (b) (-2)(-3)(-\frac{1}{2}); (c) \frac{(-3)(4)(-4)}{(-2)(-8)}.$$

$$\text{Ans. (a) } \frac{1}{3}; (b) -3; (c) 3.$$

7. If a rectangular sheet of steel has sides 3 in. and 4 in. long, what is the length of the diagonal of the sheet?

$$\text{Ans. } 5 \text{ in.}$$

8. A man has tools weighing $\frac{1}{2}$ lb., $\frac{7}{8}$ lb., 1.14 lb., and $1\frac{5}{11}$ lb. respectively. What is the total weight of the tools?

$$\text{Ans. } 6.26 \text{ lb.}$$

9. If 2% of 20,000 manufactured pieces are faulty, and 90% of the faulty pieces are thrown away, how many pieces are discarded?

Percentage

$$\text{Ans. } 360.$$

10. If 15 out of every 500 pistons crack under test, what percentage of the pistons crack?

$$\text{Ans. } 3\%.$$

11. Extract the square roots of the following numbers, using 6 digits in the answer.

Roots and powers

$$(a) 0.000307; (b) 108,000; (c) 1634.15.$$

$$\text{Ans. (a) } 0.017521; (b) 328.634; (c) 40.4246.$$

12. Simplify the expressions:

$$2^3 \times 2^4 \times 2^8; \quad a^2 a^5; \quad (x^2 y^3)(x^2 y^3) \quad \text{Ans. } 2^{15}; a^7; x^4 y^6.$$

13. Multiply $3x^2 + x^3 - 1$ by $7 + x^2 + 2x$.

$$\text{Ans. } x^5 + 5x^4 + 13x^3 + 20x^2 - 2x - 7.$$

14. Divide $1 + x^4 - 2x^2$ by $1 - x$.

$$\text{Ans. } 1 + x - x^2 - x^3.$$

Factoring

15. Factor the expressions:

$$(a) 9x^2 - y^2; \quad (b) a^2 - b^2 + a - b;$$

$$(c) x^2 - 6y + 9y^2; \quad (d) 4x^2 + 28x + 49.$$

$$\text{Ans. (a) } (3x - y)(3x + y); (b) (a - b)(a + b + 1); (c) (x - 3y)^2; (d) (2x + 7)^2.$$

16. Reduce the fractions:

$$(a) \frac{a^2 - 25}{a^2 + 10a + 25}; \quad (b) \frac{x^2 - x}{x^3 - x^2}; \quad (c) \frac{24a^3b^2(x+1)^3}{15a^2b^4(x+1)^2}.$$

$$\text{Ans. (a) } \frac{a-5}{a+5}; \quad (b) \frac{1}{x}; \quad (c) \frac{8a(x+1)}{5b^2}.$$

Ratios

17. If 7 is to 40 as x is to 100, what is x ?

$$\text{Ans. } 17\frac{1}{2}.$$

18. What is the ratio 5 : 200 when expressed in per cent? Also for 4 : 130; 7 : 80; and $2\frac{1}{2} : 10\frac{1}{2}$.

$$\text{Ans. } 2\frac{1}{2}\%; 3.08\%; 8.75\%; 23.8\%.$$

19. If one plane can travel at 280 m.p.h. and another at 400 m.p.h., what is the ratio of their speeds expressed as a per cent?

Ans. The slower plane can travel 70% as fast as the top speed of the other.

20. The intensity I of illumination on a flat surface due to a small light source varies inversely as the square of the distance d from the light source to the surface. If the intensity is 20 foot-candles at a distance of 10 ft., what is the intensity when d is 30 ft.?

$$\text{Ans. } \frac{20}{9} \text{ foot-candles.}$$

21. The force F of attraction between two electrical charges e_1, e_2 varies directly as the product of these charges, and inversely as the square of the distance d they are apart. If $F = 10$ when $e_1 = 2, e_2 = 3$, and $d = 5$, what is the constant k of proportionality, and what is the force when $e_1 = 1, e_2 = 10$, and $d = 1$?

$$\text{Ans. } k = \frac{125}{3}, F = \frac{1250}{3}.$$

22. If the compressive force of an ordinary spring is 100 lb. when the spring is 2 in. long, and 200 lb. when the spring is 1 in. long, what is this force when the spring is $1\frac{3}{4}$ in. long?

$$\text{Ans. } 125 \text{ lb.}$$

Rounded numbers

23. Divide 7 by 11 and round off the answer to 4 decimal places.

$$\text{Ans. } 0.6364.$$

24. Round off the following numbers by dropping the last three digits:

$$0.007893; 78,345; 405.01; 404.99; 1.0001; 4342; 18,500; 9500.$$

$$\text{Ans. } 0.008; 78,000; 410; 400; 1.0; 4000; 18,000; 10,000.$$

25. Write the following as multiples of powers of 10: 100,000; 3 million; 18,000.

$$\text{Ans. } 10^5; 3 \times 10^6; 18 \times 10^3.$$

26. If $A = 15, B = 25$, and A is accurate to 1% while B is accurate to 0.5%, to which number should the product AB be rounded off?

$$\text{Ans. } 380.$$

27. If a steamer is found to travel 32 nautical miles in 3 hours, values being off by at most 1.6 mi. and 1.2 min., what is the maximum possible error in the computed speed $32/3$?

$$\text{Ans. } \text{About } 5.67\%.$$

SOLUTION OF EQUATIONS

In the expression $25x$ the number 25 is fixed and is therefore called a *constant*. The symbol x may have different values, and is called a *variable*. In the expression $-6xy^2$ the number -6 is a constant, while x and y are variables. If in $25x$ we replace 25 by a we obtain ax . We still say that a is a constant and x is a variable. *Variables and constants*

Usually the initial letters of the alphabet, such as a , b , and c , are reserved for constants, and the last letters of the alphabet, such as x , y , and z , for variables. In the expression ab^2x^3z the symbols a and b would be called constants, and x and z variables.

Because of its association with straight lines, to be explained in detail later, an equation of the form *Linear equations*

$$(21) \quad ax + by + c = 0$$

is called a *linear equation*. Any equation reducible to the form of equation 21 by the algebraic operations of addition, subtraction, multiplication, and division (division by zero excluded) is also linear. The following are examples of linear equations:

$$2x + 5 + 4y = 0, \quad 6x + 7 = 4x - 2 + x, \quad x + y = 2x - 7y.$$

To solve a linear equation, such as

$$(22) \quad 4x - 7 + 5x = 9 - 4x + 2x + 6$$

you will find it convenient to follow the steps listed below.

1. Collect the terms in x on each side of the equation, and do the same for the constants. You obtain *Method of solving linear equations*

$$9x - 7 = 15 - 2x.$$

2. Using addition and subtraction *change the equation into the form* $ax = b$ with a term in x on one side, and a constant on the other. For the example under consideration you can do this by first adding $2x$ to both sides, and then by adding 7 to both sides. You have $11x = 22$.

3. Finally, *divide both sides of the equation by the same number to leave only x on one side*. This yields

$$x = \frac{22}{11} = 2.$$

The solution of equation 22 is thus $x = 2$. In carrying through the steps above, the same operations were performed on the left of the equation as on the right. This must always be done to preserve the equality.

*Checking
equations*

To be sure that no error has been made in the solution of an equation, the result should be checked by substituting the solution in the given equation. In the case above, this substitution yields:

$$8 - 7 + 10 = 9 - 8 + 4 + 6; \text{ or } 11 = 11.$$

Since this is an obvious equality, the solution for x is correct.

As another example, let us solve for x in the equation

$$\frac{a}{x} + b = \frac{a + 3b}{x} + 4.$$

Multiplying both sides by x we obtain the linear equation

$$a + bx = a + 3b + 4x.$$

Subtracting $4x$ and a from both sides we have

$$(b - 4)x = 3b.$$

Dividing both sides by the coefficient of x produces

$$x = \frac{3b}{b - 4},$$

which is the desired solution.

Substitution of this solution for x in the given equation yields

$$\frac{\frac{a}{\frac{3b}{b-4}}}{\frac{3b}{b-4}} + b = \frac{a + 3b}{\frac{3b}{b-4}} + 4.$$

Combining terms on the left and right according to the laws for fractions, we have

$$\frac{ab - 4a + 3b^2}{3b} = \frac{ab - 4a + 3b^2}{3b},$$

which results in an identity and completes the check.

*Engineering
example*

Linear equations in one variable occur often in practice. One such example follows. How many tons of steel scrap, 99% iron, must be added to 10 tons of steel, 95% iron, to obtain a steel 98% iron, and how many tons will there be in the final steel billet? To answer these questions let x be the number of tons of steel which is 99% iron. There are then $10 + x$ tons of the final product. The amount of iron in x tons of 99% steel plus the amount of iron in 10 tons of 95% steel equals the amount of iron in $10 + x$ tons of 98% steel. We have the equation

$$\frac{99}{100}x + \frac{95}{100}(10) = \frac{98}{100}(10 + x).$$

Multiplying both sides by 100 simplifies this equation into

$$99x + 950 = 980 + 98x.$$

Subtracting $98x + 950$ from both sides brings the equation into the form

$$x = 30;$$

from which

$$10 + x = 40.$$

Hence, there are 40 tons of the final product. The checking of the solution will be left to you.

GRAPHICAL AIDS FOR SOLVING EQUATIONS

In Fig. 225 there is pictured a Cartesian coordinate system for the plane. The horizontal line is called the *x axis*, and the vertical line the *y axis*. Numbers are associated with points on these axes as in Fig. 223. The number connected in this manner with a point on the *x axis* is called the *x coordinate* of the point, and is understood to be a value of the variable *x*. In the same way *y* coordinates are associated with the points on the *y axis*. In Fig. 225 an arbitrary point *P* has been chosen in the plane. Perpendiculars have been dropped from *P* to the *x* and *y* axes, giving points P_x and P_y respectively. The *x* and *y* coordinates of the point *P* are by definition the *x* and *y* coordinates of the points P_x and P_y respectively. It is customary to indicate a point by its coordinates, for example, *P* by (x, y) . Points $(-2, 3)$, $(-4, -5)$, and $(4, -3)$ are *plotted*, that is, located, on the plane of Fig. 225.

The coordinate axes divide the plane into four parts, called *quadrants*. The quadrant in which the points with positive coordinates lie is termed the *first quadrant*. In Fig. 225 this is labeled I. The second, third, and fourth quadrants are labeled II, III, and IV respectively as shown.

The Cartesian coordinate system of the plane gives a unique "name" to each point, namely the pair of coordinates (x, y) of the point. The point $(0, 0)$, denoted by *O* in Fig. 225, is called the *origin* of the coordinate system. The arrows on the axes in

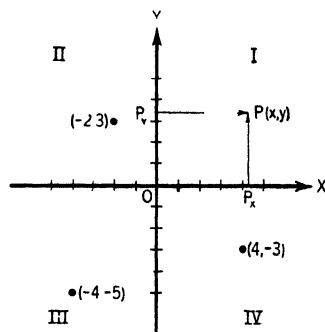


FIG. 225. CARTESIAN COORDINATE SYSTEM IN THE PLANE

Fig. 225 designate the positive direction on each of these lines, the *positive direction* being the direction of increasing coordinates.

Graphs

If, in the equation

$$(23) \quad y = x + 2,$$

we assign the value 1 to x , we obtain 3 for y . Other pairs of values of x and y which satisfy this equation can be obtained as shown in the following table:

| x | y |
|-----|-----|
| 1 | 3 |
| 0 | 2 |
| -1 | 1 |
| -2 | 0 |

Equation 23 connects a given value of x with one and only one value of y . Hence the equation represents only certain pairs of values (x, y) , and certain points in the plane of Fig. 227. Here the

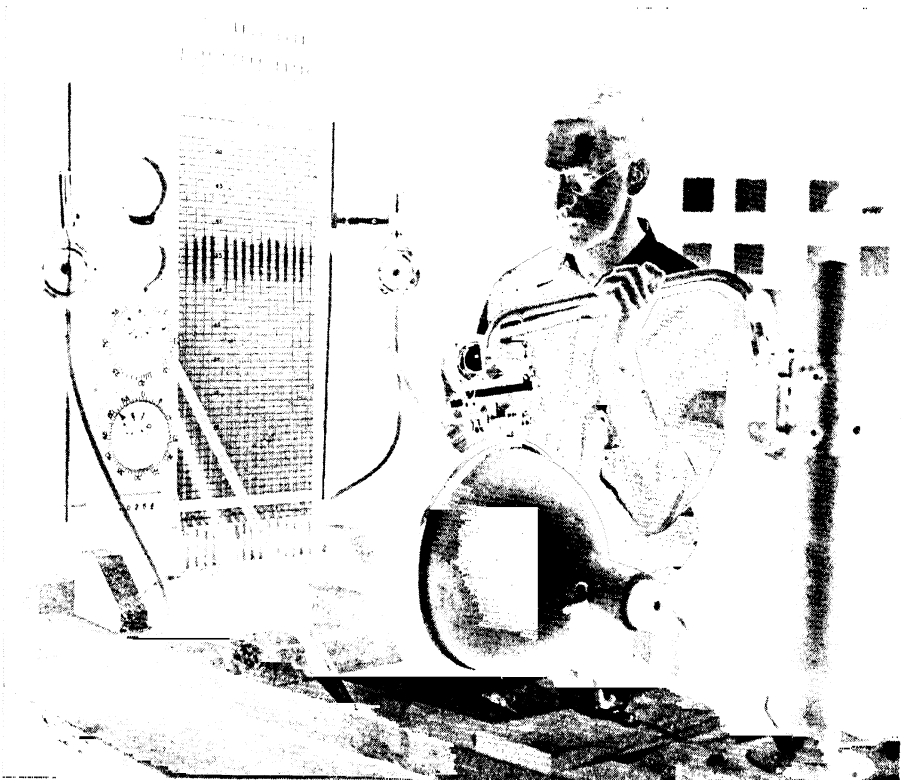


FIG. 226. COORDINATE SYSTEM IN USE EXPERIMENTALLY

points, whose coordinates are the pairs of values from the above table, have been plotted, and a straight line has been drawn through them. We say that equation 23 is the equation of the corresponding straight line of Fig. 227.

We shall refer to equation 23 as the *line l* of Fig. 227.

It will be shown later that a linear equation in x and y yields a straight line when "plotted" on a plane with a Cartesian coordinate system, and conversely, the points of a straight line on a Cartesian coordinate system satisfy a linear equation. The points where a straight line cuts the coordinate axes are called the *intercepts* of the line. The intercepts of the line of equation 23 are the points $(0, 2)$ and $(-2, 0)$ respectively. Since a straight line is determined or located by any two points on the line, we can locate a straight line by means of its intercepts.

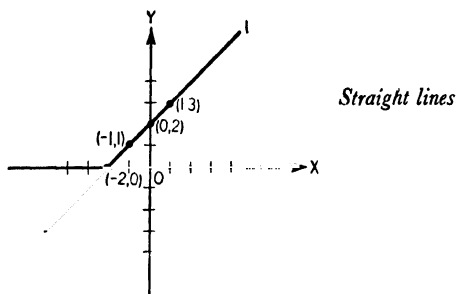


FIG. 227. POINTS ON A LINE l

SOLVING SIMULTANEOUS EQUATIONS

If we set y equal to zero in the equation

$$(24) \quad 2x + 3y = 4,$$

we find that x is 2. This value 2 is said to be the x -intercept of the line represented by equation 24. The y -intercept of this line is $\frac{4}{3}$.

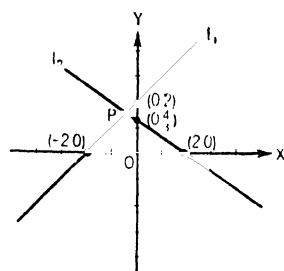


FIG. 228. INTERSECTION OF TWO LINES AS THE SOLUTION OF LINEAR EQUATIONS

By using these intercepts and drawing a line through these points, we obtain the line l_2 in Fig. 228, which line represents equation 24. Similarly, the line l_1 represents equation 23. The lines l_1 and l_2 intersect in a point designated by P . Since P is on the line l_1 the coordinates x, y of P satisfy equation 23. Similarly, x and y satisfy equation 24. Equations 23 and 24 determine P when taken simultaneously. They are therefore called *simultaneous equations*.

If you substitute $x + 2$ for y (which comes from equation 23) into equation 24, you obtain an equation in x only which is

$$2x + 3(x + 2) = 4.$$

Straight lines

Intersecting lines

Solution by substitution

Solution of this equation by the usual methods gives $x = -\frac{2}{5}$. This must be the x -coordinate of the point P in Fig. 228 since it is obtained by simultaneous solution of equations 23 and 24. Substitution of this value of x in equation 23 yields $y = \frac{8}{5}$, the other coordinate of P . The point of intersection of the lines 23 and 24 is the point $(-\frac{2}{5}, \frac{8}{5})$.

The x -coordinate of a point P is often termed the *abscissa* of P , whereas the y -coordinate is called the *ordinate*.

*Solution by
elimination*

A second method of solving simultaneous linear equations will be illustrated with an example. The given equations are

$$\begin{aligned} 3x + 3y &= 2, \\ 4x - 5y &= 7. \end{aligned}$$

Multiplying the first equation by 4 and the second equation by 3 produces the new equations

$$\begin{aligned} 12x + 12y &= 8, \\ 12x - 15y &= 21. \end{aligned}$$

Subtracting both sides of the second equation from the corresponding sides of the first equation yields the equation

$$27y = -13$$

in the variable y . Solution of this equation for y gives $y = -\frac{13}{27}$. Substituting this value of y in the first equation produces an equation in x only

$$3x - \frac{39}{27} = 2.$$

The solution for x gives us $x = \frac{31}{27}$.

To be sure that the computation is correct you should, of course, substitute the solutions for x and y in the original equations.

Application

Two weights A and B attached to a bar balance each other when the bar is supported at a point P for which the ratio of the distance from A to P to the distance from P to B is 3:4. (See Fig. 229.) Together A and B weigh 300 lb. The problem is to determine the weights of A and B individually.



FIG. 229. WEIGHTS A AND B ON LEVER WITH FULCRUM AT P

*Setting up
equations*

We let x be the weight of A and y the weight of B . By the law of the lever from mechanics the weight of A is to the weight of B as the distance from B to P is to the distance from A to P . There follows our first equation,

$$\frac{x}{y} = \frac{4}{3} \quad \text{or} \quad 3x = 4y.$$

Since A and B together weigh 300 lb. we have the second of these simultaneous equations,

$$x + y = 300.$$

Solution of the first equation for x and substitution in the second gives an equation in y alone

$$\frac{4y}{3} + y = 300.$$

Solution of this equation for y gives $y = \frac{900}{7}$. By the first equation we then get $x = \frac{1200}{7}$.

The methods of solution above can often be employed for *equations in several variables*. We shall treat the system of three variables x, y, z defined by the following equations *Equations in three variables*

$$\begin{aligned} 3y + 2x + 7 &= 0, \\ z + 3x + 4 &= y - 1, \\ 5 - 4x + y &= z. \end{aligned}$$

It is convenient to arrange the terms so that terms in the same variable occur in the same column. This rearrangement gives

$$\begin{aligned} 2x + 3y &= -7, \\ 3x - y + z &= -5, \\ -4x + y - z &= -5. \end{aligned}$$

Multiplying the first equation by 6, the second by 4, and the third by 3 brings these into the system *Equalizing coefficients*

$$\begin{aligned} 12x + 18y &= -42, \\ 12x - 4y + 4z &= -20, \\ -12x + 3y - 3z &= -15, \end{aligned}$$

where the coefficient of x is the same in each equation except for sign. Subtracting the second from the first gives us

$$22y - 4z = -22.$$

Addition of the second to the third yields

$$-y + z = -35.$$

The solution of the system has now been simplified to

Solution

$$\begin{aligned} 11y - 2z &= -11, \\ -y + z &= -35, \end{aligned}$$

which can be treated like the pairs of simultaneous equations above. After the values of y and z are determined from these equations their values may be put back into any one of the three original equations to obtain the value of x .

SOLVING QUADRATIC EQUATIONS

*Single variable
squared*

For a simple equation of the type $y^2 = 17$ we do not need any special methods because the answer is seen at once to be $y = \pm \sqrt{17}$. Here the $\sqrt{17}$ means the square root obtained by the root-extraction process described earlier in this chapter. Approximately, $\sqrt{17} = 4.1$. Since the equation $y^2 = 17$ has two roots, we say that 17 has two square roots, one positive and the other negative. When we write $\sqrt{17}$ without a plus or minus sign in front, we mean the plus square root whereas $-\sqrt{17}$ designates the minus square root, or approximately -4.1 . As a consequence $\sqrt{9}$ is 3, while $-\sqrt{9}$ is -3 .

*Inequality
sign*

The symbol \neq means "not equal to." Thus $a \neq 2$ means "a not equal to 2."

*The quadratic
equation*

An equation which is of the form

$$(25) \quad ax^2 + bx + c = 0, \quad (a \neq 0)$$

or one that can be reduced to this form by rearrangement of terms, is called a *quadratic equation*. Evidently the value of a should not be zero because a quadratic equation must have terms in the square of the variable. A quadratic equation has no terms of "higher power" than the square. The expression "higher power" refers to the fact that x^m is a higher power than x^n if m is greater than n . For example, x^4 is a higher power of x than x^2 or x^3 .

*Solution by
completing the
square*

We shall derive a simple formula, called the *quadratic formula*, by means of which you will be able quickly to solve quadratic equations. To derive this formula we describe the general *method of completing the square*.

1. Transpose the term c in equation 25 to the right by subtracting c from both sides of the equation, giving

$$ax^2 + bx = -c.$$

2. Divide both sides by the coefficient a to get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

*Completing
square*

3. Add $[\frac{1}{2}(b/a)]^2$, or $b^2/4a^2$, to both sides to obtain

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

4. Rewrite the left side to show that it is a perfect square.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$

5. Collect the terms on the right, which gives

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

6. Extract the square root of both sides. To perform this step go back to rule 16 under factoring. With this rule you can solve the equation $x^2 = 4$, for example. To do this you first subtract 4 from both sides obtaining $x^2 - 4 = 0$. Then factor the left-hand member to get $(x - 2)(x + 2) = 0$. Since *a product of numbers can vanish (become zero) only if at least one of the factors vanishes*, the solution of the quadratic equation $x^2 - 4 = 0$ reduces to the solution of two linear equations, namely,

$$x - 2 = 0, \quad x + 2 = 0.$$

From these equations $x = 2$ and $x = -2$. For simplicity of notation these two solutions are written as $x = \pm 2$, meaning *x is plus or minus 2*. This method shows that if a is a positive number, the solution of $x^2 = a$ is $x = \pm \sqrt{a}$, where by \sqrt{a} we mean the ordinary positive square root of a . From step 5 you have by this procedure

$$\left(x + \frac{b}{2a}\right)^2 = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

7. Finally, subtract $b/2a$ from both sides, getting

$$(26) \quad x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}},$$

which is the solution. There are two roots (they may be coincident) of the quadratic equation 25, namely

$$-\frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}}; \quad \text{and} \quad -\frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

Since

$$\left(\sqrt{\frac{a}{b}}\right)^2 = \frac{a}{b} \quad \text{and} \quad \left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{a}{b},$$

Square root of quotient

we have the rule

$$(27) \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

which applies to the numbers a and b . Using this rule the solution of equation 25 becomes

$$(28) \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Quadratic formula

This is the quadratic formula.

By means of the quadratic formula you can at once write down the solution of

$$2x^2 + 8x + 3 = 0.$$

Here $a = 2$, $b = 8$, and $c = 3$. The solution is

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \times 2 \times 3}}{2 \times 2},$$

or

$$x = \frac{-8 \pm \sqrt{64 - 24}}{4} = \frac{-8 \pm \sqrt{40}}{4} = -2 + \frac{1}{4}\sqrt{40} \text{ or } -2 - \frac{1}{4}\sqrt{40}.$$

We see therefore that there are always two values of x , equal or unequal, that will satisfy a quadratic equation in x . Often in practical engineering problems one of these solutions is the correct one while the other is impractical or unreasonable. For example, if x is the speed of a car in miles per hour and the two values of x are found to be 32 and 45,200, it is evident that $x = 32$ is the correct solution.

Example of a quadratic equation

A simple problem in shop production involving a quadratic equation will be solved. A sheet of metal is 24 by 36 in. It is desired

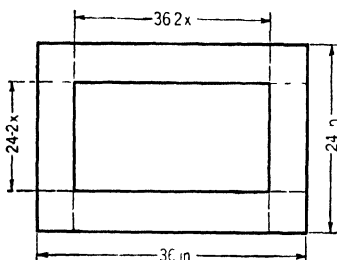


FIG. 230. SHEET OF METAL WITH A BORDER OF WIDTH x INCHES

to remove a uniform border from the metal to obtain a smaller sheet with area 400 sq. in. What must be the width of the border?

Let x designate the width of the border as shown in Fig. 230. The reduced sheet has dimensions $24 - 2x$ and $36 - 2x$. It follows that

$$(24 - 2x)(36 - 2x) = 400.$$

Multiplying out the indicated product on the left, dividing the equation by 4, and collecting all terms on one side, we have

$$x^2 - 30x + 116 = 0.$$

Quadratic equation

Solution by the quadratic formula gives

$$x = 15 \pm \frac{1}{2}\sqrt{436}.$$

To an accuracy expressed by three significant digits

$$\sqrt{436} = 20.9.$$

The solutions for x are thus approximately

$$x = 25.4; 4.6.$$

The first answer is impossible, and is therefore thrown away. The second solution is the desired one; whence the border must be 46 in. in width.

If the term $b^2 - 4ac$ is negative, the quadratic formula has no significance when we restrict ourselves to real numbers. In 1572 Raphael Bombelli of Bologna, Italy, published a memoir in which he endeavored to show that square roots of negative numbers can be given a meaning which is useful and logical. Later such numbers were called *imaginary*. *Imaginary numbers*

Meaning is given to imaginary numbers such as $\sqrt{-5}$ by adopting the convention that whenever $(\sqrt{-5})^2$ occurs we may replace this term by -5 . For example, $(\sqrt{-5})^4$ is $(\sqrt{-5})^2(\sqrt{-5})^2$, or $(-5)(-5)$, which is $+25$.

If we are willing to employ imaginary numbers, the quadratic formula is still valid when the term $b^2 - 4ac$ is negative. The solution of

$$x^2 + x + 1 = 0$$

for example, gives

$$x = \frac{-1 \pm \sqrt{-3}}{2}.$$

Numbers of the form $a + b$, where a is real and b is the square root of a negative number, or zero, are *complex numbers*. Complex numbers, like rational and real numbers, form a system because the common operations of addition, subtraction, multiplication, and division (division by zero excluded) with these complex numbers always again yield complex numbers. *Complex numbers*

Simple physical interpretations for complex numbers can be given. These numbers have been employed with considerable success in many branches of engineering, especially in treating equations describing motions of material things, such as the rotation of a flywheel in a diesel engine, the flow of water through a pipe, the flight of a shell from a gun, and the transmission of electricity along a wire. *Some uses of complex numbers*

Let us consider equations which are sums of terms set equal to zero, where each term is a real number times a power of a variable, say x^n . Linear and quadratic equations in x are examples of such equations. Carl Friedrich Gauss (1777-1855) proved that each such equation has a complex root. This statement means that all algebraic equations in a single variable raised to any powers can be solved by the use of complex numbers.

*Graphs of
quadratic
equations in
two variables*

The equation

$$u^2 + 2u + v^2 = 4$$

contains squares of the variables u and v , but no higher powers, and is an example of a *quadratic equation in two variables*. By giving values to u and solving the equation for v , you can derive the following table of pairs of values of u and v . As an example of the

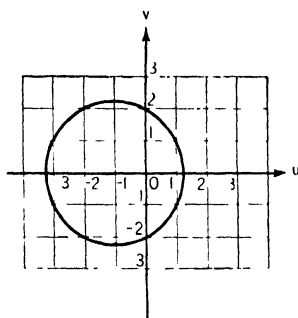


FIG. 231. A CIRCLE AS THE GRAPH OF A QUADRATIC EQUATION

| u | v |
|-----|----------------|
| 0 | ± 2 |
| 1 | ± 1 |
| 2 | Not real |
| -1 | $\pm \sqrt{5}$ |
| -2 | ± 2 |
| -3 | ± 1 |
| -4 | Not real |

computation involved, when $u = 2$ the equation becomes

$$4 + 4 + v^2 = 4 \quad \text{or} \quad v = \pm \sqrt{-4}$$

so that v is imaginary. In Fig. 231 the points which go with the values in the table are plotted on a u, v Cartesian coordinate system. Through these points a curve is drawn. This is the curve of the given quadratic equation. By analytic geometry we can later show that this is a circle.

Problems

1. Plot the points $(1, -2)$, $(-3, -3)$, $(5, -9)$, and $(-1, 4)$ on a Cartesian coordinate system. In what quadrants do these points lie?

2. Graph the line represented by the equation $2x = 4 - 3y$ by plotting some points and then drawing the line through these points. What are the intercepts of the line?
Ans. The intercepts are $(2, 0)$ and $(0, \frac{4}{3})$.

3. Graph the curve $x^2 = 2y$.

Simultaneous equations

4. Graph the lines $x = 5 - 3y$, $2x + 5y = 2$.
Find the point of intersection, and check by solving the equations algebraically.
Ans. $(-19, 8)$.

5. What is the solution of the equations $x + y = 2$, $2x = 1 - 2y$?

Ans. There is none. Why?

6. Solve the equations below when y and z are variables.

$$2z = 2y - 4,$$

$$z + 2 = y.$$

Ans. These are equations of the same line. Any pair of values of (y, z) which satisfy one equation also satisfy the other.

7. Solve the equations below for a , b , and c .

$$\begin{aligned}2a + 3b &= 14 + c, \\4 + c - 3a &= -b, \\a + b + c &= 4,\end{aligned}$$

Ans. $a = 2, b = 3, c = -1$.

8. Solve the equations below for x .

Quadratic equations

(a) $\frac{1}{x} + 3 = \frac{x-1}{2x}, \quad (x \neq 0)$

(b) $\frac{x^2-1}{(x+1)^2} = 2, \quad (x \neq -1)$

(c) $3x^2 = 6 - 15x,$

(d) $\frac{3}{x} + x = -1. \quad (x \neq 0)$

Ans. (a) $-\frac{3}{5}$; (b) -3 ; (c) $\frac{-5 \pm \sqrt{33}}{2}$; (d) $\frac{-1 \pm \sqrt{-11}}{2}$.

9. The perimeter of a certain rectangle is 100 ft. and the altitude is three fourths of the base. Find the dimensions of the rectangle.

Linear equations in applications

Ans. $1\frac{5}{7}$ ft., $2\frac{6}{7}$ ft.

10. A weight of 200 lb. is placed on a lever 10 ft. from the point of support. How far from the point of support on the other side must a weight of 100 lb. be placed to produce equilibrium, that is, to balance the first load?

Ans. 20 ft.

11. How many gallons of a solution containing 80% alcohol should be added to 10 gal. of a 30% solution, to give a mixture which is 40% alcohol?

Hint. Let x equal the gallons of 80% solution; then $10 + x$ equals the gallons of 40% solution. Alcohol in final mixture equals $0.80x + (0.30)10$ or $0.8x + 3$. But this is also expressed as $0.40(10 + x)$. Hence, you want to solve the equation $0.8x + 3 = 0.4x + 4$.

Ans. $2\frac{1}{2}$ gal.

12. How many ounces of pure silver must be added to 100 oz. of silver, 50% pure, to give a mixture 60% pure?

Ans. 25 oz.

13. A man has one investment yielding 4%, another giving 6%, and a third producing 7%. The total amount of money invested is given to be \$50,000, and the annual interest is \$2800. What is the amount of money invested at each rate if the amount invested at 6% equals the sum of the amounts invested at the other rates?

Ans. \$25,000 at 6%; \$15,000 at 4%; \$10,000 at 7%.

14. A tank can be emptied by means of either of two orifices in the bottom or by both. If by opening the first orifice alone the tank can be emptied in 3 hours, and by opening the second alone the tank can be emptied in 4 hours, in how many hours can the tank be emptied when both orifices are used?

Hint. In t hours the first orifice empties t thirds of the tank, and the second orifice empties t fourths.

Ans. 1 hour, $42\frac{2}{3}$ min.

15. Two automobiles traveling in opposite directions at 50 and 70 m.p.h., respectively, started from the same place at the same time. After how many hours will they be 300 miles apart?

Linear equations, speed problems

Hint. In a time t hours, one automobile travels $50t$ miles, and the other $70t$ miles.

Ans. 2 hours, 30 min.

16. A boat traveling at constant speed down a river takes one hour to cover 10 miles, and 3 hours to travel the same distance going against the stream. What is the speed of the boat in still water, and the speed of the stream?

Hint. Let x be the speed of the boat in still water, and y the speed of the stream in miles per hour. The speed of the boat downstream relative to the shore is $x + y$ m.p.h., and against the stream $x - y$ m.p.h.

Ans. Boat speed: $6\frac{2}{3}$ m.p.h.; stream speed: $3\frac{1}{3}$ m.p.h.

17. A freight train leaves Chicago for New York at a speed of 35 m.p.h. Three hours later an express train leaves Chicago for New York at a speed of 75 m.p.h. When will the express train catch up with the freight train?

Hint. Let t be the time in hours from the departure of the freight until the express train overtakes the freight. The distance the freight covers in the time t is $35t$ (rate times time). This is equal to the distance $75(t - 3)$ covered by the express in the time t .

Ans. 5 hours, $37\frac{1}{2}$ min.

*Quadratic
equations in
applications*

18. A rectangular piece of property is 40 ft. long and 30 ft. wide. The property is increased to a larger rectangle, of double the original area, by adding a border of uniform width. How wide is the border?

Ans. $\frac{-35 + \sqrt{2425}}{2}$ or 7.12 ft.

19. The diagonal of a certain square is 2 ft. longer than a side of the square. Find the length of a side.

Ans. $2 + 2\sqrt{2}$ or 4.83 ft.

EXPONENTS AND RADICALS

You have seen how powers of numbers arise in mathematical considerations. To use these powers with efficiency you will need some rules for manipulation, which will be outlined. When we write a power as a^n the number n is the *exponent* while a is the *base*. In the power 12^6 the exponent is 6 and the base 12.

*Fractional
exponents*

Thus far we have treated only powers with positive integral exponents. It is convenient, however, to assign a meaning to a power with exponent n , where n is any real number. In fact, for many mathematical studies it is necessary to use a^n where n is an imaginary number.

We shall restrict ourselves to real exponents. We begin with reciprocals of integers, such as $\frac{1}{2}$ and $\frac{1}{4}$. Since the law (12), $a^n \times a^m = a^{n+m}$, concerning products of powers of the same base is valid for positive exponents which are positive integers, it is logical to restrict the meaning of all exponents so that this rule of multiplication (12) is always satisfied. For example, the power $2^{\frac{1}{2}}$ must be such that

$$2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2} + \frac{1}{2}} = 2,$$

from which it is clear that

$$2^{\frac{1}{2}} = \sqrt{2}.$$

*Equivalence
with roots*

You will recall that the n th root of a number a is a number, *Existence of roots*
written $\sqrt[n]{a}$, such that

$$(29) \quad (\sqrt[n]{a})^n = a.$$

The *cube root* of 2 written $\sqrt[3]{2}$, that is the third root, is a number such that

$$(\sqrt[3]{2})^3 = 2.$$

In more advanced mathematics it is proved that there are three cube roots of 2, two of which are imaginary. When we use the term *cube root* in what follows we mean the *real* cube root; in fact, we shall restrict ourselves to real roots only, unless there is no real root of the given number.

There is always a positive n th root of a positive number no matter what integer is assigned to n . Examples of positive roots of positive numbers follow: *Roots of positive and negative numbers*

$$\sqrt[4]{16} = 2; \quad \sqrt[3]{64} = 4.$$

There is a negative n th root of a negative number when n is odd. Examples are

$$\sqrt[3]{-64} = -4, \quad \sqrt[5]{-32} = -2.$$

However, there are no real even roots of a negative number. Thus there is no real square root of -4 , no real fourth root of -5 , and so forth.

Since by application of rule 12 a product of n factors each equal to $a^{\frac{1}{n}}$ is a , we have

$$(30) \quad (a^{\frac{1}{n}})^n = a \quad \text{and} \quad a^{\frac{1}{n}} = \sqrt[n]{a}.$$

More generally, since by rule 12 a product of n factors each equal to a number $a^{\frac{m}{n}}$ is a^m and $(\sqrt[n]{a^m})^n$ is a^m , we write

$$(31) \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

For example,

$$8^{\frac{3}{4}} = \sqrt[4]{8^3} = \sqrt[4]{64} = 4.$$

In the following illustration one power of 2 is divided by another: *Quotients of powers*

$$\frac{2^5}{2^2} = \frac{\cancel{2} \times \cancel{2} \times 2 \times 2 \times 2}{\cancel{2} \times \cancel{2}} = 2 \times 2 \times 2 = 2^3.$$

This example is an illustration of the law

$$(32) \quad \frac{a^n}{a^m} = a^{n-m} \quad (a \neq 0)$$

when n is greater than m . Formula 32 is proved by noting that when m a 's are divided out of the numerator and denominator of the left member of equation 32, the denominator becomes 1 and the $n-m$ factors equal to a remain in the numerator.

Zero
exponent

We shall define the rest of our exponents in such a way that equation 32 is satisfied. In particular, consider the quotient $3^2/3^2$. Subtracting exponents this fraction becomes 3^0 , whereas direct division gives 1. Therefore, $3^0 = 1$, and since any number except 0 might have been used in place of the number 3, we conclude that

$$(33) \quad a^0 = 1 \quad (a \neq 0).$$

The restriction that a be different from zero is made because division by zero is not permitted, and rule 33 is based on division with some power of a . It is customary to say that 0^0 "has no meaning."

Negative
exponents

By making use of rule 12 we may write

$$a^n a^{-n} = a^{n-n} = a^0 = 1.$$

But we also find that

$$a^n \left(\frac{1}{a^n} \right) = \frac{a^n}{a^n} = 1.$$

We therefore identify

$$(34) \quad a^{-n} = \frac{1}{a^n}.$$

Examples follow:

$$\begin{aligned} \left(\frac{1}{2}\right)^{-3} &= 8; & 2^{-2} &= \frac{1}{2^2} = \frac{1}{4}; & (-3)^{-3} &= \frac{1}{(-3)^3} = \frac{1}{-27} = -\frac{1}{27}; \\ 4^{-\frac{1}{2}} &= \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2}. \end{aligned}$$

We have defined a^n where n is any rational number. A definition of a^n where n is irrational, such as $n = \pi$, will not be given, although a number such as 2^π is real and can be determined to any number of decimal places.

Rules for
exponents

A summary of useful rules for handling exponents follows.

$$(35) \quad \text{Rule for multiplication:} \quad a^n a^m = a^{n+m}.$$

$$(36) \quad \text{Rule for division:} \quad \frac{a^n}{a^m} = a^{n-m}.$$

$$(37) \quad \text{Rule for finding a power of a power:} \quad (a^n)^m = a^{nm}.$$

$$(38) \quad \text{Rule for finding a power of a product:} \quad (ab)^n = a^n b^n.$$

$$(39) \quad \text{Rule for finding a power of a quotient:} \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

The rules above will be applied to some examples, which follow:

$$\begin{aligned} x^4 x^{10} &= x^{14}; & 2^{25} \times 2^{100} &= 2^{125}; & 3^2 \times 3^3 \times 3^4 &= 3^9; \\ (a^4)^3 &= a^{12}; & (2b)^5 &= 2^5 b^5 = 32 b^5; & \frac{x^{12}}{x^2} &= x^{10}; \\ \frac{2^5 \times 3^{12}}{2^8 \times 3^4} &= \frac{3^8}{2^3}; & \left(\frac{2^4}{3^4}\right)^3 &= \frac{2^9}{3^{12}}; & (a^2 b c^4)^2 &= a^4 b^2 c^8. \end{aligned}$$

You will observe that a product or quotient of two powers can be combined into one only if the *bases are the same*, in which case rule 35 or 36 applies, or if the *exponents are the same*, when rules 38 and 39 can be used. It is impossible to combine $2^3 \times 3^2$ by multiplying the bases or adding the exponents, a mistake often made by the novice.

A common error

The symbol $\sqrt[n]{a}$ is called a *radical*. It has the same value as $a^{\frac{1}{n}}$. A radical is a root written as indicated. The number n is called the *index* of the radical. By convention, the index is not written in the case of square roots.

Radicals

Some properties of radicals may be obtained from the Rules for Exponents above.

Rules of radicals

$$\begin{aligned} (40) \quad & (\sqrt[n]{a})^n = (a^{\frac{1}{n}})^n = a. & \text{(From 37)} \\ (41) \quad & \sqrt[n]{a^m} = a^{\frac{m}{n}} = (\sqrt[n]{a})^m. & \text{(From 37)} \\ (42) \quad & \sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}. & \text{(From 38)} \end{aligned}$$

[Except when n is even and a, b are both negative]

$$\begin{aligned} (43) \quad & \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}. & \text{(From 39)} \\ (44) \quad & \sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a} = \sqrt[n]{\sqrt[m]{a}}. & \text{(From 37)} \end{aligned}$$

The first property (40) is merely the definition of the n th root of a . The first equality of (41) is the general law connecting roots and fractional exponents.

By rule 38 for exponents $a^{\frac{1}{n}} b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$. Since $a^{\frac{1}{n}} = \sqrt[n]{a}$, we have $\sqrt[n]{a} \sqrt[n]{b} = \sqrt[n]{ab}$. These rules of radicals and exponents break down when a and b are both negative, and n is even. For example, by definition

Limitation on the root or power of a product

$$\sqrt{-2} \sqrt{-2} = -2.$$

If we apply rule 42 for radicals we have

$$\sqrt{-2} \sqrt{-2} = \sqrt{(-2)(-2)} = \sqrt{4} = +2.$$

We have arrived at a contradiction, whence the above restriction on a, b is needed in the statement of the third property (42). The same restriction of course applies to rule 38 for exponents when a and b are negative, and the exponent is one of the common fractions ($\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$) having an even number as a denominator.

To use rules 42 and 43 it is necessary to have the *same index* on each radical. Thus, to multiply $\sqrt{2}$ by $\sqrt[3]{2}$, we first change $\sqrt{2}$ to $\sqrt[6]{8}$ and $\sqrt[3]{2}$ to $\sqrt[6]{4}$, so that

$$\sqrt{2} \sqrt[3]{2} = \sqrt[6]{8} \sqrt[6]{4} = \sqrt[6]{32}.$$

*Examples of
radicals*

Examples of the above properties of radicals follow.

$$\begin{aligned} (\sqrt[3]{4})^3 &= 4; & \sqrt[3]{8^2} &= (\sqrt[3]{8})^2 = 2^2 = 4; & \sqrt[3]{2} \sqrt[3]{4} &= \sqrt[3]{8} = 2; \\ \sqrt[3]{3} \sqrt[3]{4} &= \sqrt[3]{27} \sqrt[3]{256} = \sqrt[3]{6912}; & \frac{\sqrt[3]{-24}}{\sqrt[3]{3}} &= \sqrt[3]{-8} = -2; \\ \sqrt[3]{\sqrt{8}} &= \sqrt{\sqrt[3]{8}} = \sqrt{2}; & \frac{\sqrt{8}}{\sqrt{2}} &= \sqrt{4} = 2. \end{aligned}$$

*Simplification
of radicals*

The third property (42) is particularly helpful in treating radicals. For example,

$$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \sqrt{2} = 3 \sqrt{2}.$$

Let us suppose that you wish to use a numerical value for the fraction $\frac{1}{\sqrt{2}}$. An obvious method is to extract the root of 2, and then divide the result, say 1.414, into 1. It is much simpler to follow another procedure where the fraction is first *rationalized*, that is, the denominator is freed of radicals. This can be done in the example at hand by multiplying above and below by $\sqrt{2}$ as shown.

*Rationalization
of fractions*

$$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.414 \cdots}{2} = 0.707 \cdots$$

In the following illustrations fractions have been rationalized.

$$\begin{aligned} \frac{1}{\sqrt{3}} &= \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}; & \frac{1}{\sqrt[3]{9}} &= \frac{1}{\sqrt[3]{9}} \frac{\sqrt[3]{3}}{\sqrt[3]{3}} = \frac{\sqrt[3]{3}}{\sqrt[3]{27}} = \frac{\sqrt[3]{3}}{3}; \\ \frac{1}{1-\sqrt{2}} &= \frac{1}{(1-\sqrt{2})(1+\sqrt{2})} = \frac{1+\sqrt{2}}{1-2} = \frac{1+\sqrt{2}}{-1} = -(1+\sqrt{2}). \end{aligned}$$

Care must be exercised in treating sums of radicals, for such sums cannot be combined into one unless the terms can be reduced

to expressions in the *same radical*. We cannot combine $\sqrt{2} + \sqrt{3}$ into, for example, $\sqrt{5}$ or $\sqrt{6}$. The terms

$$\sqrt{12} + 2\sqrt{3} - \sqrt{1200}$$

can be united into one term by removing factors from the radicals with the aid of property 42 for radicals, giving $2\sqrt{3} + 2\sqrt{3} - 20\sqrt{3}$, that is, $-16\sqrt{3}$. *Addition of radicals*

To simplify

$$\sqrt[9]{\frac{a}{b}} + 3\sqrt{a^3b} + \sqrt{\frac{16a^7}{b^3}} + \sqrt[4]{\frac{4b^5}{a}}$$

we remove from each radical the factors which are perfect squares to obtain

$$\frac{3\sqrt{ab}}{b} + 3a\sqrt{ab} + \frac{4a^3}{b^2}\sqrt{ab} + \frac{2b^2\sqrt{ab}}{a}.$$

(Combining terms yields

$$\frac{(3ab + 3a^2b^2 + 4a^4 + 2b^4)}{ab^2}\sqrt{ab}.$$

LOGARITHMS

In view of rules 35 and 36 for multiplication and division of powers, it is a simple matter to multiply or divide numbers when they are written as powers of the same base. In fact, then *multiplication reduces to addition and division reduces to subtraction*. To Henry Briggs (1556–1631) there occurred the idea of writing the positive real numbers as powers of 10, and of working with the corresponding exponents instead of the numbers. Briggs constructed tables of these exponents, so that to multiply two numbers, such as 179 and 43.85, one need only look up in these tables the exponents 2.25285 and 1.64197 (since $179 = 10^{2.25285}$ and $43.85 = 10^{1.64197}$), then add these exponents to get 3.89482, and finally look up in the tables the number which corresponds to $10^{3.89482}$. This number is found to be 7849.2, which is therefore the desired product of 179 and 43.85. *Logarithms as exponents*

If a number m is written as $m = a^n$, we say that n is the *logarithm of m to the base a* . For example, 2 is the logarithm of 100 to the base 10 since $10^2 = 100$, and 5 is the logarithm of 32 to the base 2 since $2^5 = 32$. The expression “logarithm” is often abbreviated to “log.”

Natural logarithms

John Napier (1550–1617) is probably the inventor of logarithms. It is after him that logarithms to a base called e , arising automatically in the branch of mathematics termed the *calculus*, are called *Napierian logarithms*. They are also termed *natural logarithms*. But in what follows “logarithm” will mean *logarithm to the base 10* as in the first example above.

Decimals and logarithms

If $a = 10^n$, $b = 10^m$, rule 35 implies that $ab = 10^{n+m}$. Since $n = \log a$, $m = \log b$ and $n + m = \log ab$, we have

$$(45) \quad \log ab = \log a + \log b.$$

Thus the *logarithm of a product is the sum of the logarithms of the factors*. Since the logarithm of 10 is 1 and the logarithm of 2 is 0.30103 (from tables), the logarithm of 20, that is, 2×10 , is 1.30103. The decimal parts of the logarithms of 2 and 20 are the same. Likewise you can verify that the logarithm of 200 is 2.30103, the logarithm of 2000 is 3.30103, and so on.

The number 0.2 is $2 \times \frac{1}{10}$. The logarithm of $\frac{1}{10}$ is -1 since $10^{-1} = \frac{1}{10}$. It follows that the logarithm of 0.2 is $-1 + 0.30103$. This logarithm is usually written as $9.30103 - 10$. Similarly, the logarithm of 0.02 is $-2 + 0.30103$, written as $8.30103 - 10$, and the logarithm of 0.002 is $-3 + 0.30103$, written as $7.30103 - 10$.

Mantissa and characteristic

From the argument above you can see that the decimal part of the logarithm of any number having 2 as the only nonzero digit is always 0.30103. This decimal part, termed the *mantissa* of the logarithm, corresponds to the digit 2, whereas the rest of the logarithm, called the *characteristic*, relates to the position of the decimal point.

The logarithms of 721.1, 72.11, 7.211, 0.7211, 0.07211, 0.007211, and 0.0007211 are 2.85800, 1.85800, 0.85800, 9.85800 - 10, 8.85800 - 10, 7.85800 - 10, and 6.85800 - 10 respectively. The mantissa of each of these numbers is 85800. The characteristic is 2, 1, 0, 9 - 10, 8 - 10, 7 - 10, and 6 - 10 respectively. It will prove convenient to use 9 - 10, 8 - 10, and so on, as characteristics in place of negative numbers.

Since the *mantissa* is independent of the position of the decimal point in the given number, tables of logarithms actually give only mantissas, the determination of the *characteristic* being left to the computer.

Finding the mantissa in the tables

The log tables given at the back of this book are five-place tables, that is, five places of the mantissa are given. This corresponds to four significant digits in the number whose mantissa is given in

the tables. The first three digits of this number are to be found in the column headed N, and the last digit in the row containing the same letter N. The mantissa of the number is to be found in the row containing the first three digits of the number and the column headed at the top by the remaining digit. Thus, to get the mantissa of 2,437,000 you use only the four digits 2437. You locate 243 in the column headed N, and 7 in the row of N. In the same row as the one in which 243 occurs and in the column headed 7 you will find 686, which are the last three digits of the mantissa. The first two will be found above and at the left, namely 38, so that the mantissa of 2,437,000 is 38686. The first two digits of the mantissa are the same for several successive mantissas in the tables, and therefore these digits are not constantly repeated, in fact, they are given only once. Thus all of the mantissas after 38021 in the tables up to 39005 have 38 as the first two digits.

*Turn to
page 528
of the tables*

A star in front of the last three digits of a mantissa means that the mantissa's two beginning digits are the next first two digits occurring in the tables. The use of these stars saves printing space. In particular, the mantissa 39005 comes from *005 and the 39 of 39094. The mantissa preceding 38021 is thus 38003, not 37003 since the digits *003 are starred.

*Starred values
in tables*

From the examples above you will see that the *characteristic of a number bigger than or equal to 1 is one less than the number of digits to the left of the decimal place*. The characteristic of 2,437,000 is 6. Therefore, $\log 2,437,000 = 6.38686$.

*Computation
of character-
istic*

To get the logarithm of 89.01 you must first look up 890 in the column headed N, and 1 in the row marked N. The mantissa of the number is 94944. The characteristic of the number is 1. The desired logarithm is thus 1.94944.

From the discussion of characteristic above you will conclude that the *characteristic of a positive number less than 1 is $-n$, where n is one more than the number of zeros between the decimal point and the first nonzero digit to the right of the decimal point*. The characteristic of 0.68433 is thus -1 , or $9 - 10$, and the characteristic of 0.00038 is -4 , or $6 - 10$. The logarithm of 0.000008901 is $4.94944 - 10$.

By means of the tables headed *proportional parts* you can use five digits of the given number instead of four. These tables are based on the fact that for small differences in the number the differences in the mantissas of the numbers are proportional to the differences in the numbers. Thus the number 34586 is six-tenths

*Interpolating
log tables*

of the way between 34580 and 34590. The mantissa of 34586 is therefore six-tenths of the way between the mantissa 53882 of 34580 and the mantissa 53895 of 34590. The mantissa of 34586 is thus 53890. The following relation is approximately true.

$$\frac{53890 - 53882}{53895 - 53882} = \frac{6}{10}.$$

The mantissa 53890 can be obtained directly from a table headed *proportional parts*. The difference between the mantissas of 34580 and 34590 is 13. If you locate the fifth digit 6 of 34586 to the left of the proportional-parts column headed 13 you will find 7.8, which is 0.6×13 , in the same row. The entry 7.8 when rounded off to 8 gives the correction which must be added to the mantissa 53882 of 34580 to obtain the mantissa 53890 of 34586.

Antilogarithms

A number whose logarithm is n is termed the *antilogarithm* of n . Thus the antilogarithm of 0.30103 is 2. The process of finding the antilogarithm of a number by means of the tables is, of course, the opposite to that described above for determining the logarithm of a number. For example, let us find the antilogarithm of $8.46752 - 10$. The mantissa 46752 is between the entries 46746 and 46761 in the tables on page 529. Since the difference between 46746 and 46761 is 15, we must use the table of proportional parts headed 15. The difference between the mantissas 46746 and 46752 is 6. We look up this difference among the logarithmic corrections in this table. The corresponding antilogarithmic correction is 4. Since the mantissa 46746 belongs to the sequence of digits 2934, the mantissa 46752 belongs to 29344. From the characteristic $8 - 10$ we find that the antilogarithm of $8.46752 - 10$ is 0.029344. That is, the logarithm of 0.029344 is $8.46752 - 10$.

Example of multiplication with logs

To multiply 534.87 by 0.046832 we look up the logarithms of these numbers in the tables. They are 2.72825 and $8.67054 - 10$ respectively. Addition yields $11.39879 - 10$, which we abbreviate to 1.39879. The antilogarithm of 1.39879 is 25.049, which is the desired product.

Logarithm of quotient

If $a = 10^n$, $b = 10^m$, rule 36 implies that $a/b = 10^{n-m}$. Since $n = \log a$, $m = \log b$, $n - m = \log a/b$, we have

$$(46) \quad \log \frac{a}{b} = \log a - \log b.$$

It follows that the *logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator*.

To find the quotient of the number 0.046832 by the number 534,870,000 we write the logarithm of 0.046832, given above, as $18.67054 - 20$. We subtract as indicated, the logarithm of 534,870,000 being 8.72825.

$$\begin{array}{r} 18.67054 - 20 \\ 8.72825 \\ \hline 9.94229 - 20 \end{array}$$

The antilogarithm of $9.94229 - 20$ is 0.000,000,000,087,556, or $87,556 \times 10^{-15}$.

With $a = 10^n$, so that n is the logarithm of a , rule 37 implies *Logarithm of a power* that $a^m = 10^{nm}$, so that $\log a^m = nm$. But, since $n = \log a$, we have the rule which follows:

$$(47) \quad \log a^m = m \log a.$$

Thus the *logarithm of a power of a number is the exponent times the logarithm of the number.*

To find the value of $(534.87)^{\frac{1}{3}}$ we first multiply the logarithm of 534.87, which is 2.72825, by $\frac{1}{3}$ to get 2.04619. The antilogarithm of 2.04619 is 111.22, so that

$$(534.87)^{\frac{1}{3}} = 111.22.$$

With logarithms you can extract roots of numbers which *25th root* you could not otherwise obtain in reasonable time. Thus to get the real 25th root of 24, you write $\sqrt[25]{24} = 24^{\frac{1}{25}}$, to make use of rule 47. Multiplying the logarithm of 24 which is 1.38021 by $\frac{1}{25}$, you obtain 0.055208, whose antilogarithm is 1.1355. Therefore, $\sqrt[25]{24} = 1.1355$.

The slide rule, covered in a later chapter, enables you to add *Slide rule* and subtract logarithms mechanically. The scales on the slide rule which you ordinarily use for multiplication of numbers are called *logarithmic scales*, or simply *log scales*. The distance from the left end of the log scale to the number 2 on this scale is not 2 units as on an ordinary rule, but rather the logarithm 0.30103 of 2 (for an appropriate unit of measure), the distance from the left end to the number 3 is the logarithm 0.47712 of 3, and so forth.

When we multiply 2 by 3 on the slide rule we add the distance *Multiplication with the slide rule* 0.30103 on one scale to the distance 0.47712 on another by laying off these distances end to end. The result is a distance 0.77815, marked 6 on the scale where the answer is read. The logarithm of 6 is 0.77815. Thus the addition of logarithms is accomplished on the slide rule by the addition of distances.

We have used five-place logarithms for purposes of explanation. Actually, the slide rule is accurate to three places only. If you need more accuracy you will have to employ some other method such as the use of logarithmic tables.

Problems

1. Write the following as powers of 2:

$$32; 16; \frac{1}{4}; 1; 256; \sqrt[3]{4}; \sqrt[5]{2}.$$

$$\text{Ans. } 2^5; 2^4; 2^{-2}; 2^0; 2^8; 2^{\frac{2}{3}}; 2^{\frac{1}{5}}.$$

Powers and roots

2. Perform the indicated operations:

$$\begin{aligned} & \text{(a) } a^4 a^6; \text{ (b) } x^m x^{2-m}; \text{ (c) } (2x)^4; \text{ (d) } (y^4)^4; \text{ (e) } (2^4)^n; \text{ (f) } aa^2 a^3; \text{ (g) } (-3z^2)^3; \\ & \text{(h) } x^2 (xy)^3; \text{ (i) } \frac{x^6}{x^2}; \text{ (j) } \frac{a^6}{a^{10}}; \text{ (k) } \left(\frac{-2a}{b}\right)^3; \text{ (l) } \left(\frac{3a^{\frac{1}{2}}}{y^{\frac{2}{3}}}\right)^3; \text{ (m) } \frac{2^0 a^3}{a^{\frac{1}{2}}}; \text{ (n) } 8^{\frac{3}{2}}; \\ & \text{(o) } (.001)^{\frac{2}{3}}; \text{ (p) } \left(-\frac{1}{8}\right)^{\frac{4}{3}}; \text{ (q) } \sqrt[4]{\sqrt{64}}; \text{ (r) } \sqrt[3]{8a^4}. \end{aligned}$$

$$\text{Ans. (a) } a^{10}; \text{ (b) } x^2; \text{ (c) } 16x^4; \text{ (d) } y^{16}; \text{ (e) } 2^{4n}; \text{ (f) } a^6; \text{ (g) } -27z^6; \text{ (h) } x^5 y^3;$$

$$\text{(i) } x^4; \text{ (j) } a^{-4}; \text{ (k) } -\frac{8a^3}{b^3}; \text{ (l) } \frac{27a}{y^2}; \text{ (m) } a^{\frac{3}{2}}; \text{ (n) } 4; \text{ (o) } 0.01; \text{ (p) } \frac{1}{128};$$

$$\text{(q) } \sqrt[4]{8}; \text{ (r) } 2a\sqrt[3]{a}.$$

3. Find the value of each of the radicals:

$$\sqrt{36}; \sqrt[4]{81}; \sqrt[5]{32}; \sqrt[3]{.0009}; \sqrt[4]{-32}; \sqrt{\frac{1}{9}}; \sqrt[3]{-8}.$$

$$\text{Ans. } 6; 3; 2; 0.03; -2; \frac{1}{3}; -2.$$

4. Find the value of each of the following expressions by changing each negative power to a positive one, and each fractional power to a radical

$$9^{\frac{1}{2}}; (-2)^{-4}; 6^{-1}; (.0001)^{\frac{1}{4}}; 27^{-\frac{1}{3}}.$$

$$\text{Ans. } 3; -\frac{1}{8}; \frac{1}{6}; .1; \frac{1}{3}.$$

5. Rationalize the denominators of the fractions and simplify:

$$\frac{1}{\sqrt{5}}; \frac{1}{\sqrt{8}}; \sqrt{\frac{2}{3}}; \sqrt{\frac{1}{4}}; \frac{1}{1+\sqrt{3}}; \sqrt{\frac{3}{2ab^4}}.$$

$$\text{Ans. } \frac{\sqrt{5}}{5}; \frac{\sqrt{2}}{4}; \frac{\sqrt{6}}{3}; \frac{\sqrt{2}}{2}; \frac{1}{2}(\sqrt{3}-1); \frac{\sqrt{6ab}}{2ab^2}.$$

6. Simplify the expressions:

$$\begin{aligned} & \text{(a) } \sqrt[3]{8a^3 b^9}; \text{ (b) } \sqrt{45} + \sqrt{20}; \text{ (c) } \sqrt{\frac{1}{2}} - \sqrt{8}; \text{ (d) } 2\sqrt{x} - \sqrt{4x} + \sqrt{9x}; \\ & \text{(e) } \sqrt{12} + \sqrt{18}; \text{ (f) } 3x\sqrt{16x} - \sqrt{81x^3}; \text{ (g) } \sqrt{2x^3} \sqrt{2x}; \text{ (h) } \sqrt{\sqrt{16x^2}}; \\ & \text{(i) } \frac{\sqrt{2}}{\sqrt[3]{4}}; \text{ (j) } \sqrt{2} \sqrt[3]{3}. \end{aligned}$$

$$\text{Ans. (a) } 2ab^3; \text{ (b) } 5\sqrt{5}; \text{ (c) } -\frac{3}{2}\sqrt{2}; \text{ (d) } 3\sqrt{x}; \text{ (e) } 2\sqrt{3} + 3\sqrt{2}; \text{ (f) } 3x\sqrt{x};$$

$$\text{(g) } 2x^2; \text{ (h) } \sqrt[3]{4x}; \text{ (i) } \frac{\sqrt[3]{32}}{2}; \text{ (j) } \sqrt[6]{72}.$$

Logarithms

7. Find the logarithms of
- $81, \frac{1}{8}, 243, \frac{1}{27}, 1$
- to the base 3.

$$\text{Ans. } 4; -2; 5; -3; 0.$$

8. What are the characteristics of the logarithms of the following numbers?

$$3,743.4; 1,000,000; 0.001; 0.236$$

$$\text{Ans. } 3; 6; 7-10; 9-10.$$

9. If the mantissas of 25.84, 0.5801, 1342 are 41229, 76350, 12775 respectively, what are the logarithms of the following?

$$25.84; 0.5801; 1342; 25.84 \times 0.5801; 25 \times 84 \times 0.5801 \times 1342;$$

$$\frac{25.84}{1342}, \frac{1342}{.5801}, \frac{25.84 \times 1342}{.5801}; (1342)^4; \sqrt[5]{1342}.$$

$$\text{Ans. } 1.41229; 9.76350 - 10; 3.12775; 11.17579 - 10; 6.21347; 8.28454 - 10; 3.36125; 4.77654; 12.51100; 0.62555.$$

10. By means of the tables find the logarithms of the following numbers:

Log tables

$$8,234; .00068; 568,430; 7.6835; 0.89999.$$

$$\text{Ans. } 3.91561; 6.83251 - 10; 5.75467; 0.88555; 9.95424 - 10.$$

11. By means of the tables find the antilogarithms of the numbers:

$$8.66666 - 10; 2.34156; 0.11111; 9.12315 - 10.$$

$$\text{Ans. } 0.046415; 219.56; 1.2916; 0.13288.$$

12. By means of logarithms compute the values of the following expressions to five significant digits.

$$(a) \sqrt[30]{3.6875}; \quad (b) \frac{37.687 \times 0.04351}{376.88}; \quad (c) (3.1416)^7; \quad (d) 10\frac{7}{8};$$

$$(e) \sqrt[3]{(1.1212)^2}; \quad (f) \left(\frac{56,792}{400,150} \right)^4.$$

$$\text{Ans. } (a) 1.0445; (b) 3020.3; (c) 1.0793; (d) 0.0043509;$$

$$(e) 7.4990; (f) 0.00040573.$$

13. The resistance of air to a certain ball varies directly as the three-halves power of the speed for the range of speeds of this problem. When the speed of the ball is 2500 ft. per minute, the air resistance is exactly 125 lb. By the use of logarithms find the air resistance of the ball when the speed of the ball is 3,341.5 ft. per minute.

$$\text{Ans. } 193.16 \text{ lb.}$$

TRIGONOMETRIC FUNCTIONS OF ANGLES

The word *trigonometry* comes from the Greek and means “the measure of triangles.” Angles have already been used in the treatment of the Pythagorean theorem. For trigonometric studies it will be necessary to give meaning to the “size” of angles, and to the concepts of “positiveness” and “negativeness” of angles.

Starting with a ray in the position OX of Fig. 232, and keeping the initial point O of the ray fixed, we generate an angle by rotating this ray in the plane to a position OA . In generating the angle, the direction of rotation is for convenience kept constant, that is, the ray is rotated clockwise, or counterclockwise. By *clockwise* is meant the direction in which the hands of a clock revolve.

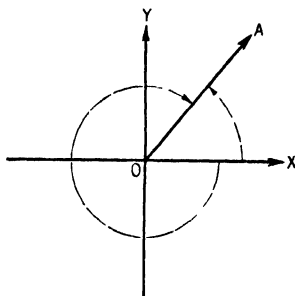


FIG. 232. TWO ANGLES TERMINATING AT OA

*Positive and
negative angles*

If the angle is generated by rotating the ray counterclockwise, it is said to be a *positive angle*, and if the ray is rotated clockwise, it generates a *negative angle*. In Fig. 232 two angles are indicated, both having OX as the initial side and OA as the terminal side. The angle obtained by the smaller rotation is positive, and the other is negative.

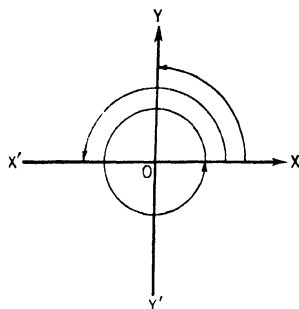


FIG. 233. ANGLES OF 90° , 180° , AND 360°

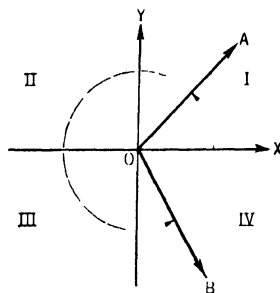


FIG. 234. ADDITION OF ANGLES XOA AND AOB

The initial and terminal sides of the angle may be identical. In Fig. 233 such an angle is indicated, formed by one complete revolution. Such an angle is said to be an angle of 360 degrees, written 360° . One half of this complete rotation produces a *straight angle* (of 180°); and one quarter of a complete rotation generates a *right angle*, that is, an angle of 90° . These are also illustrated in Fig. 233.

*Addition of
angles*

Two angles are added by placing the initial side of the second angle on the terminal side of the first. The initial side of the sum is the initial side of the first angle, and the terminal side of the sum is the terminal side of the second angle. (See Fig. 234.) If both of the given angles are generated by the process of rotating rays counterclockwise, the sum of these angles is understood to be generated by rotating a ray continuously in the same manner. In Fig. 235 we have added the positive angle XOA to the negative angle AOB to obtain the negative angle XOB . The ray was rotated first positively (counterclockwise) and then negatively (clockwise). Addition of an angle of 400° to an angle of -700° is an angle of -300° . Evidently an angle may exceed one revolution which is 360° as shown in Fig. 236.

*Minutes and
seconds*

The right angle is divided into 90 equal parts, each part being equal to one degree. Each degree is the sum of 60 equal parts,

termed *minutes*, and each minute is subdivided into 60 equal parts, called *seconds*. One minute is written as $1'$, and one second as $1''$. One minute is $60''$, and one degree is $60'$.

In the United States Army the unit of angular measure is the *mil*. *Mils* This is $\frac{1}{6400}$ of one revolution, or approximately $\frac{1}{1000}$ of a radian.

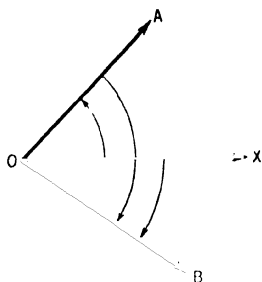


FIG. 235. ADDITION OF A NEGATIVE ANGLE TO A POSITIVE ONE

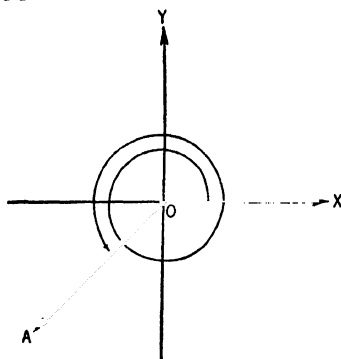


FIG. 236. ANGLE OF MORE THAN ONE REVOLUTION

In Figs. 232–236 angles have been drawn with OX as the initial side. This was done for convenience, although the initial side can be taken anywhere. When the initial side of the angle is taken to be OX , the angle is said to be in *standard position*. *Standard position of angles*

If the terminal side of an angle is in a given quadrant, the angle is said to be in that quadrant. As shown in Fig. 234 the angle 45° is an angle in the first quadrant, whereas -60° is in the fourth. The angle of 0° , which has the same initial and terminal sides, is in the first quadrant. *Quadrants*

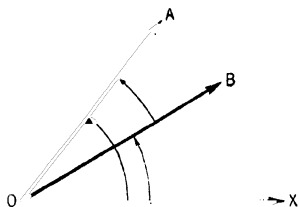


FIG. 237. SUBTRACTION OF ANGLES

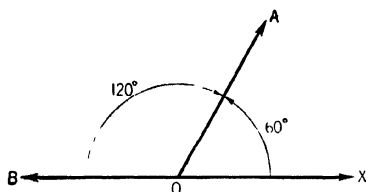


FIG. 238. SUBTRACTION OF -120° FROM 60°

From the addition of angles the concept of subtraction follows immediately. In Fig. 237 the angle XOB is the angle XOA minus the angle BOA . It will be seen that we have the following rule: *Subtraction of angles*

To subtract an angle BOA from an angle XOA , place the terminal side of BOA on the terminal side of XOA . The *initial side of the new angle* representing the difference of these angles is the initial side OX of the angle XOA , and the terminal side of the new angle is the initial side OB of the angle BOA . In Fig. 238 the angle -120° is subtracted from 60° to yield the angle XOB of angular measure 180° .

Examples of the addition and subtraction of angles follow.

*Examples of
addition and
subtraction of
angles*

EXAMPLE 1. *Addition.*

$$\begin{array}{r} 20^\circ 50' 37'' \\ 400^\circ 40' 34'' \\ \hline 420^\circ 90' 71'' \end{array}$$

Since $71''$ is $1' 11''$ and $90'$ is $1^\circ 30'$, the answer should be written as $421^\circ 31' 11''$.

EXAMPLE 2. *Subtraction.*

Suppose that you wish to subtract $87^\circ 30' 30''$ from $135^\circ 14' 22''$. Since $30''$ cannot be subtracted from $22''$ to give a positive result, one minute is borrowed from the $14'$ and added to the $22''$ to give $82''$. The larger angle is now written as $135^\circ 13' 82''$. For the same reason that $22''$ was changed to $82''$, one degree is borrowed from 135° and added to $13'$ so that the larger angle becomes $134^\circ 73' 82''$. The subtraction proceeds as indicated.

$$\begin{array}{r} 134^\circ 73' 82'' \\ 87^\circ 30' 30'' \\ \hline 47^\circ 43' 52'' \end{array}$$

*Negative
result*

EXAMPLE 3. *Subtraction.*

The result of subtracting $347^\circ 2' 50''$ from $13^\circ 8' 10''$ is negative. The angle $13^\circ 8' 10''$ can therefore be subtracted from $347^\circ 2' 50''$ and a negative sign placed before the answer. We write $347^\circ 2' 50''$ as $346^\circ 62' 50''$, and subtract as shown.

$$\begin{array}{r} 346^\circ 62' 50'' \\ - 13^\circ 8' 10'' \\ \hline 333^\circ 54' 40'' \end{array}$$

Hence, the answer is $-333^\circ 54' 40''$.

ANGLE, ARC, AND RADIAN

Arc of circle

When you move from A to B on a circle as in Fig. 239, we say that you *generate an arc* of the circle. The circumference of a circle

covers an arc of one revolution. If you move halfway around a circle, you generate an arc of a half revolution, and if you go around a circle twice, you generate an arc of two revolutions. These examples serve to show that an arc may not only be part of the circumference of a circle, but even more than the entire circumference.

In Fig. 239 an angle AOB is drawn with vertex at the center of a circle of radius r . When the angle AOB is generated, the side OA is rotated counterclockwise to the position OB as indicated. In this generation the point A sweeps over an arc of the circle, called the arc *subtended* by the angle AOB . This arc is called the *arc* AB . The arc AB is said to have the same angular measure as the angle AOB . An angle of 50° with vertex at the center of a circle subtends an arc of 50° on the circumference of the circle. The complete circumference of a circle is an arc of 360° .

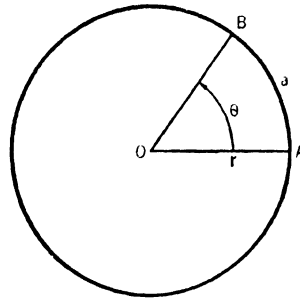


FIG. 239. ARC AB SUBTENDED BY THE ANGLE AOB

We recall that the length of the circumference of a circle is π times the length of the diameter. The idea of the length of an arc has thus already risen in this chapter. We let a and b denote arcs on a circle. If the number of degrees in the arc a is k times the number of degrees in the arc b , the arc a is k times as long as b . The circumference of a circle is thus four times as long as an arc of the circle subtended by an angle of 90° with vertex at the center of this circle. In a circle of radius 1 in. the circumference is 2π in.; for a radius of 10 in. the length of this arc is 20π in.

If the angle AOB in Fig. 239 subtends an arc AB whose length equals the radius of the circle, the angle AOB is said to be an angle of *one radian*, and the arc AB is said to be an arc of one radian. For most mathematical considerations the radian is a more convenient unit of measure than the degree. For this reason radian measure is said to be the natural system of measuring angles.

An angle of 360° is an angle of 2π radians since the circumference of a circle of radius r is $2\pi r$. It follows that an angle of 1° is an angle of $\pi/180$ radians. The quotient $\pi/180$ is approximately 0.01745, so one degree is 0.01745 radian.

In changing from degrees, minutes, and seconds to radians with-

out the use of tables for this purpose, it is convenient to change minutes and seconds to decimal parts of a degree. One minute is $(\frac{1}{60})^\circ$, or $0.0166 \dots^\circ$, and one second is $(\frac{1}{3600})^\circ$, or $0.000277 \dots^\circ$. Let us change $514^\circ 15' 30''$ to radians. We have

$$15' = 0.25^\circ; \quad 30'' = 0.0083^\circ$$

so that

$$514^\circ 15' 30'' = 514.258^\circ.$$

Multiplying 514.258° by 0.01745 we obtain 8.9738 radians as an approximate equivalent to $514^\circ 15' 30''$.

*Applications of
radian measure*

Using the notation θ for the measure of the angle AOB in radians, r for the length of the radius OA , and a for the length of the arc AB as shown in Fig. 239, we have the relation

$$\theta = \frac{a}{r}.$$

The Greek letter θ is read "theta." The angle θ in radians is the length a of the arc divided by the length r of the radius, or in usual terminology the angle is the arc divided by the radius.

EXAMPLE 1.

Length of arc

Consecutive spokes make an angle of 10° at the center of a wheel of diameter 4 ft. The problem is to find the length of the arc subtended by the angle.

The angle 10° is first changed to $\pi/18$ radians. Multiplying $\pi/18$ by the radius 2 (ft.), the length of the arc is found to be $\pi/9$ ft. To obtain the answer to four figures you can divide 9 into 3.1416, or you can change 10° to 0.1745 radian and multiply this figure by 2. By either method the answer is 0.349 ft.

EXAMPLE 2.

Linear speed

A pulley 8 in. in diameter attached to an electric motor is revolving at an *angular speed* of 1800 revolutions, that is $1800 \times 2\pi$ or 3600π radians, per minute. A belt is attached to the pulley. We shall find the speed at which a point on the belt is traveling, that is, the speed of the belt. The speed of a body traveling along a straight line is called a *linear speed*, which we wish to compute here.

In 1 min. the pulley rotates through 3600π radians. A point on the circumference 4 in. from the center travels 3600π times 4, or $14,400\pi$ in. per minute. Since the belt is always in contact with the pulley, a point on the belt travels at the same speed as a point on the circumference of the pulley. The speed of the belt

is thus $14,400\pi$ in. per minute. This amounts to 3770 ft. per minute. Multiplying by 60, the number of minutes in an hour, and dividing by 5280, namely the number of feet in a mile, we find that the belt speed is approximately 43 m.p.h.

EXAMPLE 3.

A locomotive travels 110 m.p.h. One wheel of the locomotive is 6 ft. in diameter. We shall determine the angular speed of the wheel in revolutions per minute, in radians per second, and in degrees per second. *Angular speed*

Multiplying 110 (miles per hour) by 5280 and dividing by 60 we find that the speed of the locomotive is 9680 (feet per minute). The circumference of the wheel is π times the diameter or 6π ft. Since a point on the circumference of a wheel travels at the same speed as the locomotive, the angular speed of the wheel in revolutions per minute (abbreviated r.p.m.) is $9680/6\pi$, or approximately 514 r.p.m. Each revolution is 360° and there are 60 sec. in a minute. Hence, the angular speed is also $514 \times 360 \div 60$ or 3084 degrees per second. Since there are 2π radians in one revolution and 60 sec. in 1 min., the speed in radians per second is $\frac{514 \times 2\pi}{60}$, or 53.7.

The above examples illustrate the use of radian measure in the treatment of circles. Since circles play a dominant role in both mathematics and engineering, radian measure is of considerable importance.

SIMILAR TRIANGLES

In the treatment of similar triangles to follow, the angles are measured counterclockwise so that they are all positive. The angular measure of each is taken to be the smallest positive measure the angle can have. *Measuring angles*

In Fig. 240 the angles A , B , C of the triangle ABC are equal respectively to the angles A' , B' , C' of the triangle $A'B'C'$. The triangles are therefore said to be *similar*.

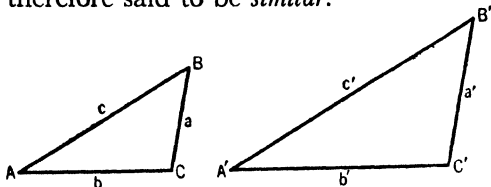


FIG. 240. SIMILAR TRIANGLES

*Proportionality
of sides*

We shall show that corresponding sides of similar triangles are proportional, that is, for Fig. 240,

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

*Sum of the
angles in a
triangle*

A rectangle has four right angles formed by its perpendicular sides. These four right angles add up to 360° . From Fig. 221 it appears that the sum of the angles A , B , and C in the right triangle ABC is one-half of the sum of the angles of the rectangle. It follows that the angles in a right triangle add up to 180° . Any triangle can be cut up into two right triangles as shown in Fig. 219. The sum of the angles in the right triangles ABD and DBC of the left figure is 360° . The angle ADC is a *straight angle* having a measure of 180° . This angle is not one of the angles of the triangle ABC , although it is the sum of two angles of the right triangles ABD and DBC . The angles in the whole triangle ABC therefore add up to an angle of 180° . Hence, *the sum of the angles in any triangle equals a straight angle*.

Parallel lines

Two lines in the same plane which never meet are said to be parallel. We cut two lines l and l' by a line m , as shown in Fig. 241.

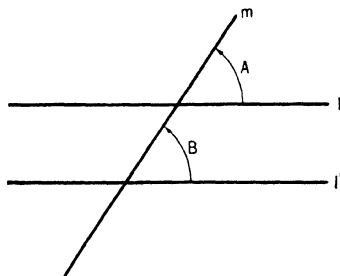


FIG. 241. PARALLEL LINES CUT BY A THIRD LINE

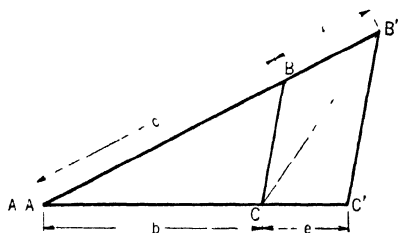


FIG. 242. SUPERIMPOSED SIMILAR TRIANGLES

From propositions of geometry it can be proved that the lines l and l' are parallel if and only if the angles A and B are identical.

The triangles of Fig. 240 are superimposed as in Fig. 242 so that A and A' coincide, whereas the side AB lies on $A'B'$ and AC on $A'C'$. Because the angles C and C' are equal, the sides BC and $B'C'$ are parallel. We draw the dotted lines CB' and BC' .

The triangles ABC and CBC' have bases AC and CC' respectively and the same altitude. Hence it follows that

$$\frac{\text{Area of } ABC}{\text{Area of } CBC'} = \frac{b}{e}.$$

Similarly,

$$\frac{\text{Area of } ABC}{\text{Area of } CBB'} = \frac{c}{f}.$$

The triangles CBC' and CBB' have the common base CB and the same altitude. These triangles therefore have the same area. *Proportionality of sides*
In the equations above, b/e and c/f are now seen to be equal to the same quantity; hence, we write

$$(48) \quad \frac{b}{e} = \frac{c}{f}.$$

Adding 1 to both sides (e/e on the left and f/f on the right) we have

$$\frac{b}{e} + \frac{e}{e} = \frac{c}{f} + \frac{f}{f}; \quad \text{or} \quad \frac{b+e}{e} = \frac{c+f}{f}; \quad \text{which with (48) yields} \quad \frac{b'}{e} = \frac{c'}{f}$$

because $b' = b + e$ and $c' = c + f$. Hence, the following is proved:
If two triangles are similar, corresponding sides are proportional. Conversely, it can be proved that if corresponding sides of two triangles are proportional, the triangles are similar. To say that two triangles are similar is the same as saying that they have the same shape.

FUNCTIONS OF ANGLES

An angle with positive measure less than 90° is termed an *acute angle*. Given any acute angle A , a right triangle can be constructed with A as one of its angles. The measure of the remaining acute angle in this triangle is 90° minus the measure of the angle A . All angles of the right triangle are thus determined by A . As a result, the shape of a right triangle is fixed by the measure of one of its acute angles. In this sense an acute angle A corresponds uniquely to all similar right triangles containing A . Any property which all of these triangles possess in common is thus a property of A . In particular, the ratio of one side of a right triangle containing the angle A to another side of this triangle is the same as the ratio of corresponding sides of any other similar right triangle containing the angle A . These ratios are the *trigonometric functions* of the angle A . They are called *functions* because they depend for their values on the measure of the angle A .

Suppose we let θ be an acute angle in a right triangle. The trigonometric functions of θ are defined to be:

Trigonometric
functions of
acute angles

$$(49) \quad \text{sine of } \theta = \sin \theta = \frac{\text{side opposite}}{\text{hypotenuse}}$$

$$(50) \quad \text{cosine of } \theta = \cos \theta = \frac{\text{side adjacent}}{\text{hypotenuse}}$$

$$(51) \quad \text{tangent of } \theta = \tan \theta = \frac{\text{side opposite}}{\text{side adjacent}}$$

$$(52) \quad \text{cotangent of } \theta = \text{ctn } \theta = \frac{\text{side adjacent}}{\text{side opposite}}$$

$$(53) \quad \text{secant of } \theta = \sec \theta = \frac{\text{hypotenuse}}{\text{side adjacent}}$$

$$(54) \quad \text{cosecant of } \theta = \text{cosec } \theta = \frac{\text{hypotenuse}}{\text{side opposite}}$$

Here \sin is the abbreviation of sine; the word "cotangent" is often shortened to *cot* instead of *ctn*. In the right triangle of Fig. 220 we have

$$\sin A = \frac{a}{c}.$$

Functions
of 45°

A right triangle with equal legs has acute angles each equal to 45° . Such a triangle is shown in Fig. 243. In this case the side

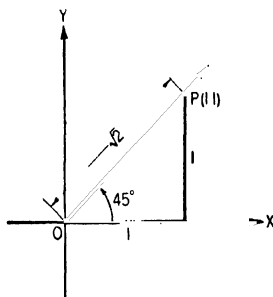


FIG. 243. ANGLE OF 45°

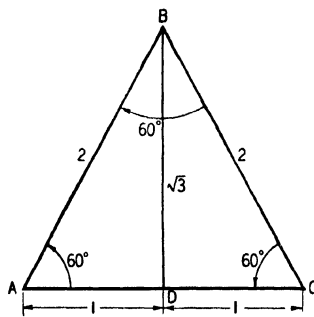


FIG. 244. EQUILATERAL TRIANGLE

opposite the indicated angle of 45° is 1, the side adjacent to this angle is 1, and the hypotenuse is $\sqrt{2}$. We therefore have the following:

$$\sin 45^\circ = \frac{\sqrt{2}}{2}; \quad \tan 45^\circ = 1; \quad \sec 45^\circ = \sqrt{2};$$

$$\cos 45^\circ = \frac{\sqrt{2}}{2}; \quad \text{ctn } 45^\circ = 1; \quad \text{cosec } 45^\circ = \sqrt{2}.$$

A triangle with equal angles is called an *equilateral triangle*. (See Fig. 244.) Such a triangle also has equal sides. Since the angles add up to 180° , each angle is one of 60° . We will let such a triangle have a side 2 units in length, as shown in Fig. 244. By

the Pythagorean theorem, applied to the right triangle ABD , the altitude BD of the triangle has length $\sqrt{3}$. The angle ABD is one-half of the angle ABC , and is therefore an angle of 30° . To obtain

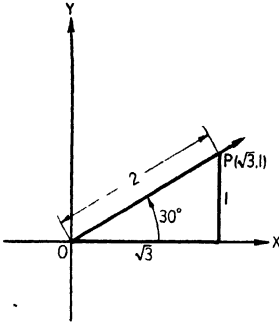


FIG. 245. ANGLE OF 30°

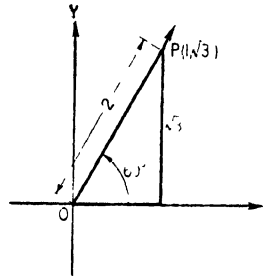


FIG. 246. ANGLE OF 60°

the trigonometric functions of 30° we may therefore use a right triangle with legs 1, $\sqrt{3}$, and hypotenuse 2, as shown in Fig. 245. We can see that:

$$\begin{array}{lll} \sin 30^\circ = \frac{1}{2}; & \tan 30^\circ = \frac{\sqrt{3}}{3}; & \sec 30^\circ = \frac{2\sqrt{3}}{3}; \\ \cos 30^\circ = \frac{\sqrt{3}}{2}; & \cot 30^\circ = \sqrt{3}; & \operatorname{cosec} 30^\circ = 2. \end{array}$$

*Functions
of 30°*

You can readily verify from Fig. 246 that

$$\begin{array}{lll} \sin 60^\circ = \frac{\sqrt{3}}{2}; & \tan 60^\circ = \sqrt{3}; & \operatorname{cosec} 60^\circ = \frac{2\sqrt{3}}{3}; \\ \cos 60^\circ = \frac{1}{2}; & \cot 60^\circ = \frac{\sqrt{3}}{3}; & \sec 60^\circ = 2. \end{array}$$

*Functions
of 60°*

The trigonometric functions of a nonacute angle θ cannot be obtained in the manner considered above because there is no right triangle with θ as one of its angles. For this reason the angle θ is placed in standard position on a Cartesian coordinate system as shown in Fig. 247. We select any point P on the terminal side of θ . We let (x, y) be the coordinates of P , and r the distance from the origin O to P . The definitions of the trigonometric functions of θ follow:

*Trigonometric
functions of
nonacute
angles*

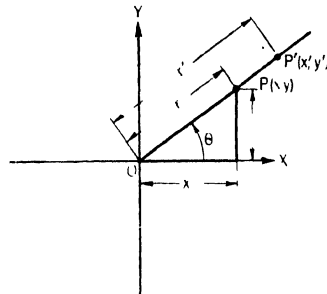


FIG. 247. ANGLE IN STANDARD POSITION ON A CARTESIAN COORDINATE SYSTEM

$$(55) \quad \sin \theta = \frac{\text{ordinate}}{\text{distance } r} = \frac{y}{r}$$

$$(56) \quad \cos \theta = \frac{\text{abscissa}}{\text{distance } r} = \frac{x}{r}$$

*Principal
functions of
angles*

$$(57) \quad \tan \theta = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x}$$

$$(58) \quad \text{ctn } \theta = \frac{\text{abscissa}}{\text{ordinate}} = \frac{x}{y}$$

$$(59) \quad \sec \theta = \frac{\text{distance } r}{\text{abscissa}} = \frac{r}{x}$$

$$(60) \quad \text{cosec } \theta = \frac{\text{distance } r}{\text{ordinate}} = \frac{r}{y}$$

The values of the trigonometric functions depend on θ only, not on the position of the point P on the terminal side of θ .

In Figs. 243–246 angles discussed above have been placed in standard position. From the coordinates of the points P in these figures and definitions 55 to 60, the trigonometric functions of these angles may be computed.

*Functions
of 0°*

We use the point $(1, 0)$ as in Fig. 248 to get the trigonometric functions of 0° . It follows from definitions 55, 56, 57, and 59 that $\sin 0^\circ = \frac{0}{1} = 0$; $\cos 0^\circ = \frac{1}{1} = 1$; $\tan 0^\circ = \frac{0}{1} = 0$; $\sec 0^\circ = \frac{1}{1} = 1$. Since division by 0 is not allowed, $\text{ctn } 0^\circ$ and $\text{csc } 0^\circ$ have no meaning. Sometimes the sign for infinity, ∞ , is written for these functions, meaning that as an acute angle of x° gets smaller and smaller,

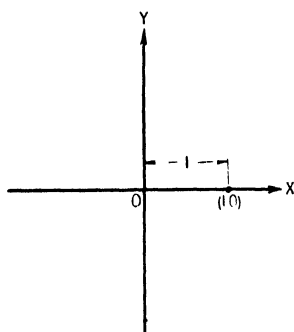


FIG. 248. ANGLE OF 0°

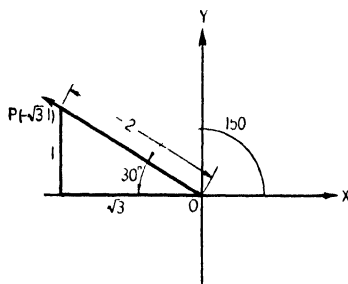


FIG. 249. ANGLE OF 150°

$\text{ctn } x^\circ$ and $\text{cosec } x^\circ$ get larger and larger; in fact as x goes to 0, the functions $\text{ctn } x^\circ$ and $\text{cosec } x^\circ$ increase beyond all bounds.

The study of the trigonometric functions of an angle over 90° in standard position can always be reduced to that of the acute

*Functions
of 150°*

angle whose terminal side is identical with that of the given angle, and whose initial side is along the one half of the x axis. Thus if 150° is subtracted from 180° , the acute angle 30° of Fig. 249 is obtained. The right triangle of Fig. 245 containing 30° is used in Fig. 249 to obtain the trigonometric functions of 150° . From the point $P(-\sqrt{3}, 1)$ on the terminal side of 150° and from definitions 55 to 60 we have the following results:

$$\begin{aligned} \sin 150^\circ &= \frac{1}{2}; & \tan 150^\circ &= -\frac{\sqrt{3}}{3}; & \sec 150^\circ &= -\frac{2\sqrt{3}}{3}; \\ \cos 150^\circ &= -\frac{\sqrt{3}}{2}; & \csc 150^\circ &= 2; & \cot 150^\circ &= -\sqrt{3}; \end{aligned}$$

The sides and angles of a triangle are called the *parts* of the triangle. Trigonometric functions are particularly useful in finding unknown parts of a triangle. We can illustrate with an example.

Solution of triangles in surveying

EXAMPLE. To find a leg of a right triangle.

A man whose eye level is 6 ft. above the ground stands 150 ft. from a building. The measure of the *angle of elevation* of the building is exactly 30° , which is the angle between the horizontal line of sight and the line of sight from the man to the top of the building. How high is the building?

We can let y be the height of the building, $y - 6$ being the vertical distance from the level of the man's eye to the top of the building, as shown in Fig. 250. By definition

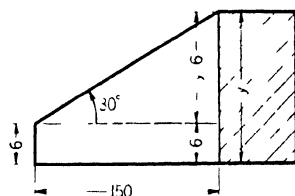


FIG. 250. MAN ABOUT 6 FEET HIGH AND BUILDING y FEET HIGH

$$\tan 30^\circ = \frac{y - 6}{150}; \quad y - 6 = 150 \tan 30^\circ; \quad \text{or} \quad y = 150 \tan 30^\circ + 6.$$

Substituting the value of $\tan 30^\circ$, which is $\sqrt{3}/3$ or 0.577, we have

$$y = 150 \times 0.577 + 6 = 92.55.$$

The height of the building is therefore 92 ft. 7 in.

The trigonometric functions of an angle are not independent. In fact, all of them are determined by any one trigonometric function and the quadrant in which an angle lies, when placed in standard position. For example, let us suppose that θ is an angle lying in the third quadrant for which $\sin \theta = -\frac{1}{3}$. We draw a circle of radius 3 with center at the origin as shown in Fig. 251. We also construct the line $y = -1$ parallel to the x axis and one

Dependence of trigonometric functions

unit below. The point P where this circle and line intersect in the third quadrant is a point on the terminal side OP of θ . By the Pythagorean theorem, applied to the right triangle formed by dropping a perpendicular from P to the x axis, the point P has coordinates $(-2\sqrt{2}, -1)$. The trigonometric functions of θ are therefore:

$$\cos \theta = -\frac{2\sqrt{2}}{3}; \quad \csc \theta = 2\sqrt{2}; \quad \operatorname{cosec} \theta = -3.$$

$$\tan \theta = \frac{\sqrt{2}}{4}; \quad \sec \theta = -\frac{3\sqrt{2}}{4}.$$

Signs

The point P' of Fig. 251 is on the terminal side OP' of an angle in the fourth quadrant also having the sine $-\frac{1}{3}$. Since a trigonometric

Third and
fourth
quadrant Δ

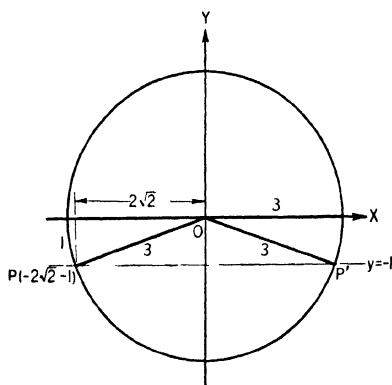


FIG. 251. ANGLE θ DETERMINED BY ITS SINE AND QUADRANT

function has the same plus or minus sign in two quadrants, an angle with one known trigonometric function may be located as lying in either of two possible quadrants. For example, since the coordinate x has the same sign as y in the first and third quadrants, the tangent is positive for all angles lying in these quadrants. It follows that any angle for which the tangent is positive lies in the first or third quadrant.

Then, if this same angle is known to have a negative sine, it must be a third-quadrant angle because only angles in the third and fourth quadrants have negative sines.

Trigonometric
identities

From the definitions of the trigonometric functions we have:

$$\begin{aligned} \operatorname{cosec} \theta &= \frac{1}{\sin \theta}; & \sin \theta &= \frac{1}{\operatorname{cosec} \theta}; & \tan \theta &= \frac{1}{\operatorname{ctn} \theta}; & \tan \theta &= \frac{\sin \theta}{\cos \theta}; \\ \sec \theta &= \frac{1}{\cos \theta}; & \cos \theta &= \frac{1}{\sec \theta}; & \operatorname{ctn} \theta &= \frac{1}{\tan \theta}; & \operatorname{ctn} \theta &= \frac{\cos \theta}{\sin \theta}. \end{aligned}$$

The above relations are true for any value of θ , and are therefore called *identities*. To abbreviate notations we write $(\sin \theta)^2 = \sin^2 \theta$, also $(\cos \theta)^2$ as $\cos^2 \theta$, and so on. Since $\sin \theta = y/r$ and $\cos \theta = x/r$ we have

$$\sin^2 \theta + \cos^2 \theta = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2}.$$

By the Pythagorean theorem $x^2 + y^2 = r^2$, whence we have proved that

$$(61) \quad \sin^2 \theta + \cos^2 \theta = 1.$$

In the same way we can show that

$$(62) \quad 1 + \tan^2 \theta = \sec^2 \theta; \quad 1 + \cot^2 \theta = \operatorname{cosec}^2 \theta.$$

The above identities are called the *fundamental trigonometric identities* because they are used so often, but there are countless other identities involving the trigonometric functions of θ .

The fundamental identities 61 and 62 are useful in simplifying trigonometric expressions. For example, we may write *Simplifying expressions*

$$(\sec^2 \theta - 1) \operatorname{cosec}^2 \theta = (\tan^2 \theta) \operatorname{cosec}^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} \times \frac{1}{\sin^2 \theta} = \frac{1}{\cos^2 \theta} = \sec^2 \theta.$$

In carrying out computations it is much easier to use $\sec^2 \theta$ than the original expression.

These identities can also be used to find the trigonometric functions of an angle, given one such function of the angle. As an illustration let us suppose that θ is an angle in the fourth quadrant with $\operatorname{cosec} \theta = -\frac{5}{3}$. Since the sine is the reciprocal of the cosecant, we have $\sin \theta = -\frac{3}{5}$. From $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$ and the fact that the cotangent is negative in the fourth quadrant, we have $\cot \theta = -\sqrt{\frac{25}{9} - 1} = -\sqrt{\frac{16}{9}} = -\frac{4}{3}$. Since the tangent is the reciprocal of the cotangent we have $\tan \theta = -\frac{3}{4}$. From the relationship $\tan \theta = \frac{\sin \theta}{\cos \theta}$ we determine that $\cos \theta = \frac{\sin \theta}{\tan \theta} = \frac{-\frac{3}{5}}{-\frac{3}{4}} = \frac{4}{5}$. *Computing trigonometric functions*

From the relation $\sec \theta = 1/\cos \theta$, we have $\sec \theta = \frac{5}{4}$.

When an angle θ is small, the value of θ in radians and the values of $\sin \theta$ and $\tan \theta$ are approximately the same in the sense that as θ goes to 0, the fractions $\sin \theta/\theta$ and $\tan \theta/\theta$ go to 1. Thus the sine and tangent of one-tenth radian equal $\frac{1}{10}$ to an accuracy that is satisfactory for most engineering calculations. A particularly important application is to the angle of deflection or slope of a loaded beam. This angle is usually small and so its sine and tangent are equal to its angular value in radians to reasonable accuracy. *Sines and tangents of small angles*

1. Perform the indicated operations graphically, using a protractor:

$$90^\circ + 405^\circ; 90^\circ - 135^\circ; 315^\circ - 90^\circ; 180^\circ - 45^\circ; 270^\circ + 135^\circ.$$

2. Change $190^\circ 40' 20''$ to its decimal equivalent.

$$\text{Ans. } 190.6722 \dots^\circ.$$

3. Perform the indicated computations:

$$400^\circ 57' 38'' + 210^\circ 33' 50''; 100^\circ 13' 9'' - 45^\circ 40' 40''; 10^\circ 15' 50'' - 100^\circ 20' 10''.$$

$$\text{Ans. } 611^\circ 31' 28''; 54^\circ 32' 29''; -90^\circ 4' 20''.$$

Problems

4. Change 87.345° to degrees, minutes, and seconds. *Ans.* $87^\circ 20' 43''$.

5. In what quadrants do the following angles lie?

50° , 400° , 700° , 200° , -200° , -50° , -500° .

Ans. I, I, IV, III, II, IV, III.

Radians

6. The radius of the earth is approximately 4000 miles. Two cities on the earth's surface are 1000 miles apart, this distance being measured on the surface. How many degrees, minutes, and seconds are they apart, and what is the difference in their sun times if one is directly east of the other?

Hint. Twenty-four hours corresponds to 360° .

Ans. $14^\circ 19' 26''$; 57 min. 18 sec. in solar time.

7. Change the following angles to radians: 37.125° ; $300^\circ 10' 40''$.

Ans. 0.648; 5.239.

8. Change these angles to radians:

100° ; -47° ; 30° ; 45° ; 60° ; 90° ; 120° ; -135° ; -150° .

Ans. 0.55π ; $-.26\pi$; $\frac{\pi}{6}$; $\frac{\pi}{4}$; $\frac{\pi}{3}$; $\frac{\pi}{2}$; $\frac{2\pi}{3}$; $-\frac{3\pi}{4}$; $-\frac{5\pi}{6}$.

9. Change the following angles from radians to degrees, minutes, and seconds:

7π radians; 3 radians; $\frac{\pi}{7}$ radians.

Ans. 1260° ; $171^\circ 53' 14''$; $25^\circ 41' 51''$.

10. The bob of a pendulum 1 ft. long swings through an arc of 2 in. Through what angle does the pendulum swing? *Ans.* $\frac{1}{6}$ radian, or $9^\circ 32' 57''$.

Revolving machines

11. The rim speed of a flywheel is 100 ft. per sec. Find the angular speed of the flywheel in radians per second if its diameter is 3 ft. *Ans.* $66\frac{2}{3}$ radians per second.

12. A wheel in a slow-moving mechanism revolves with an angular speed of 100° per minute. Find the rim speed in feet per second if the radius of the wheel is 2 in. *Ans.* 0.00485 ft. per sec.

13. A belt in contact with a pulley travels with a speed of 10 m.p.h. The radius of the pulley is 3 in. Find the angular speed of the pulley in revolutions per minute, and in radians per second. *Ans.* 560 r.p.m.; $58\frac{2}{3}$ radians per second.

Trigonometric functions

14. Find the trigonometric functions of 90° , 120° , 135° , 180° , 225° , and 270° .

| | Angle | Sin | Cos | Tan | Ctn | Sec | Cosec |
|-------------|-------------|-----------------------|-----------------------|-------------|-----------------------|-------------|-----------------------|
| <i>Ans.</i> | 90° | 1 | 0 | No meaning | 0 | No meaning | 1 |
| | 120° | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $-\sqrt{3}$ | $-\frac{\sqrt{3}}{3}$ | -2 | $\frac{2\sqrt{3}}{3}$ |
| | 135° | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | -1 | -1 | $-\sqrt{2}$ | $\sqrt{2}$ |
| | 180° | 0 | -1 | 0 | No meaning | -1 | No meaning |
| | 225° | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | 1 | 1 | $-\sqrt{2}$ | $-\sqrt{2}$ |
| | 270° | -1 | 0 | No meaning | 0 | No meaning | -1 |

15. An angle θ is an acute angle of a right triangle having a hypotenuse of 4 ft. The side adjacent to the angle is 3 ft. Find the trigonometric functions of the angle.

$$\text{Ans. } \sin \theta = \frac{\sqrt{7}}{4}; \cos \theta = \frac{3}{4}; \operatorname{cosec} \theta = \frac{4\sqrt{7}}{7}; \sec \theta = \frac{4}{3};$$

$$\tan \theta = \frac{\sqrt{7}}{3}; \operatorname{ctn} \theta = \frac{3\sqrt{7}}{7}$$

16. An angle θ is in the third quadrant. Given that $\sin \theta = -\frac{5}{13}$, find the other trigonometric functions of θ .

$$\text{Ans. } \cos \theta = -\frac{12}{13}; \tan \theta = \frac{5}{12}; \operatorname{ctn} \theta = \frac{12}{5}; \sec \theta = -\frac{13}{5}; \operatorname{cosec} \theta = -\frac{13}{5}.$$

17. By the fundamental identities simplify the expression

$$\frac{\cos \theta}{1 - \sin \theta} - \frac{1 - \sin \theta}{\cos \theta}.$$

Hint. Multiply the first fraction above and below by $1 + \sin \theta$. *Ans.* $2 \tan \theta$.

18. The line of sight of a man to an airplane makes an angle of 45° with horizontal. The airplane is 2 miles up. How far is the airplane from the man? *Solving triangles*

$$\text{Ans. } 2\sqrt{2} \text{ miles, or approximately 2.83 miles.}$$

19. A sheet of metal is in the shape of a right triangle with a 30° angle. The side adjacent to the angle is 30 in. long. What is the area of the metal sheet?

$$\text{Ans. } 260 \text{ sq. in., or } 1.80 \text{ sq. ft.}$$

20. What are the plus or minus signs of trigonometric functions of angles in the fourth quadrant? *Properties of trigonometric functions*

$$\text{Ans. } \sin -; \cos +; \tan -; \operatorname{ctn} -; \sec +; \operatorname{cosec} -.$$

21. Prove that $1 + \cot^2 \theta = \sec^2 \theta$.

22. What is the range of values of the sine, cosine, tangent, and secant?

Ans. The sine and cosine are between -1 and 1 ; the tangent can be any real number; and the secant is greater than 1 or equal to 1 , or less than or equal to -1 .

23. What trigonometric functions of an acute angle θ increase with θ ?

Ans. Sine, tangent, secant.

USE OF TRIGONOMETRIC TABLES

You have already learned how to compute the trigonometric functions of certain angles, such as 30° , 45° , and 60° . It would be both tedious and difficult for you to derive the value of a trigonometric function of an angle each time you had to use it. For this reason tables of values of trigonometric functions have been computed. Such tables will be found at the back of this book. These tables can be constructed from exact formulas derived from the calculus for each trigonometric function. These formulas enable one to find a trigonometric function of any angle to any desired accuracy. As an example of such an expression we have for θ in radians *Trigonometric tables*

$$\sin \theta = \theta - \frac{\theta^3}{1 \times 2 \times 3} + \frac{\theta^5}{1 \times 2 \times 3 \times 4 \times 5} - \frac{\theta^7}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} + \dots$$

The first extensive tables of trigonometric functions were developed by the Germans in the fifteenth and sixteenth centuries, prominent among the Germans being John Mueller, called *Regiomontanus* (1436–1476).

*Explanation of
trigonometric
tables*

With values of the angles in degrees given at the tops of the pages of the tables go the minute column at the left and the trigonometric function headings at the top. For example, $\sin 26^\circ 17'$ is 0.44281. On page 560 corresponding to 26° we locate 17 in the minute column, and the entry 0.44281 is found in the same row as 17 but under the column headed Sin. Similarly, we find $\tan 26^\circ 17'$ to be 0.49387.

*Complementary
angles*

Two angles which add up to 90° are said to be *complementary*. The angles 30° and 60° are complementary. The sine of 30° is the cosine of 60° . Because of this property the cosine is said to be the *cofunction* of the sine. Similarly, the cofunctions of the cosine, tangent, cotangent, secant, and cosecant are the sine, cotangent, tangent, cosecant, and secant respectively. From the definitions of the trigonometric functions of acute angles in right triangles you can easily see that the *function of an angle is the cofunction of its complement*. The tangent of 20° is thus the cotangent of 70° . The origin of the prefix “co” in cosine, cotangent, and cosecant is therefore clear.

*Tables for
angles over 45°*

Since a trigonometric function of an angle between 45° and 90° is the cofunction of the complement, the complement being between 0° and 45° , separate tables are not given for the acute angles between 45° and 90° . Instead the angles between 45° and 90° will be found listed at the bottoms of the tables. With these angles go the headings at the bottom and the minute columns on the right.

For example, consider the angle $63^\circ 43'$ whose complement is $90^\circ - (63^\circ 43') = 26^\circ 17'$. $\cos 63^\circ 43'$ will be found on page 560 listing 63° at the bottom and 26° at the top. Since we are interested in a cosine we look in the column headed Cos at the bottom and in the horizontal row which has 43 in the minute column on the right. The value of the cosine of $63^\circ 43'$ is 0.44281. This is the same value obtained for $\sin 26^\circ 17'$ found above. These cofunctions of complementary angles should be identical.

Interpolation

The tables are graduated in minutes. An angle given to a fraction of a minute, such as $26^\circ 17' 20''$, will not be found in these tables. It is therefore necessary to *interpolate*. It will be observed

that for small differences in the angles near a given angle the differences in the angles are proportional to the differences in the sines of these angles. For angles near $26^\circ 17'$ the difference in the sine is 0.00026 for a difference of $1'$ in the angle. Thus the sine of $26^\circ 18'$ is 0.00026 more than the sine of $26^\circ 17'$. See page 560.

To get the sine of $26^\circ 17' 20''$ we observe that this angle is $\frac{20}{60}$ or one-third of the way between $26^\circ 17'$ and $26^\circ 18'$. The same is true of the sine. We therefore take one-third of 0.00026, namely 0.00009, and add this to 0.44281, the sine of $26^\circ 17'$. Hence we find that

$$\sin 26^\circ 17' 20'' = 0.44290.$$

From the definitions of sine and tangent, or from the tables, it can be seen that for acute angles the *sine and tangent increase with the angle*. The interpolation method described above for the sine thus works equally well for the tangent. As an illustration we shall find $\tan 75^\circ 13' 15''$. From the tables (page 554) we have

Sine and tangent increase with angle

$$\tan 75^\circ 13' = 3.7893; \quad \tan 75^\circ 14' = 3.7938.$$

The difference between the entries is 45 in the last two places. The given angle is $\frac{15}{60}$ or one-fourth of the way between $75^\circ 13'$ and $75^\circ 14'$. We take one-fourth of 45, namely 11, and add this to the last two places of $\tan 75^\circ 13'$ to obtain

$$\tan 75^\circ 13' 15'' = 3.7893 + 0.0011 = 3.7904.$$

The *cosine and cotangent of an acute angle decrease with an increase in this angle*, so the method of interpolation used above must be modified for these functions. The small differences in angles near a given angle are still proportional to the corresponding differences in the cosines and cotangents of these angles. However, when the correction is obtained it must be subtracted from the entry corresponding to the smaller angle.

Interpolation for cosine and cotangent

For example, let us find $\cos 14^\circ 30' 10''$. From the tables (page 554) we have

$$\cos 14^\circ 30' = 0.96815; \quad \cos 14^\circ 31' = 0.96807.$$

The difference between the entries is 8 in the last place. Since the given angle is $\frac{10}{60}$ or one-sixth of the way between $14^\circ 30'$ and $14^\circ 31'$ we take one-sixth of 8, namely 1, and subtract this from $\cos 14^\circ 30'$. The result is

$$\cos 14^\circ 30' 10'' = 0.96814.$$

As a second example let us find $\text{ctn } 80^\circ 11' 23''$. From the tables we obtain

$$\text{ctn } 80^\circ 11' = 0.17303; \quad \text{ctn } 80^\circ 12' = 0.17273.$$

The difference between the entries is 30 in the last two places. We take $\frac{23}{30}$ of 30, namely 12, and subtract this result from the last two places of $\text{ctn } 80^\circ 11'$. Thus we obtain

$$\text{ctn } 80^\circ 11' 23'' = 0.17303 - 0.00012 = 0.17291.$$

*Functions of
angles over 90°*

Except for a plus or minus sign, the trigonometric function of an angle greater than 90° is equal to the same trigonometric function of the acute angle formed by the terminal side of the given angle and the x axis. For example,

$$\sin 225^\circ = -\sin 45^\circ; \quad \cos 300^\circ = \cos 60^\circ; \quad \tan 100^\circ = -\tan 80^\circ.$$

*Finding angles
from functions*

The tables can also be used to determine angles from their functions. We shall illustrate the process with acute angles.

EXAMPLE 1.

*Angle from its
sine*

Given $\sin \theta = 0.50423$, we shall find θ , where θ is acute. From the tables we have

$$\sin 30^\circ 17' = 0.50428$$

$$\sin \theta = 0.50423$$

$$\sin 30^\circ 16' = 0.50403.$$

The middle sine is $\frac{20}{5}$, that is, four-fifths of the way between the lower sine and the upper sine. The same ratio applies to the angles, so that the angle correction is

$$\frac{4}{5} \times 60'' = 48''.$$

Adding $48''$ to $30^\circ 16'$, we have $\theta = 30^\circ 16' 48''$.

EXAMPLE 2.

*Angle from its
cosine*

Given $\cos \theta = 0.51700$, we shall find an acute angle θ having this cosine. From the tables we have

$$\cos 58^\circ 53' = 0.51678$$

$$\cos \theta = 0.51700$$

$$\cos 58^\circ 52' = 0.51703.$$

The middle cosine is $\frac{3}{5}$ of the way between the lower cosine and the upper cosine. Since the same ratio applies to the angles, we compute the angle correction to be

$$\frac{3}{5} \times 60'' = 7''.$$

Adding $7''$ to $58^\circ 52'$ we have $\theta = 58^\circ 52' 7''$.

The procedure for interpolating the value of an angle from the table of tangents or cotangents is the same as that used above for the sine and cosine respectively.

SOLVING RIGHT TRIANGLES

We shall illustrate the solution of right triangles with the aid of the tables, using the notation of Fig. 220.

EXAMPLE 1. To find the acute angles and hypotenuse, given the legs.

Legs given

$$\text{Given: } \begin{cases} a = 1.2301 \\ b = 4.6702 \end{cases}$$

$$\text{To find: } \begin{cases} c = 4.8297 \\ A = 14^\circ 45' 21'' \\ B = 75^\circ 14' 39'' \end{cases}$$

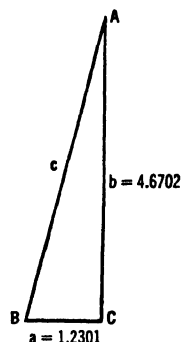


FIG. 252. RIGHT TRIANGLE OF EXAMPLE 1

$$\tan A = \frac{a}{b}, \quad c = \frac{a}{\sin A}, \quad B = 90^\circ - A.$$

Formulas

$$\tan A = \frac{a}{b} = 0.26339$$

$$90^\circ = 89^\circ 59' 60''$$

$$A = 14^\circ 45' 21''$$

Computations

From the tables

$$B = 90^\circ - A = 75^\circ 14' 39''$$

$$\tan 14^\circ 46' = 0.26359$$

$$\sin 14^\circ 46' = 0.25488$$

$$\tan A = 0.26339$$

$$\sin A = 0.25470$$

$$\tan 14^\circ 45' = 0.26328$$

$$\sin 14^\circ 45' = 0.25460$$

$$\frac{1}{3} \times 60'' = 21''$$

$$A = 14^\circ 45' 21''$$

$$c = \frac{1.2301}{.25470} = 4.8297$$

$$a^2 = c^2 - b^2 = (c - b)(c + b)$$

Check

$$1.2301^2 = (4.8297 - 4.6702)(4.8297 + 4.6702)$$

$$1.5131 = 1.5123 \text{ (approximately).}$$

The solutions have been recorded in the "To find" section. It is a good idea to enter the results here as they are found. The formula used for the *check* comes from the Pythagorean theorem for $a^2 = c^2 - b^2$ is a direct consequence of $a^2 + b^2 = c^2$.

*Hypotenuse
and a leg
given*

EXAMPLE 2. Given a leg and the hypotenuse of a right triangle, to find the remaining leg and the acute angles.

$$\text{Given: } \begin{cases} c = 1.2326 \\ b = 0.1473 \end{cases}$$

$$\text{To find: } \begin{cases} a = 1.2238 \\ A = 83^\circ 8' 12'' \\ B = 6^\circ 51' 48'' \end{cases}$$

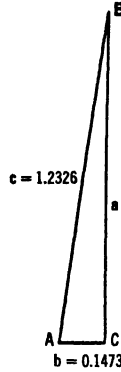


FIG. 253. RIGHT TRIANGLE OF EXAMPLE 2

Formulas

$$\cos A = \frac{b}{c}, \quad a = b \tan A, \quad B = 90^\circ - A.$$

$$\cos A = \frac{b}{c} = 0.11950$$

Computations

From the tables

$$\cos 83^\circ 9' = 0.11927$$

$$\cos A = 0.11950$$

$$\cos 83^\circ 8' = 0.11956$$

Correction ratio is $\frac{6}{29}$.

$$\text{Correction} = \frac{6}{29} \times 60'' = 12''$$

$$A = 83^\circ 8' + 12'' = 83^\circ 8' 12''$$

$$90^\circ = 89^\circ 59' 60''$$

$$A = 83^\circ 8' 12''$$

$$B = 90^\circ - A = 6^\circ 51' 48''$$

$$\tan 83^\circ 9' = 8.3245$$

$$\tan 83^\circ 8' = 8.3041$$

$$0.0204$$

$$\text{Correction for } 12'' = \frac{12}{60} \times 0.0204 = 0.0041$$

$$\tan A = 8.3041 + 0.0041 = 8.3082$$

$$a = b \tan A = 0.1473 \times 8.3082 = 1.2238$$

Check

$$a^2 = c^2 - b^2 = (c-b)(c+b)$$

$$1.2238^2 = (1.2326 - 0.1473)(1.2326 + 0.1473)$$

$$1.4977 = 1.4977.$$

Applications

The solution of the right triangle comes into engineering problems in numerous ways of which only a few examples can be given.

*Angles of
elevation and
depression*

The acute angle from the horizontal at an observer to a point P above the observer is called the *angle of elevation* of P at the location of the observer. Similarly, the acute angle from the horizontal at an observer to a point P below the observer is termed the *angle of depression* of P at the location of the observer.

EXAMPLE 1.

The base of a warehouse is at the same level as the base of a skyscraper. (See Fig. 254.) The angles of elevation and depression of the top and base of the skyscraper from the top of the warehouse are $70^\circ 30' 14''$ and $24^\circ 10' 15''$ respectively. We shall find the height of the skyscraper. *Surveyor's problem*

Referring to Fig. 254, we have

$$\begin{aligned}x &= 200 \tan 70^\circ 30' 14'', \\y &= 200 \tan 24^\circ 10' 15''.\end{aligned}$$

From the tables, with the aid of interpolation

$$\begin{aligned}\tan 70^\circ 30' 14'' &= 2.8245, \\ \tan 24^\circ 10' 15'' &= 0.44881.\end{aligned}$$

Substituting the values of these tangents, we have

$$x = 564.90, \quad y = 89.76.$$

The height $(x + y)$ of the skyscraper is therefore 654 ft. 8 in.

In the example above, the angles were given to $1''$, which is much more accuracy than is obtainable with ordinary surveying instruments. An accuracy of $1'$ would be adequate for most problems of this type.

If a railroad rises x ft. for every 100 ft. along the horizontal, the railroad is said to have a *grade* of x per cent. The angle of inclination of the railroad is the acute angle between the line of the rails and the horizontal. The grade is therefore the tangent of the angle of inclination of the railroad.

Grade and angle of inclination

EXAMPLE 2.

We assume that the angle of inclination of a railroad is 5° and find the grade of the road. The grade is $\tan 5^\circ$. By the tables, $\tan 5^\circ = 0.08749$. The grade is therefore approximately 9 per cent.

We recall that a *parallelogram* is a figure with four sides, opposite sides being parallel. The line segment drawn from a vertex of a parallelogram to an opposite vertex is a *diagonal of the parallelogram*.

Resultant velocity

The *velocity* of a body in motion is its speed and direction. The velocity is usually represented by an arrow whose length is a convenient measure of the speed of the body, and whose direction is the direction of motion of the body. Thus the velocity of a body

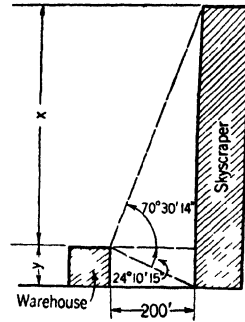


FIG. 254. FINDING THE HEIGHT OF THE SKYSCRAPER

traveling northeast (N.E.) at 10 m.p.h. is represented by an arrow

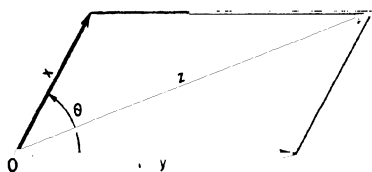


FIG. 255. RESULTANT z OF VELOCITIES x AND y

ten units long pointing northeast. If a body is traveling at y miles per hour in one direction and x miles per hour in another, the *resultant velocity* of the body is the diagonal z of the parallelogram as shown in Fig. 255. This statement will not be proved.

EXAMPLE 3.

Airplane's velocity

We suppose that an airplane is flying toward the South Pole from the equator at a speed of 400 m.p.h., but is being blown eastward by a hurricane of speed 150 m.p.h. We shall find the actual speed of the plane relative to the surface of the earth, and the direction of travel.

We can solve a right triangle here since the vector arrows 400 and 150 units in length, representing the given velocities, are perpendicular to each other, the first being directed south and the other east, as shown in Fig. 256. The hypotenuse x of the triangle in Fig. 256 has the length $\sqrt{400^2 + 150^2}$, or about 427, which follows from the Pythagorean theorem. The speed of the plane relative to the earth is thus 427 m.p.h. From Fig. 256 we have

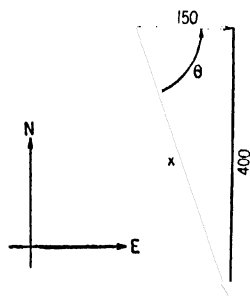


FIG. 256. VELOCITY x OF AIRPLANE HEADING SOUTH AND DRIFTING IN A WIND BLOWING EAST

$$\tan \theta = \frac{150}{400} = 2.6667.$$

From the tables we find that θ is approximately $69^\circ 27'$. The plane's line of flight thus makes an angle of $69^\circ 27'$ with the east direction.

Resultant force

Forces are pictured in the same manner as velocities, and are "added" in the same way. Thus if x and y in Fig. 255 are forces acting on a body, the resultant action of these forces on the body is properly represented by the arrow z .

EXAMPLE 4.

Two men are assumed to be pushing a rock, one with a force of 50 lb. toward the east, and the other with a force of 60 lb. toward the north. We shall find the resultant force.

You are referred to Fig. 257. We have

$$\tan \theta = \frac{60}{50} = 1.2000.$$

From the tables we have $\theta = 50^\circ 12'$ approximately. We thus have the direction of the resultant force. Now the resultant

$$x = \frac{50}{\cos \theta}.$$

By the tables $\cos \theta = 0.64011$, whence x is about 78 lb. We may, therefore, for practical purposes replace the two men by one man exerting a force of 78 lb. making an angle of $50^\circ 12'$ with the east direction, obtaining thus the same physical effect with one man as with the two given men. The resultant force of 78 lb. can be checked by the Pythagorean theorem since $\sqrt{50^2 + 60^2}$ must also equal the magnitude 78 of the resultant force in lb.

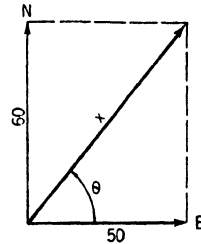


FIG. 257. FORCES ACTING ON ROCK

Many-sided figures can often be solved by means of right triangles as the following example shows. Pentagon

EXAMPLE 5.

A heavy pentagonal sheet of metal (5 equal sides and angles) 5 ft. on a side is to be flame-cut from a circular piece of metal plate. How large must the circle be?

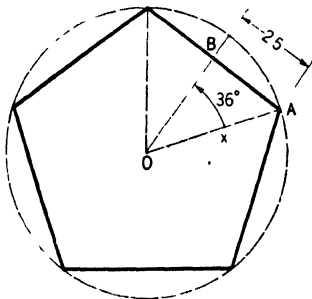


FIG. 258. PENTAGON

We let x be the radius of the circle passing through the vertices of the pentagon as shown in Fig. 258. From the center of the pentagon a perpendicular is dropped to a side of the pentagon meeting this side at a point B . The angle AOB is one-tenth of a revolution, or 36° . From the figure we have

$$x = \frac{2.5}{\sin 36^\circ}.$$

By the tables $\sin 36^\circ$ is 0.58779. It follows that the circle must have a radius of 4.25 ft., or 4 ft. 3 in.

SOLVING OBLIQUE TRIANGLES

We shall use the expression *oblique triangle* in referring to a triangle that is not a right triangle. Using the notation of Fig. 259 *Sine and cosine laws*

we have two laws by means of which we can solve all such triangles from known parts. These laws follow.

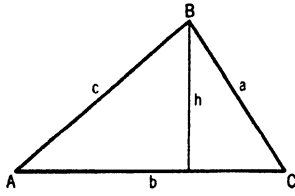


FIG. 259 OBLIQUE TRIANGLE

Sine law:

$$(63) \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Cosine law:

$$(64) \quad a^2 = b^2 + c^2 - 2bc \cos A.$$

$$(65) \quad b^2 = a^2 + c^2 - 2ac \cos B.$$

$$(66) \quad c^2 = a^2 + b^2 - 2ab \cos C.$$

The proofs of these laws will be omitted. They follow from properties of right triangles into which oblique triangles can be subdivided by drawing altitudes.

EXAMPLE 1.

*Using the
cosine law*

A face of a metal tank is an oblique triangle with sides 2, 4, and 5 ft. respectively. We wish to find the area of this metal sheet.

Using the notation of Fig. 259 we let

$$a = 2, \quad b = 4, \quad c = 5.$$

From the cosine law

$$\cos A = \frac{4^2 + 5^2 - 2^2}{2 \times 4 \times 5} = \frac{37}{40} = 0.925.$$

By the tables $A = 22^\circ 20'$.

With AC as the base, the altitude h of the triangle is such that

$$h = c \sin A.$$

Substituting the value of c , and the value of $\sin A$ from the tables, we have $h = 1.90$.

Now we may write

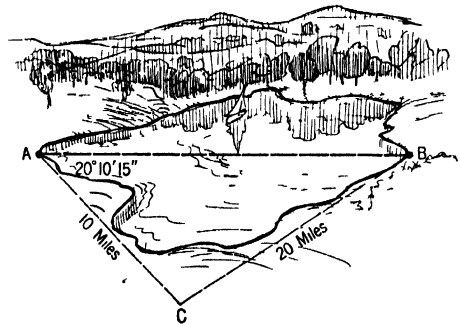
$$\frac{1}{2} hb = \frac{1}{2} \times 1.90 \times 4 = 3.80,$$

so that the area of the sheet is 3.80 sq. ft.

EXAMPLE 2.

*Using the
sine law*

Points A and B are on opposite sides of a lake as shown in Fig. 260. An observer at the position C

FIG. 260. POINTS A AND B ON OPPOSITE SIDES OF A LAKE

is 10 miles from A and 20 miles from B . The angle A is one of $20^\circ 10.5'$. We shall find the distance that the locations A and B are apart.

By the sine law

$$\sin B = \frac{10 \times \sin 20^\circ 10.5'}{20} = 0.17246.$$

By the aid of the tables we find that $B = 9^\circ 56'$. Subtracting the sum of the angles A and B from 180° , we have $C = 149^\circ 53.5'$.

By the sine law

$$AB = \frac{20 \sin 149^\circ 53.5'}{\sin 20^\circ 10.5'} = \frac{20 \times 0.5016}{0.3449} = 29.0.$$

Thus we have found that the side AB is 29 miles in length.

1. Find the following trigonometric functions:

$$\sin 27^\circ 14' 42''; \cos 60^\circ 10' 15''; \tan 75^\circ 1.2'; \operatorname{ctn} 30^\circ 30' 30''.$$

$$\text{Ans. } 0.45780; 0.49742; 3.7373; 1.6971.$$

2. Find the acute angles having the following functions:

$$\tan \theta = 0.11745; \sin \theta = 0.91124; \cos \theta = 0.87666; \operatorname{ctn} \theta = 7.7777.$$

$$\text{Ans. } 6^\circ 41' 58''; 65^\circ 40' 40''; 28^\circ 45' 30''; 7^\circ 19' 35''.$$

3. Find the following trigonometric functions:

$$\sin 400^\circ 10' 15''; \cos -331^\circ 19' 47''; \tan -10^\circ 10' 10''; \operatorname{ctn} 800^\circ 45' 50''.$$

$$\text{Ans. } 0.64507; 0.47976; -0.17656; 0.16261.$$

4. Find θ for which:

$$(a) \sin \theta = -0.31416, \text{ where } \theta \text{ is in the third quadrant.}$$

$$(b) \cos \theta = -0.91222, \text{ where } \theta \text{ is in the second quadrant.}$$

$$\text{Ans. (a) } 198^\circ 18' 37''; (b) 155^\circ 48' 50''.$$

5. Solve the right triangle of Fig. 220, where $a = 124$, $b = 178$.

$$\text{Ans. } c = 216.93; A = 34^\circ 51' 35''; B = 55^\circ 8' 25''.$$

6. Solve the right triangle of Fig. 220, where $a = 0.041$, $c = 0.062$.

$$\text{Ans. } c = 0.0465; A = 41^\circ 23' 52''; B = 48^\circ 36' 8''.$$

7. Solve the right triangle of Fig. 220, where $A = 27^\circ 10' 28''$, $b = 13.58$.

$$\text{Ans. } c = 29.73; a = 26.45; B = 62^\circ 49' 32''.$$

8. Solve the right triangle of Fig. 220, where $A = 79^\circ 10.8'$, $c = 2340$.

$$\text{Ans. } a = 2298.4; b = 439.29; B = 10^\circ 49' 12''.$$

9. Solve the oblique triangle of Fig. 259, where $a = 13.5$, $b = 14.7$, $c = 16.8$.

$$\text{Ans. } A = 50^\circ 12' 47''; B = 56^\circ 47' 44''; C = 72^\circ 59' 29''.$$

10. Solve the oblique triangle of Fig. 259, where $B = 37^\circ 14.1'$, $a = 138$, $c = 176$.

$$\text{Ans. } b = 106.52; A = 51^\circ 37' 18''; C = 91^\circ 8' 36''.$$

11. A roadbed has a grade of 2.2%. What is the angle of inclination of the road, and what is the rise at this grade in 30 miles?

$$\text{Ans. } 1^\circ 15' 37''; 3480 \text{ ft.}$$

*Angles of
elevation and
depression*

12. From the top of a hill 52 ft. 3 in. above the surrounding plain the angle of depression to a surveyor's bench mark on the plain is $5^\circ 53'$. How far is the bench mark from the top of the hill in the horizontal direction?

$$\text{Ans. } 507.06 \text{ ft.}$$

13. An airplane is directly above an observer B . An observer A is 2000 ft. from B on the same level as B . At A the angle of elevation of the plane is $80^\circ 14'$. How far is the plane above C ?

$$\text{Ans. } 11,620 \text{ ft.}$$

14. From a point 900 ft. above a lake the angle of depression to a point on the near shore is $11^{\circ} 40'$, and to a point directly beyond on the opposite shore is $5^{\circ} 14'$. What is the width of the lake between the points mentioned? *Ans.* 5468 ft.

15. The angle of elevation of a cloud from a point due south is 40° , and from another point 2 miles due east of the first point the angle of elevation is 35° . Find the height of the cloud.

Hint. Let x be the distance the first point is south of the cloud, y the height of the cloud, and obtain two linear equations from right triangles that can be constructed conveniently. *Ans.* 3.03 miles.

Velocity problems

16. If an airplane is traveling north at 200 m.p.h., but is drifting east, due to the wind, at 60 m.p.h., what is the velocity of the airplane?

Ans. 209 m.p.h.; the angle the direction of travel of the plane, relative to the earth, makes with the north is $16^{\circ} 42'$.

17. A boat is traveling across a stream at 6 m.p.h., and is drifting downstream at 5 m.p.h. What is the velocity of the boat?

Ans. 7.8 m.p.h.; the angle the direction of travel of the boat makes with the downstream direction is $50^{\circ} 12'$.

18. A cutting tool on a machine is moving upward at 3 ft. per sec., and sideways at 4 ft. per sec. What is the velocity of the cutting tool?

Ans. 5 ft. per sec.; the direction of motion is $53^{\circ} 7'$ with the vertical.

19. A helicopter is traveling south at 50 m.p.h. A wind from the northwest is blowing at 40 m.p.h. What direction does the wind appear to have to the pilot of the helicopter?

Hint. You must solve an oblique triangle. *Ans.* Toward $52^{\circ} 5'$ east of north.

20. Wind is known to be blowing at 20 m.p.h. from due north. A plane is traveling east. To the pilot the wind appears to be traveling at a speed of 170 m.p.h. What is the speed of the plane? *Ans.* 169 m.p.h.

Forces

21. If a force of 500 lb. acting east is applied to a boat, and a force of 1000 lb. is applied, acting to the north, what is the combined effect of these two forces expressed as a single force? *Ans.* 1118 lb.; $63^{\circ} 26'$ north of east.

Miscellaneous problems

22. If a sheet of plastic is in the form of a parallelogram with sides equal to 5 ft. and 14 ft. respectively, and an angle between two sides is $87^{\circ} 27'$, what is the area of the plastic sheet? *Ans.* 70 sq. ft.

23. A hexagon is a figure with six equal sides and six equal angles. The head of a bolt is in the shape of a hexagon. If the area of the head is to be 1 sq. in., what must be the length of a side? *Ans.* $\frac{5}{8}$ in.

24. A pole 30 ft. long is resting against a wall at an angle of $20^{\circ} 13.6'$. The bottom of the pole is drawn 1 ft. farther away from the wall. How far down the wall will the top of the pole fall? *Ans.* 3.28 ft.

25. Points A and B are on opposite sides of a lake as shown in Fig. 260. The point C , however, is 4.6 miles from A and 6.9 miles from B . The angle CAB is $18^{\circ} 10' 40''$. Find the distance from A to B . *Ans.* 11.1 miles.

POINTS AND STRAIGHT LINES

The use of the coordinate system made possible great strides in geometry. The field of geometry in which a coordinate system is used is called *analytic geometry*. By means of coordinate systems the study of geometry reduces to algebra, so that any statement in algebra can be transformed into a geometrical statement which can be pictured. The remainder of this chapter will be devoted almost entirely to the applications of analytic geometry.

We wish to consider any two points P_1 and P_2 in the plane with coordinates (x_1, y_1) and (x_2, y_2) respectively. A horizontal line through P_1 and a vertical line through P_2 intersect at a point P as shown in Fig. 261. Except for a possible change of sign, the dis-

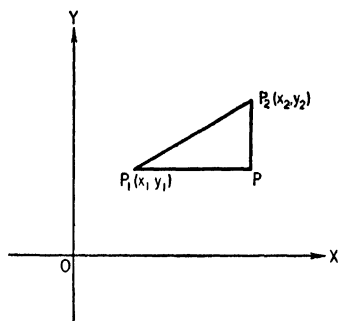


FIG. 261. DISTANCE P_1P_2 BETWEEN POINTS

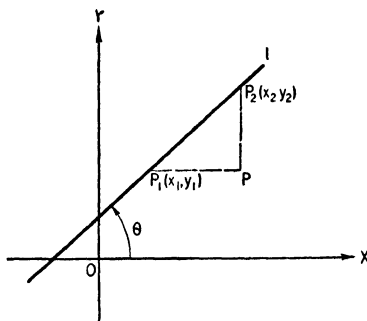


FIG. 262. ANGLE OF INCLINATION θ

tance PP_1 is always $x_2 - x_1$, whereas PP_2 is $y_2 - y_1$. By the Pythagorean theorem the distance P_1P_2 is given by

$$(67) \quad P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Formula 67 gives the *distance between any two points* on the plane.

If any coordinate is negative, such as x_1 or x_2 , it is introduced in equation 67 with its proper sign. For example, the distance between the point $(3, -4)$ and the point $(5, 7)$ is approximately 11.18, arising from

$$\sqrt{[7 - (-4)]^2 + (5 - 3)^2} = \sqrt{11^2 + 2^2} = \sqrt{125} = 5\sqrt{5}.$$

To be able to measure the direction of a straight line in a plane we use the smallest positive angle which this line makes with the positive direction of the x axis. In Fig. 262 we have drawn a line l for which this angle is θ . The angle θ is called the *angle of inclination* of the line. If the line does not intersect the x axis, this angle is 0° .

*Distance
between points*

*Negative
coordinates*

*Angle of
inclination*

The angle can be any positive angle up to, but not including 180° . Two lines have the same direction if and only if they have the same angle of inclination. Parallel lines, therefore, have the same direction.

Slope of line

In practice it is more convenient to work with a trigonometric function of θ than with θ itself. The function in common use is $\tan \theta$, called the *slope* of the line. For different directions $\tan \theta$ has different values. Since the slope of a line increases without bounds as the line approaches parallelism with the y axis, it is customary to say that the slope of a line parallel to the y -axis is "infinite," and to denote it by the symbol ∞ .

Two points determine a straight line, and thus also the slope of the line. We can therefore express the slope of the line in terms of coordinates of two points on the line.

Formula for slope

We let P_1 and P_2 be two points on the line l in Fig. 262. The distance P_1P and PP_2 are $x_2 - x_1$ and $y_2 - y_1$ respectively. From the definition of $\tan \theta$ we find that the slope m of l is given by

$$(68) \quad m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1}.$$

You can easily verify that equation 68 holds for any line l and any two points on l .

A line passing through the points $(-2, -3)$, $(4, 5)$ has the slope m , where

$$m = \frac{5 - (-3)}{4 - (-2)} = \frac{5 + 3}{4 + 2} = \frac{8}{6} = \frac{4}{3}.$$

In Fig. 263 we have drawn perpendicular lines l_1 and l_2 with angles of inclination θ_1 and θ_2 respectively. From the figure we see that

$$\theta_2 = \theta_1 + 90^\circ.$$

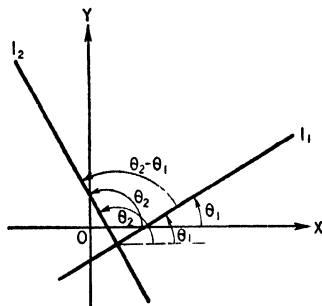


FIG. 263. PERPENDICULAR LINES

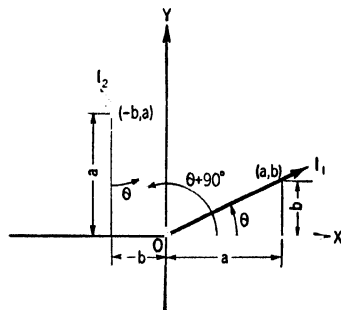


FIG. 264. ANGLE $90^\circ + \theta$

From Fig. 264

$$\tan(\theta + 90^\circ) = \frac{a}{-b} = -\cot \theta = -\frac{1}{\tan \theta},$$

Perpendicular lines

whence

$$\tan \theta_2 = -\frac{1}{\tan \theta_1}.$$

If m_1 and m_2 designate the slopes of l_1 and l_2 respectively, we have

$$m_2 = -\frac{1}{m_1},$$

or simply

$$(69) \quad m_1 m_2 = -1.$$

It can be shown easily that equation 69 applies to any two perpendicular lines l_1 and l_2 .

We have proved that *two lines are perpendicular if and only if the product of their slopes is -1 .*

We let θ be the angle formed by rotating a line l_1 counterclockwise about its point of intersection with a line l_2 until l_1 and l_2 coincide as shown in Fig. 265. The slopes m_1 and m_2 of l_1 and l_2 respectively are related to θ by the formula

Angle between two lines

$$(70) \quad \tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

The proof of this formula will be omitted.

We consider the example where two lines l_1 and l_2 intersect at the point $(1, 1)$, while l_1 passes through $(3, 4)$ and l_2 through $(2, 6)$. By equation 68 the slopes m_1 and m_2 of l_1 and l_2 are given by

$$m_1 = \frac{4 - 1}{3 - 1} = \frac{3}{2}; \quad m_2 = \frac{6 - 1}{2 - 1} = \frac{5}{1} = 5.$$

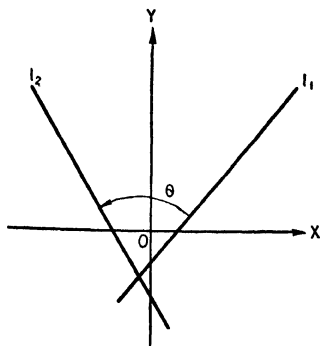


FIG. 265. ANGLE θ BETWEEN TWO LINES

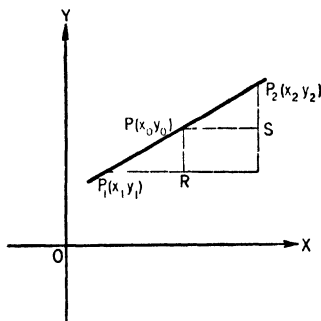


FIG. 266. MIDPOINT P OF LINE SEGMENT P_1P_2

In view of equation 70 we conclude that the tangent of the angle θ between these lines is

$$\tan \theta = \frac{5 - \frac{3}{2}}{1 + 5 \times \frac{3}{2}} = \frac{\frac{7}{2}}{1 + \frac{15}{2}} = \frac{\frac{7}{2}}{\frac{17}{2}} = \frac{7}{17}.$$

*Mid-point of
line segment*

In Fig. 266 we have arbitrarily taken two points P_1 and P_2 in the plane, and located the mid-point P of the line segment joining P_1 and P_2 . The coordinates of these points are chosen as indicated. Horizontal and vertical lines are drawn through these points to yield the points R and S as shown. Since P is the mid-point of P_1P_2 , the triangles P_1PR and PP_2S are identical. The distance P_1R is the same as the distance PS . It follows that

$$P_1R = \frac{x_2 - x_1}{2}.$$

Adding P_1R to x_1 we have

$$x_0 = x_1 + \frac{x_2 - x_1}{2}.$$

Thus

$$(71) \quad x_0 = \frac{x_1 + x_2}{2}; \quad \text{also,} \quad y_0 = \frac{y_1 + y_2}{2}.$$

The same type of proof applies to any other pair of points P_1, P_2 .

The *coordinates of the mid-point* of a line segment are thus the averages of the corresponding coordinates of the end points of the line segment.

The mid-point of the line segment joining $(-4, -8)$ with $(1, 7)$ is thus $(-\frac{3}{2}, -\frac{1}{2})$.

Problems

1. The positions of three cities are plotted on a map and found to have the coordinates $(2, -3)$, $(7, 1)$, and $(-1, 8)$ respectively, the unit of measure being 1 mile. Find the distances between the cities. *Ans.* 6.4, 10.6, and 11.4 miles.

2. Four roads plotted on a map intersect at the points $(-2, -1)$, $(3, 3)$, $(-1, 2)$, $(4, 6)$. Prove by the distance formula 67 that these points are vertices of a parallelogram.

Hint. The opposite sides of a parallelogram are of the same length.

3. Three roads plotted on a map intersect at the points $(0, 4)$, $(1, -1)$, and $(3, 2)$. Prove by the distance formula 67 and the Pythagorean theorem that these points are vertices of a right triangle.

4. Find the slope and angle of inclination of each of the lines through the following pairs of points:

(a) $(-4, -1)$, $(5, -4)$; (b) $(4, 7)$, $(-3, 11)$.

Ans. (a) $-\frac{1}{3}$, $18^\circ 26' 5''$; (b) $-\frac{4}{7}$, $29^\circ 44' 42''$.

5. Three mountain peaks plotted on a map are at the points $(-5, -3)$, $(7, -1)$, and $(4, 5)$. Find the angles of the triangle with these points as vertices.

Ans. $72^\circ 53' 49''$, $74^\circ 55' 53''$, $32^\circ 10' 18''$.

6. By the use of slopes of sides prove that the figure of problem 2 is a parallelogram.

7. By the use of slopes of sides prove that the figure of problem 3 is a right triangle.

8. The ends of a rod are located on a Cartesian coordinate system, and are found to be at the points $(-4, -5)$, $(2, 7)$. Find the mid-point of the rod. *Mid-points of lines*

Ans. $(-1, 1)$.

9. The point $(3, 4)$ is the mid-point of a rod joining $(-5, 1)$ with another point. Find the coordinates of the other point. *Ans.* $(11, 7)$.

10. A beam indicated on a drawing has slope 2. Find the slopes of the other beams which make angles of 45° with this beam. *Ans.* $\frac{1}{3}$, -3 .

11. A searchlight beam directed upwards, when plotted, passes through the points $(1, 5)$ and $(3, 4)$. A second beam in the same vertical plane has exactly 30° as its angle of inclination (with the horizontal). Find the acute angle between these beams. *Ans.* $56^\circ 33' 58''$.

12. A light ray, when plotted, passes through the points $(1, 5)$ and $(8, 4)$. A second ray passes through $(-1, 3)$ and $(6, 8)$. Find the acute angle between these rays. *Ans.* $43^\circ 40' 4''$.

EQUATIONS OF STRAIGHT LINES

By definition *an equation of a straight line is an equation satisfied by the coordinates of each point on the line, and not satisfied by the coordinates of any point off the line*. There are different standard formulas for equations of a straight line depending on the data given. These are called standard forms.

A line is determined by its slope m and one point (x_1, y_1) on the line. If (x, y) designates any other point on the line, by formula 68 *Point-slope form*

$$m = \frac{y - y_1}{x - x_1},$$

provided $x \neq x_1$. Clearing, we have an equation of the line expressed in *point-slope* form,

$$(72) \quad y - y_1 = m(x - x_1).$$

If a line l passes through the point $(2, 3)$ with slope 5, the points (x, y) on the line satisfy the equation 72 of the line. With the given coordinates and slope substituted, we obtain the expression

$$y - 3 = 5(x - 2),$$

or, after simplification,

$$5x - y - 7 = 0.$$

Line from equation

Any equation of the first degree in x and y which can be put in the point-slope form is an equation of the line with slope m passing through the point (x_1, y_1) . Suppose, for example, that $2y - x - 4 = 0$ is the given equation. By algebraic operations this equation can be brought into the form

$$y - 0 = \frac{1}{2}(x + 4).$$

This equation is therefore an equation of a line with slope $\frac{1}{2}$ passing through the point $(-4, 0)$.

Two-point form

By rule 68 any point (x, y) on a line passing through the points (x_1, y_1) , (x_2, y_2) satisfies the equation

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

provided x_2 and x are different from x_1 . Multiplying by $(x - x_1)$ we have

$$(73) \quad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1).$$

The latter is called an *equation of a line in two-point form*.

For example, let us suppose that a straight line passes through the points $(-4, 5)$ and $(6, -1)$. By the two-point formula the line must satisfy the equation

$$y - 5 = \frac{-1 - 5}{6 - (-4)} [x - (-4)].$$

This equation can be simplified into

$$3x + 5y - 13 = 0.$$

Slope-intercept form

We let m be the slope of a line not parallel to the y axis, and b its y -intercept. The line thus goes through the point $(0, b)$. By the point-slope form (72) the line has the equation

$$y - b = m(x - 0).$$

We can write this equation in the *slope-intercept form*

$$(74) \quad y = mx + b.$$

Intercept form

A line which passes through $(a, 0)$ and $(0, b)$ has the x - and y -intercepts a, b respectively. By the two-point form the line has the equation

$$(75) \quad \frac{x}{a} + \frac{y}{b} = 1.$$

This equation is said to be in the *intercept form*. To use equation 75 we must have the condition $a \neq 0, b \neq 0$.

A line passes through the points (5, 7) and (3, -5). We shall find an equation of the line in intercept form. By the two-point form (73) *Example*

$$y - 7 = \frac{-5 - 7}{3 - 5} (x - 5).$$

Simplified, this equation becomes

$$6x - y = 23.$$

Further manipulation gives the intercept form where $a = \frac{23}{6}$ and $b = -23$:

$$\frac{x}{23/6} + \frac{y}{-23} = 1.$$

The equation $x = -4$ is satisfied by all points with abscissa -4 . This relationship is therefore an equation of the line parallel to the y axis, but 4 units to the left of this axis. More generally, the equation $x = a$ is an equation of a line parallel to the y axis, and $y = b$ is an equation of a line parallel to the x axis. *Lines parallel to coordinate axes*

We have shown that *each straight line has an equation of the first degree*. To prove that each equation of the first degree is an equation of a straight line we note that each first degree equation in x and y can be brought by algebraic operations into the form *General form*

$$(76) \quad Ax + By + C = 0$$

where A and B are not both zero. This is the *general form* of an equation of a straight line.

If B is zero in equation 76, we can write this equation as

$$x = -\frac{C}{A}.$$

Equivalence of equations and lines

This is an equation of a straight line parallel to the y axis.

If B is not zero in equation 76, we divide the equation by B to get

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

The latter is an equation of the straight line with slope $-A/B$ and y -intercept $-C/B$. This equation is in slope-intercept form.

We have shown that *every equation of the first degree is an equation of a straight line*, and conversely that every straight line has an equation of the first degree. The study of first-degree equations is thus identical with the study of straight lines.

Problems

1. In each case find an equation of the straight line with the properties given:

- (a) Slope $\frac{1}{2}$, passing through the point $(-3, -5)$.
- (b) Slope $-\frac{3}{4}$, passing through the origin $(0, 0)$.
- (c) Passing through the points $(-7, -1)$, $(8, -9)$.
- (d) Slope $-\frac{1}{4}$ and y -intercept 5.
- (e) Intercept on x axis -2 and y -intercept -3 .
- (f) Parallel to the line $x + y + 1 = 0$ and passing through the origin.
- (g) Perpendicular to the line $2x - 3y = 4$ and passing through $(1, 2)$.
- (h) Slope -1 and the same y -intercept as the line $2y = 3 - x$.

Ans. (a) $x - 2y = 7$; (b) $4y + 3x = 0$; (c) $8x + 15y = -71$;
 (d) $x + 4y = 20$; (e) $3x + 2y = -6$; (f) $x + y = 0$;
 (g) $3x + 2y = 7$; (h) $x + y = \frac{3}{2}$.

2. Find the slope and y -intercept of each of the lines:

- (a) $x + 2y + 1 = 0$; (b) $x - y = 1$.

Ans. (a) $-\frac{1}{2}$, $-\frac{1}{2}$; (b) 1 , -1 .

3. Find the coordinates of the mid-point of the line segment cut out on the line $5x = 8 - 4y$ by the coordinate axes. *Ans.* $(\frac{4}{5}, 1)$.

4. What is the x -intercept of a line with slope $-\frac{3}{7}$ and y -intercept $\frac{1}{2}$? *Ans.* $\frac{7}{2}$.

5. Which pairs of the following lines are parallel and which are perpendicular?

- (a) $2x + 3y + 7 = 0$; (b) $4x = 14 - 6y$; (c) $3x = 2y + 1$;
- (d) $x + 3y + 2 = 0$; (e) $3x + 2 = y$.

Ans. (a) is parallel to (b); (a) and (b) are both perpendicular to (c); and (d) is perpendicular to (e).

*Applied
problems on
lines*

6. The positions of movable joints A and B in a machine are fixed by their respective distances x and y from a fixed pivot. If x and y are related by the equation $2x + 4y = 5$ due to connections between A and B , when one of the joints is at the pivot, where is the other? Distances are in feet.

Ans. When A is at the pivot, B is $1\frac{1}{4}$ ft. away, while if B is at the pivot, the joint A is $2\frac{1}{2}$ ft. away.

7. When plotted on a map with 1 mile as the unit of measure a highway has the equation $x - 3y = 1$, and a city is at the point $(3, 5)$. How far is it in a perpendicular direction from the city to the highway?

Ans. Approximately 4.12 miles.

8. When graphed, two parallel rods in a machine yield the lines $x + 2y = 1$, $x + 2y = 10$, the unit of measure being 1 ft. How far are these rods apart?

Ans. About 4 ft.

9. The Fahrenheit and Centigrade temperature scales are related at the boiling and freezing points by $100^\circ \text{C.} = 212^\circ \text{F.}$ and $0^\circ \text{C.} = 32^\circ \text{F.}$ respectively. Find the slope m and intercept b in the equation $T_F = mT_C + b$ relating the Fahrenheit temperature T_F to the Centigrade temperature T_C . Plot this temperature line.

Ans. Slope 1.8; intercept 32.

Family of lines

10. What property do the lines represented by $x + y = k$ have in common for the different values of k ?

Ans. The x and y intercepts are equal.

THE CIRCLE

The figure formed by the points which satisfy a given condition is said to be the *locus* of these points. Thus the locus of the points which satisfy the equation $5x - 4y + 2 = 0$ is a straight line.

A circle is by definition the locus of all points equidistant, that is at the same distance, from a given point called the *center*. If a circle has its center at $(2, 4)$ and a radius 10, any point (x, y) on the circle satisfies the equation

$$\sqrt{(x - 2)^2 + (y - 4)^2} = 10,$$

because by equation 67 the left-hand expression is the distance (radius) from the point $(2, 4)$ to the point (x, y) . Squaring, we have

$$(x - 2)^2 + (y - 4)^2 = 100.$$

Since the coordinates of a point off the circle cannot satisfy this equation, this must be an equation of the given circle.

More generally, a circle with center (h, k) and radius r has the equation

$$(77) \quad (x - h)^2 + (y - k)^2 = r^2.$$

This equation of the circle is said to be in *standard form*. If the circle has its center at the origin the standard form becomes

$$(78) \quad x^2 + y^2 = r^2.$$

The equations of the circle, discussed above, are of the second degree in x and y . It will appear later that not all equations of the second degree represent circles. Exactly which second-degree equations represent circles can readily be determined, however. By expanding the equation of the circle in standard form (77) above, and rearranging terms we have

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0.$$

This is of the form

$$(79) \quad x^2 + y^2 + Dx + Ey + F = 0,$$

where D , E , and F are constants. Note that there is no term in xy .

A term involving the product xy is called a *cross-product* term. We could readily prove the following result. A *second-degree equation in x and y* (having all terms collected on one side) *is an equation of a circle if and only if there is no cross-product term, and the coefficients of the x^2 and y^2 terms are identical.*

Problems

1. Find an equation of the circle with center at
- $(-2, 4)$
- and radius 5.

$$\text{Ans. } x^2 + y^2 + 4x - 8y - 5 = 0.$$

2. Where does the circle with center at
- $(1, 1)$
- and radius 3 cut the
- x
- axis?

$$\text{Ans. } 1 \pm 2\sqrt{2}.$$

3. Find the center and radius of the circle with the equation

$$x^2 - 4x + y^2 = 6y + 3.$$

$$\text{Ans. } (2, 3), r = 4.$$

4. Find an equation of the line which passes through the centers of the circles

$$x^2 + y^2 + 2x + 1 = 2y, \quad x^2 + y^2 = 4x + 4y + 1.$$

$$\text{Ans. } x - 3y + 4 = 0.$$

Applications

5. The equation of the rim of a wheel when plotted is
- $x^2 + y^2 - 2x - 2y = 2$
- . The wheel is on a machine with an axle at
- $(-4, -4)$
- . Find the distance from the axle to the wheel.

Hint. First find the distance from the axle to the center of the wheel.

$$\text{Ans. } 5\sqrt{2} - 2.$$

6. When graphed the equations of two wheels are

$$x^2 + y^2 - 2x - 4y + 1 = 0,$$

$$x^2 + y^2 - 10x - 2y + 22 = 0.$$

How far are they apart?

Hint. First find the distance their centers are apart.

$$\text{Ans. } \sqrt{17} - 4.$$

7. Two points on a machine have the coordinates
- $(1, 2)$
- and
- $(-1, -2)$
- . One end of a rod on the machine moves so that the sum of the squares of the distances from this end to the given points is 36. Show that the locus is a circle, and find its center and radius.

$$\text{Ans. } (0, 0), r = \sqrt{13}.$$

8. One edge of a field when plotted on a map is along the
- x
- axis, and a tree in the field is at the point
- $(2, 3)$
- . A tractor moves so that the square of its distance from the tree is proportional to its distance from the line of the given edge of the field. Show that the locus is a circle.

Hint. Let the coordinates of the tractor be (x, y) . Then its distance from the x axis is y and from the tree is $\sqrt{(x-2)^2 + (y-3)^2}$.

CONIC SECTIONS

Right circular cone

On a plane, for convenience taken horizontal, we draw a circle C with center O as in Fig. 267. We choose P vertically above O , and pass through P all possible lines intersecting the circle C . Thus we generate a *right circular cone* as shown in Fig. 267. The line joining P to O is the axis of the cone. In the expression "right circular cone" the word "circular" refers to the property of C being a circle, and the word "right" indicates that the axis PO is perpendicular to the plane of C . The point P is called the *vertex* of the cone. The part of the cone above P is termed a *nappe*; similarly, for the part below. The second sketch of Fig. 267 is a cone that is obviously neither "right" nor "circular."

If we pass any plane through a right circular cone we cut out a *conic section*. This is simply a matter of definition. Planes perpendicular to the axis of the cone cut out circles, so that circles are special cases of conic sections. It is easily seen that planes not

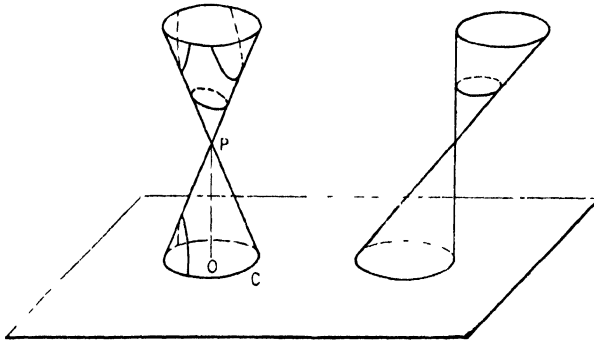


FIG. 267. CONES

perpendicular to the axis cut out conic sections which are not circles. Of course, a plane passing through the vertex perpendicular to the axis cuts through the cone at a single point only.

Conic sections were studied extensively by the ancient Greeks. The conics have numerous applications, some of which will be brought out here. They are in many respects the simplest curves after the straight line. They are of particular mathematical interest because it can be shown that the locus of the points whose coordinates satisfy an *equation of the second degree in x and y is always a conic*, provided we include the “degenerate conics” composed of two straight lines (which may be co-incident), a single point, or no real locus.

PARABOLA, ELLIPSE, HYPERBOLA

The *parabola* is the locus of all points which are the same distance from a given point as from a given line. An example of a parabola is given in Fig. 268. Here the given point marked F is the point $(2, 0)$ and the given line is the line $x = -2$. The points $(2, 4)$ and $(0, 0)$, as well as $(2, -4)$, are on this parabola because they are the same distance from the line $x = -2$ as from the point $(2, 0)$.

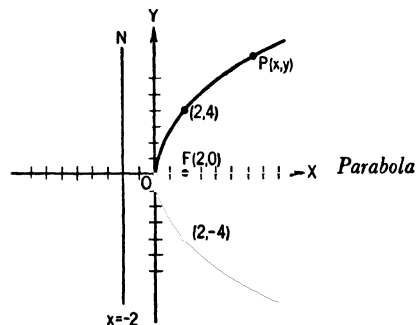


FIG. 268. PARABOLA

Functions of second degree

Following the notations of Fig. 269 we let $P(x, y)$ be a point on the parabola, d_1 being the distance from P to the line $x = -2$, and d_2 the distance from P to the point $(2, 0)$. From the figure you can see

that $d_1 = x + 2$, and from formula 67, $d_2 = \sqrt{(x - 2)^2 + (y - 0)^2}$. Since by definition we have $d_1 = d_2$, an equation of this parabola is

$$x + 2 = \sqrt{(x - 2)^2 + y^2}, \quad \text{or} \quad y^2 = 8x.$$

We return to the general treatment of the parabola. The given point is called the *focus*, and the given line the *directrix*. The point midway between the focus and the directrix is the *vertex*. The distance from the focus

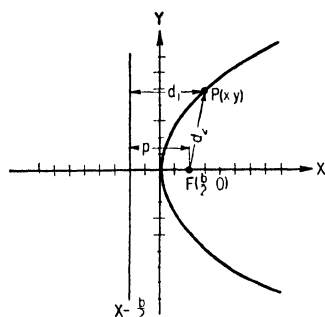


FIG 269 PARABOLA IN STANDARD POSITION

to the directrix will be denoted by p .

We shall take the focus F at the point $(\frac{p}{2}, 0)$ in Fig. 269, and the directrix D to be the line $x = -\frac{p}{2}$. The vertex is now at the origin, as in the example above. The derivation of the equation of the parabola is the same as in the example of Fig. 268. We have

$$d_1 = x + \frac{p}{2}, \quad d_2 = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2}$$

Equating the formulas for d_1 and d_2 , squaring, and simplifying we have

$$(80) \quad y^2 = 2px.$$

The latter is the *standard equation* of the parabola. The parabola of Fig. 269 is in *standard position*. Note again that $\frac{p}{2}$ is the distance from the vertex to the focus.

The line through the vertex and focus is called the *axis of the parabola*. The parabola is symmetrical with respect to its axis.

As the abscissa of the point P of Fig. 269 increases, the ordinate also increases so that P rises as we move to the right. You can see that P will rise forever as you go to the right. In a similar way the lower half of the parabola diverges steadily from the x axis.

The cable of a suspension bridge, with weight from the roadway uniformly distributed over the length of the bridge, is in the

Standard
equation of
the parabola

Axis of the
parabola

Some uses of
the parabola

form of a parabola. A bomb dropped from a plane falls approximately along a parabolic curve. When a parabola is revolved about its axis, a *paraboloid of revolution* is obtained, the "focus" of this paraboloid being the focus of the original parabola. If the reflector of a searchlight is formed as a paraboloid of revolution, with light source at the focus, the beam of light leaving the searchlight will not spread but will form a beam of parallel rays of light. The lenses of the great reflecting telescopes for observing stars are made in the shape of paraboloids of revolution. The light falling on such a lens is collected at the focus where it is recorded on a photographic plate, or observed.

A searchlight with parabolic reflector is to be 2 ft. in diameter and 10 in. deep. We shall find the position at which the light source must be placed in the searchlight. *Application to searchlights*

A cross section of the searchlight taken through the axis of the searchlight yields a parabola which we place in standard position as in Fig. 269. The equation of the parabolic section is thus (80) for some value of p . Using an inch as the unit of measure, the point (10, 12) is known to be on the parabola. Substituting the coordinates of this point in equation 80 and solving for p we obtain $p = \frac{36}{5}$. Then $\frac{p}{2} = \frac{18}{5}$. The light source must be placed at the focus $(\frac{18}{5}, 0)$, which is 3.6 in. from the vertex.

Let us suppose that a projectile is fired from a gun with a speed v , the angle of elevation of the gun being θ . We neglect such factors as air resistance and the rotation of the earth. By laws of mechanics and calculus it can be shown that the path of the projectile is part of the parabola *Application to projectiles*

$$(81) \quad y = x \tan \theta - \frac{32 x^2}{2 v^2 \cos^2 \theta}.$$

This parabola is not in standard position as you can see from Fig. 270 where we have graphed the curve.

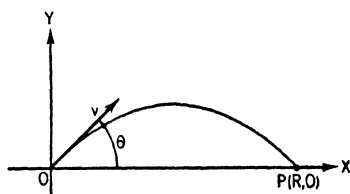


FIG. 270. PATH OF PROJECTILE

Here the gun is at the origin, the horizontal through the gun is the x axis, and the gun is pointed into the first quadrant. The distance R of Fig. 270 is called the *range* of the gun.

We shall find the range of the gun if $\theta = 45^\circ$, and the muzzle speed v is 2400 ft. per sec., as is often the case with a 75-mm. gun. Substituting $v = 2400$ and $\theta = 45^\circ$ in equation 81, we get *Range*

$$y = x - \frac{x^2}{180,000}.$$

Setting y equal to zero to get the point P of Fig. 270, we have

$$180,000x - x^2 = 0.$$

Since $x \neq 0$ at P , we can divide through by the factor x , and obtain $x = 180,000$. The range is thus 180,000 ft., or 34 miles. If we take air resistance into account the range turns out to be much less.

Ellipse

The equation

$$(82) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a is not less than b , is the standard equation of a conic, called the *ellipse*, pictured in Fig. 271. This ellipse is in *standard position*.

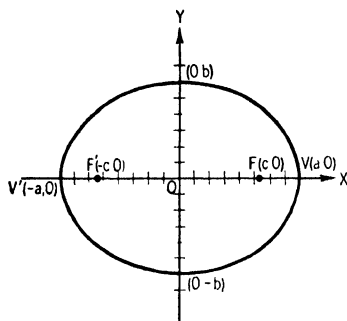


FIG. 271. ELLIPSE IN STANDARD POSITION

The ellipse may be defined by the property that it is the *locus of a point which moves in such a way that the sum of its distances from two fixed points is a constant*.

For the ellipse of Fig. 271 the constant is $2a$, and the fixed points are $F(c, 0)$ and $F'(-c, 0)$, where $c^2 = a^2 - b^2$. The points F and F' are called the *foci* of the ellipse, whereas $V(a, 0)$ and $V'(-a, 0)$ are the *vertices*. The distance $2a$ is the major axis, $2b$ the minor axis; a is the semimajor axis, and b the semiminor axis.

Example

We shall find an equation of the ellipse whose foci are $(\pm 3, 0)$, and vertices $(\pm 5, 0)$. In the notation of Fig. 271 we have $a = 5$, $c = 3$. Since $c^2 = a^2 - b^2$, it follows that $b = 4$. Substituting these values in the standard equation of the ellipse, we obtain the special equation describing this particular ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

Places where the ellipse occurs

The planets move around the sun in orbits which are elliptical. To obtain a variable rate of motion from a shaft rotating at constant speed, elliptical gears are sometimes employed. If an ellipse is rotated about an axis, an *ellipsoid of revolution* is generated. Huge elevated tanks often have ellipsoidal bottoms which serve better

than conical or hemispherical bottoms. The two "foci" of the ellipsoid of revolution are the foci of the original ellipse. A whispering gallery is in the shape of an ellipsoid of revolution. A sound leaving one focus is reflected back to the other.

Let us assume that an elliptical gear has the equation $9x^2 + 16y^2 = 144$, the unit of measure being the inch. We wish to find the lengths of the major and minor axes of the elliptical gear. Dividing by 144 we get the equation in standard form, namely,

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

It follows that the gear has a major axis of 8 in. and a minor axis of 6 in., these being the dimensions of the ellipse.

The equation

$$(83) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Hyperbola

is the *standard equation of the hyperbola (in standard position)*. The hyperbola is a curve made up of two sections, called *branches*, as shown in Fig. 272. The points $(a, 0)$ and $(-a, 0)$ are termed the

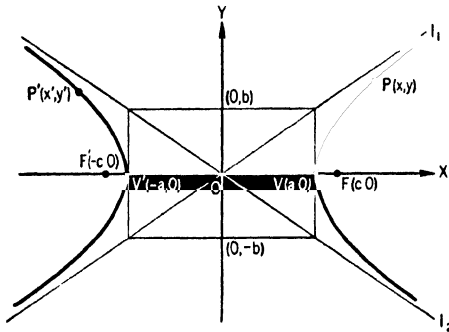


FIG. 272. HYPERBOLA IN STANDARD POSITION

vertices of the hyperbola. The line segment joining the vertices is called the *transverse axis*, and the line segment joining the points $(0, -b)$ and $(0, b)$ is called the *conjugate axis*. It follows that a and b are the semitransverse and semiconjugate axes of the hyperbola.

The hyperbola is by definition the *locus of a point which moves in such a way that the difference of its distances from two fixed points, the foci, is constant*. For Fig. 272 this constant is $2a$, and the foci are $(\pm c, 0)$.

As the abscissa of a point on the hyperbola in the first quadrant of Fig. 272 increases, the point draws closer to the line l_1 . It can be proved that by going out far enough along the hyperbola we

Asymptotes

can get as close to the line l_1 as we please. The hyperbola does not, however, cross the line l_1 . The same situation applies in the other quadrants. The lines l_1 and l_2 of Fig. 272 are called *asymptotes*, a Greek name suggested by the properties just discussed. The equations of l_1 and l_2 are

$$(84) \quad y = \frac{b}{a}x, \quad y = -\frac{b}{a}x.$$

Boyle's law

Laws of physics expressed in the form of equations, when graphed, often yield hyperbolas. Particularly well known is Boyle's law mentioned before. We recall that if p is the pressure of a gas at constant temperature and v its volume, the gas satisfies the law

$$(85) \quad pv = k,$$

where k is a constant depending on the kind and amount of gas used. The graph of equation 85 is a hyperbola, as shown in Fig. 273, with the axes as asymptotes.

Location of a gun

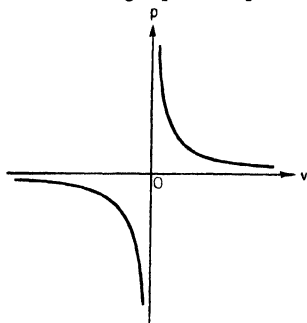


FIG. 273. HYPERBOLA FOR BOYLE'S LAW

The position of an enemy gun 10 miles away can be located within a radius of about 50 ft. by a method based on the time of travel of sound and the use of hyperbolas. Observers are placed at points F and F' a known distance $2c$ apart as in Fig. 272. We suppose that the sound of fire from the gun reaches F an interval of t seconds earlier than this sound reaches F' . If $P(x, y)$ in Fig. 272 is the position

of the gun, the distance $F'P$ is $1120t$ ft. greater than the distance FP , the speed of sound being about 1120 ft. per sec. The gun is therefore on the hyperbola of Fig. 272 where a is half of $1120t$, that is, $560t$. The gun is located in the same manner on another hyperbola, which intersects the first. The gun is therefore at one of the points of intersection of these hyperbolas. Since the position of the gun is known roughly in advance, the proper intersection is chosen immediately.

Example of gun location

We suppose that observers at F' and F are 600 ft. apart, and that the sound of gunfire reaches F at an instant one-fifth of a second earlier than F' . We shall find an equation of the locus of position of the gun.

We take one-fifth of 1120, obtaining 224. The gun is therefore

224 ft. closer to the observer at F than the observer at F' . We thus have $a = 112$. Since $c = 300$, and it can be proved that

$$(86) \quad c^2 = a^2 + b^2$$

for any hyperbola, we have $b^2 = 300^2 - 112^2 = 77,456$. By equation 83 the gun is on the hyperbola

$$\frac{x^2}{12,544} - \frac{y^2}{77,456} = 1.$$

Since the gun is nearer F than F' , the gun is on the right branch of the hyperbola.

The ellipse, hyperbola, and parabola are the only nondegenerate conics. The circle is an ellipse with equal major and minor axes. Such special cases of conics as a point, or two straight lines, are called *degenerate conics*. *Nondegenerate conics*

1. Find the distance from the vertex to the focus of the parabola $y^2 = 8x$. *Problems*

Ans. 2.

2. Find an equation of the parabola in standard position with the following properties:

(a) Focus $(3, 0)$; (b) $p = 5$; (c) Passing through the point $(5, 7)$.

Ans. (a) $y^2 = 12x$; (b) $y^2 = 10x$; (c) $y^2 = \frac{49}{5}x$.

3. Find equations of ellipses in standard position satisfying the following conditions.

(a) Foci $(\pm 2, 0)$ and vertices $(\pm 3, 0)$.

(b) Vertices $(\pm 7, 0)$ and passing through $(2, 4)$.

(c) Foci $(\pm 5, 0)$ and length of major axis equal to 6.

Ans. (a) $5x^2 + 9y^2 = 45$; (b) $16x^2 + 49y^2 = 784$; (c) $11x^2 + 36y^2 = 396$.

4. Find equations of hyperbolas in standard position satisfying the following conditions.

(a) $a = 5$, $b = 12$.

(b) Foci $(\pm 4, 0)$ and transverse axis equal to 6.

(c) Asymptotes $y = \pm 2x$, and vertices $(\pm 3, 0)$.

Ans. (a) $144x^2 - 25y^2 = 3600$; (b) $7x^2 - 9y^2 = 63$; (c) $36x^2 - 9y^2 = 324$.

5. Find the coordinates of the foci, vertices, and equations of the asymptotes of the hyperbola

$$144x^2 - 25y^2 = 169.$$

Ans. $(\pm \frac{13}{6}, 0)$; $(\pm \frac{1}{2}, 0)$; $y = \pm \frac{12}{5}x$.

6. A reflector is to be in the form of a paraboloid of revolution covered by a plane sheet of glass perpendicular to the "axis" of the paraboloid. The diameter of the glass is to be 1 ft., and the focus of the paraboloid is to be 2 in. from the glass. What must be the depth of the reflector? *Paraboloid*

Ans. 4.5 in.

Parabolas

7. A parabolic arch has a span of 120 ft. and height of 20 ft. Derive an equation for the parabola and find the heights at points 10 ft., 20 ft., and 30 ft. from the center. *Ans.* $x^2 = -180y$; 10 ft., 20 ft., and 30 ft. from center, height is $\frac{17}{9}$, $\frac{16}{9}$, and 15 ft. respectively.

8. A projectile is fired from a gun with a muzzle speed of 1000 ft. per sec. The gun is inclined at 30° to the horizontal. Find the range over a level plain neglecting air resistance. *Ans.* 26,900 ft.

9. An equation relating the time θ and volume v of a filtrate passing through a filter cloth is a parabola with the equation

$$(v + v_c)^2 = k(\theta + \theta_c),$$

where k , v_c , and θ_c are constants. Let v be in liters and θ in seconds. If $v = 0$ when $\theta = 0$, and $v = 2$ when $\theta = 53.4$, and further $v = 4$ when $\theta = 163.0$, what are the values of the constants k , v_c , and θ_c ? After having determined the values of these constants find the amount of the filtrate which will go through the filter cloth in two minutes.

Ans. $k = 0.143$, $v_c = 0.900$, $\theta_c = 5.69$; amount of filtrate is 3.34 liters.

10. For a simple pendulum the time of a half vibration in seconds is related to the length of the pendulum in feet by the equation of a parabola

$$t^2 = \frac{\pi^2}{g} l = \frac{\pi^2 l}{32} \text{ nearly}$$

[g is the acceleration of gravity].

Find the time of half vibration of a pendulum of length 3 ft. *Ans.* 0.96 sec.

11. Towers supporting a suspension bridge are 320 ft. apart and rise 80 ft. above the roadbed. The lowest point of a main cable in the shape of a parabola is 30 ft. above the roadbed. Find an equation for the curve of the cable.

Ans. One possible equation is $x^2 = 512(y - 30)$.

Ellipses

12. The earth's orbit is an ellipse with the sun at one focus, the major axis being 186×10^6 miles, and the "eccentricity" $(\sqrt{a^2 - b^2})/a$ being $\frac{1}{60}$. Find the difference between the greatest and least distance of the earth from the sun.

Ans. 3.10×10^6 miles.

13. The stone arch of a bridge has a span of 150 ft. and a height of 38 ft. The arch is in the shape of half of an ellipse. Find an equation for the ellipse, and locate the foci.

Ans. $\frac{x^2}{5625} + \frac{y^2}{1444} = 1$; foci are 64.7 ft. from center line.

Hyperbolas

14. The weight W that a simple beam of length l will support varies inversely as the length l , so that we may say $W = k/l$ for a constant k depending on the cross section of the beam, and the material. The curve representing this relation is a hyperbola. If a beam 10 ft. long will support 2 tons, how many tons will a beam 20 ft. long of the same material and cross section support? Plot the hyperbola in this case. *Ans.* 1 ton.

15. Two observers are 1000 ft. apart. The sound of a shot from a gun reaches one observer one-fifth of a second earlier than the other. Find an equation for a hyperbola on which the gun lies.

Ans. In standard position $\frac{x^2}{12,544} - \frac{y^2}{237,456} = 1$.

CURVE FITTING

The curves we have been discussing often arise in the correlation of data obtained from experiments. When the data are plotted as points on a Cartesian coordinate system, and the points lie on a curve, or approximately on a curve, it is usually easier to work with an equation of this curve than with the original points.

The process of finding a curve to represent a group of points is called *curve fitting*. If there seems to be a choice of different curves to fit a given set of points, one should naturally choose a curve with as simple an equation as possible, preferably, of course, a straight line.

Usually a curve approximating the plotted data can be drawn, and an equation of the curve can then be obtained. In Fig. 274 some points have been plotted, and a straight line has been drawn arbitrarily which approximately fits the points. This line is a reasonable choice because it passes near all of the points. Since it passes through the points $(0, 0)$ and $(5, 6)$, by the two-point formula (73) an equation of the line is

$$6x - 5y = 0.$$

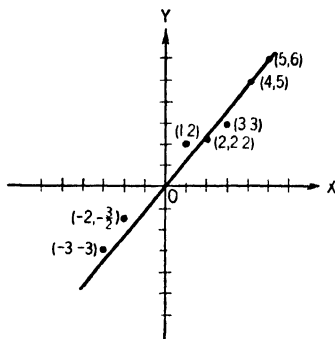


FIG. 274. LINE FITTING POINTS

The disadvantage of an arbitrary choice of a curve is that the result is not *unique*. Different persons may choose different curves to fit the same data. A more scientific procedure in general use will be described.

It is not difficult to learn how to fit straight lines to points by the method of least squares published in 1809 by Adrien Marie Legendre (1752–1831).

We suppose that (x_1, y_1) , (x_2, y_2) , etc., are a set of points to which a straight line with the equation

$$(87) \quad y = a + bx$$

is to be fitted. Remember that a is the intercept on the y axis and b is the slope of the line. For $x = x_1$ the ordinate of the line (value of y , not y_1) is $a + bx_1$. The difference between y and y_1 , that is between this value of y on the line and the ordinate y_1 of the first point, is called a *residual*, and will be designated by

$$(88) \quad r_1 = y_1 - (a + bx_1).$$

From Fig. 275 you can see that the residual r_1 is the amount whereby the straight line deviates from the point $P_1(x_1, y_1)$ in the y -direction. In the same way we form the residuals r_2, r_3 , and so on for all other plotted points. We let S^2 designate the average of the squares of the residuals. If there are n given points we have

$$(89) \quad S^2 = \frac{1}{n} (r_1^2 + r_2^2 + \cdots + r_n^2) = \frac{\Sigma r^2}{n}.$$

Here Σr^2 denotes the sum of the squares of the residuals.

Least sum of squares

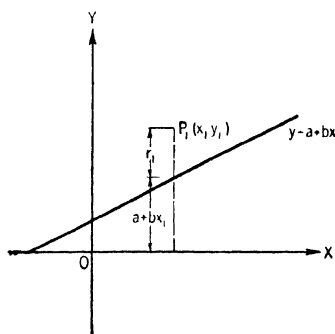


FIG. 275. RESIDUAL r_1 FROM LEAST SQUARES

By the method of least squares the straight line fitting the given points is that line for which a and b are chosen so that S^2 is as small as possible. You can see that a line found by the method of least squares can pass through all of the given points if and only if each residual vanishes, that is, $S = 0$. The smaller the residuals are, the smaller is S^2 . The quantity S^2 was defined in terms of the squares of the residuals because these are always positive even though the residuals may be negative.

The criterion of least squares, using the relation 89 given above, also applies when a parabola $y = a + bx + cx^2$ or any other more complicated curve, such as $y = a + bx + cx^2 + dx^3$, is fitted to the given points.

Fitting a line of non-zero slope

We shall treat lines of the type $y = a + bx$ for which the slope b is not zero. We let $\Sigma 1$ stand for n , the number of given points, and let Σx , Σy , Σx^2 , and Σxy denote the sums

$$\begin{aligned} & x_1 + x_2 + \cdots + x_n, \quad y_1 + y_2 + \cdots + y_n, \\ & x_1^2 + x_2^2 + \cdots + x_n^2, \quad x_1y_1 + x_2y_2 + \cdots + x_ny_n \end{aligned}$$

respectively. By the methods of the calculus we can show that S^2 takes on its minimum value when a and b satisfy the equations

$$(90) \quad \begin{aligned} a\Sigma 1 + b\Sigma x - \Sigma y &= 0, \\ a\Sigma x + b\Sigma x^2 - \Sigma xy &= 0, \end{aligned}$$

provided that these equations can be solved for a unique pair of values (a, b) . The equations 90 are two linear equations in the two unknowns a and b . The values of a and b which satisfy equa-

tions 90 are substituted in equation 87 to give the desired representative line. It is only in exceptional cases that equations 90 have no unique solution (a , b).

For the example of Fig. 274,

*Straight line
by least squares
for plotted data*

| 1 | x | y | x^2 | xy |
|----------------|-----------------|-------------------|-------------------|--------------------|
| 1 | -3 | -1 | 9 | 9 |
| 1 | -2 | -1.5 | 4 | 3 |
| 1 | 1 | 2 | 1 | 2 |
| 1 | 2 | 2.2 | 4 | 4.4 |
| 1 | 3 | 3 | 9 | 9 |
| 1 | 4 | 5 | 16 | 20 |
| 1 | 5 | 6 | 25 | 30 |
| $\Sigma 1 = 7$ | $\Sigma x = 10$ | $\Sigma y = 13.7$ | $\Sigma x^2 = 68$ | $\Sigma xy = 77.4$ |

The equations 90 for a and b are therefore

$$\begin{aligned} 7a + 10b - 13.7 &= 0, \\ 10a + 68b - 77.4 &= 0 \end{aligned}$$

Solving these equations simultaneously, we have $a = 0.4$, $b = 1.1$.

By substitution in $y = a + bx$ there follows the desired equation

$$y = 0.4 + 1.1x.$$

If a horizontal line $y = a$ is fitted to the points $(x_1, y_1), \dots, (x_n, y_n)$ by the method of least squares, we can show that

$$(91) \quad a = \frac{\Sigma y}{n}.$$

Thus a is the average of the sum of the ordinates of the given points.

For the points of Fig. 274 the average a is

$$\frac{\Sigma y}{n} = \frac{13.7}{7} = 2.0$$

correct to one decimal place. The horizontal line which best fits these points by the method of least squares is the line $y = 2.0$. Of course, it is evident that a sloping line such as the one shown in Fig. 274 is a far better representation of these points than any horizontal line. In practice the line $y = a$ should be used only if the points lie approximately along such a line.

Problems

1. When a small valve was opened, the pressure p of a gas in a cylinder in pounds per square inch was observed on a pressure gage for values of the time t at intervals of 10 sec., and the following table was obtained. Fit the lines $p = a$ and $p = a + bt$ to these points by the method of least squares.

| | | | | | | |
|-----|------|-----|-----|-----|-----|-----|
| p | 1000 | 990 | 982 | 969 | 951 | 938 |
| t | 10 | 20 | 30 | 40 | 50 | 60 |

$$\text{Ans. } p = \frac{2915}{3}; p = \frac{3047}{3} - \frac{44}{35}t.$$

2. The increasing speed v of a locomotive in miles per hour was measured at intervals of 10 sec. and the following table relating v and the time t was obtained.

| | | | | | | | | | |
|-----|---|----|----|----|----|----|----|----|----|
| v | 0 | 10 | 10 | 15 | 15 | 20 | 25 | 30 | 40 |
| t | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 |

Plot the points on graph paper and draw a line which seems to represent the points approximately. Then fit the line $v = a + bt$ to the data given, by least squares. Draw the line obtained by the method of least squares on the graph and compare with your approximate line.

$$\text{Ans. } v = \frac{4}{3} + \frac{1}{10}t.$$

3. Construct a graph on Cartesian coordinate paper for the law that the illumination from a light source varies inversely as the square of the distance from the light.

Hint. Construct a table of values for convenient choices of the distance, and plot.

FAMILIES OF CURVES AND NOMOGRAPHS

Families of curves

A relation between more than two variables cannot be expressed by a single curve. Various methods have been used to picture such a relationship. For example, an equation in three variables, say x, y , and z , is the equation of an unlimited number of curves obtained by assigning different values to z , the curves being plotted on an x, y Cartesian coordinate system. These curves form what is called a *family of curves*. This procedure is used on weather maps. On the temperature maps, which relate temperature to position on the earth's surface (position requiring two coordinates) the curves are termed *isothermal lines*, that is, lines of equal temperature. Usually these lines are so irregular that a simple equation for them cannot be obtained. They represent, however, a relation between three variables. At each point (x, y) on the isothermal line marked 60° the temperature z is supposed to be 60° at the time indicated. On pressure maps the curves of the family are lines of equal pressure, and are called *isobaric lines*.

Nomographs

The nomograph is often much more convenient than other methods of picturing a relation between variables. The ordinary

nomograph (alignment chart) consists of a series of lines (usually, but not necessarily, straight), one for each variable in the given relation. Values of the variable are marked along the line of that variable as on a scale. These markings or scales are made in such a way that by properly laying a straight edge over given values of all variables but one, the corresponding value (or values) of the remaining variable as determined by the given equation may be read off at once.

Nomographs are particularly useful where numerous computations are to be made with the same formula. Nomographs can be constructed for such expressions as *Usefulness*

$$z = 0.01 xy^2; \quad z = \frac{5x}{y}; \quad \frac{1}{z} = \frac{1}{x} + \frac{1}{y}; \quad u = x + y - z.$$

The construction of suitable nomographs depends a great deal on the ingenuity of the worker. However, an idea of how to make them can be obtained from the example which follows.

In Fig. 276 we have constructed a nomograph for the relation *Simple nomograph*

$$z = xy$$

between the variables x , y , and z . The lines A and B are parallel to C which is midway between A and B . A line perpendicular to

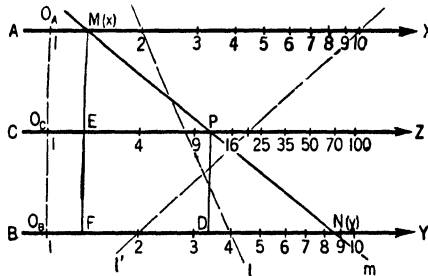


FIG. 276. NOMOGRAPH FOR $z = xy$, or $y = \frac{z}{x}$

A , B , and C cuts these lines at the “origins” O_A , O_B , and O_C respectively. The scales on A and B are logarithmic scales such as the scales on a slide rule.

A straight line perpendicular to A , B , and C cuts these lines in such a way that the scale values of the points of intersection with A and B are identical, and the point of intersection with C has a scale value equal to the square of the corresponding value on the other scales. In particular, the distance from O_A to the point *Vertical line*

marked 3 on A is $\log 3$ in an appropriate unit of measure, the distance from O_A to the point marked 4 on A is $\log 4$, and so on. The line joining the points marked 3 on A and 3 on B cuts C at the point marked 9, that is, the square of 3.

*Multiplication
and division*

It follows that to multiply 4 by 4 we need only lay a vertical line on the points marked 4 on A and B , and read off the result, 16, on C . This method of obtaining products also applies to unequal numbers, as we shall see later. Thus to multiply 2 by 4 we lay the line l on the point of A marked 2 and the point of B marked 4, as shown in Fig. 276. The line cuts the C -scale at the product 8. The line l' of Fig. 276 meets A and B at points marked 10 and 2 respectively, and cuts C at their product 20.

In general terms, to obtain the product xy of numbers x and y , we pass a straight line through the points marked x and y on A and B respectively. The value of xy is the scale value of the point where this line cuts C . Obviously, division may be performed by drawing a line from the *divisor* on A through the *dividend* on C to obtain the *quotient* read off on the scale B .

Proof

To prove the validity of the nomograph of Fig. 276 for computing the product xy , we have drawn Fig. 277. Since the scales A

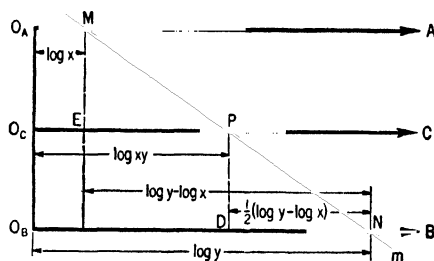


FIG. 277. NOMOGRAPH OF THE EQUATION $z = xy$

and B are *logarithmic*, the distance from the origin to M is $\log x$ and to N is $\log y$. The horizontal distance EN between these points M and N is evidently $\log y - \log x$ and one-half of this distance, DN , is $\frac{1}{2}(\log y - \log x)$. From Fig. 277

$$O_B D = O_B N - DN = \log y - \frac{1}{2}(\log y - \log x).$$

By rule 46 we have $\log y - \log x = \log y/x$, and by rule 47, $\frac{1}{2} \log y/x = \log \sqrt{y/x}$. It follows that

$$O_B D = \log y - \log \sqrt{\frac{y}{x}} = \log \frac{y}{\sqrt{\frac{y}{x}}} = \log y \sqrt{\frac{x}{y}} = \log \sqrt{xy}.$$

Since the number on the *C* scale at *P* is the square of the corresponding number on the *B* scale at *D*, we may write

$$O_C P = \log (\sqrt{xy})^2 = \log xy,$$

whence the line *m* of Fig. 276 cuts the *B* scale at a number which is the product of *x* and *y*, that is, the value *z* in the equation $z = xy$ for which the nomograph was drawn.

A nomograph such as Fig. 276 may be used effectively for calculations involving Ohm's law, $E = IR$, which states that the *potential drop* (voltage) *E* across a *resistance* *R* is the product of *R* by the *current* *I* flowing through the resistor. Letting the upper and

Nomograph for voltage

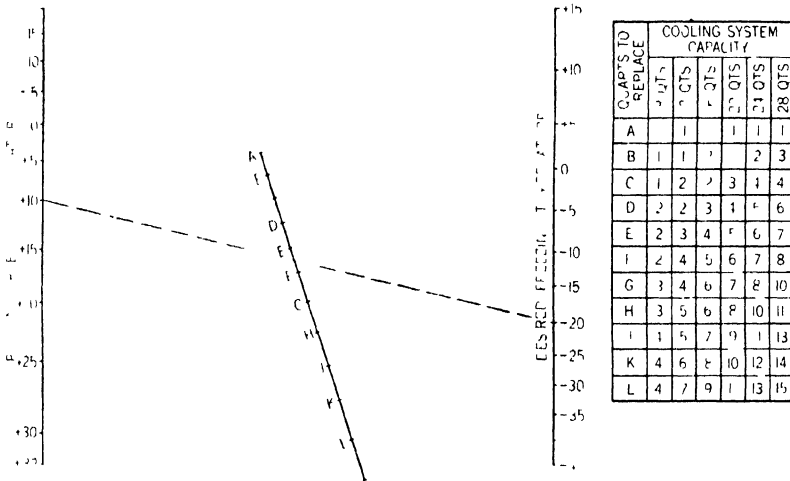


FIG. 278. NOMOGRAPH FOR DETERMINING QUANTITY OF ETHYLENE GLYCOL (PRESTONE) NEEDED TO REDUCE FREEZING POINT OF AUTOMOBILE RADIATOR SOLUTION

To use the nomograph lay the edge of a ruler from the mark on the left-hand scale representing the present protection, to the mark on the right-hand scale corresponding to the freezing protection desired. Read the first letter above or crossed by the ruler on the lettered scale in the middle of the chart. Turn to the table of Cooling System Capacity and in the column corresponding to the capacity of the radiator read the number found on the same line as the first letter appearing on or above the ruler. This number represents the quarts of solution to be drawn off from the cooling system and replaced with undiluted ethylene glycol anti-freeze.

Using the chart

Thus, if the cooling system capacity is 12 quarts, the present protection is 10° F. and protection to -20° F. is desired, lay a ruler from the +10 mark on the left-hand scale across to the -20 mark on the right-hand scale. *E* is the first letter above the ruler on the lettered scale in the middle. In the "capacity" table on the extreme right, under "12 quarts" and on the same line as *E* is the number 3, which is the number of quarts to be drawn off and replaced with undiluted ethylene glycol anti-freeze to decrease the freezing point to -20° F.

lower lines of the nomograph of Fig. 276 represent I and R respectively, we see that the middle line represents their product E . Knowing any two of these variables, the third is found by laying a straight edge across the chart.

*Nomograph
for an anti-
freeze solution*

The nomograph of Fig. 278 is a more complex type, used for computing the amount of antifreeze needed to reduce the freezing temperature of a radiator solution for any desired protection. Its use is explained underneath the chart.

SCOPE OF MATHEMATICS

In the present chapter we have given the mathematics needed for the understanding and solution of a great many engineering problems. This by no means exhausts the body of mathematics needed for the comprehension and development of scientific fields. In fact, most of the mathematics required for the treatment of engineering problems is still undiscovered, although new mathematical principles are found daily by research mathematicians.

*Applied
mathematics*

Unhappily, to understand one branch of mathematics you must have studied other branches of mathematics as well. In the mathematical approach to scientific problems the applied mathematician can therefore be of great assistance. That much useful work can be done with mathematics alone is emphasized by the branch of science called *astronomy*. Since relatively indirect methods must be used to measure the properties of stars and planets, the astronomer has had to rely mainly on mathematical reasoning to explain the movements and structure of heavenly bodies.

With mathematical reasoning and experimental analysis the dream world of the radio, the telephone, the airplane, and the automobile has become a reality.

The Slide Rule

THE ENGINEER'S COAT OF ARMS

Most engineering problems require numerical calculations and the slide rule is commonly used for making these calculations. It is an instrument with which you can multiply and divide quickly and accurately; with which you can find a power or root of a number easily; and with which you can find the sine, cosine, or tangent of an angle or the logarithm of a number in less time than you can look it up in a set of tables. The engineer's slide rule is usually straight and consists of three parts as shown in Fig. 279. There is the *frame* or stock; the movable part in the center which is called the *slide*; and the glass or *runner*. The runner has a central vertical line called the *hairline*. This line is used for locating a number on the slide rule. Each row of marks on the slide rule is called a *scale*. *What it is*

The slide rule is based on logarithms, invented by John Napier and made public in 1614. In 1620, Edmund Gunter invented the logarithmic line or scale, which was simply a scale with lengths proportional to logarithms. The first slide rule, with two or more logarithmic scales contained in a single instrument, appeared about 1630, invented independently by Edmund Wingate and William Oughtred. Since that time many additions and improvements have been made to the slide rule but the essential form remains unchanged. New types of scales have been added for special calculations; better methods of marking the scales on the slide rule make it easier to read; and improved workmanship has increased its accuracy and ease of operation. *Its evolution*

Slide rules for calculation are made in lengths from 5 to 20 in. The shorter ones can be carried more conveniently in a pocket or brief case and are easier to use, while the longer ones are more accurate. Most people use the 10-in. rule which combines the good points of the others. Slide rules of 6- or 8-ft. length that you may have seen in the classroom are for demonstration purposes only. *Short and long rules*

Read these pages with your slide rule at hand and actually

work through the examples with it. Then, as you finish a paragraph, do the problems at its end. The problems will furnish the practice you need.

THE MANNHEIM SLIDE RULE

Construction

About 1850, Amédée Mannheim, a French artillery officer and later Professor of Mathematics at L'École Polytechnique in Paris, designed the slide rule which bears his name, and which, with some additions, is still in wide use. One improved version is the Polyphase slide rule, a trade name for the standard Keuffel & Esser Company rule. The front of this slide rule is shown in Fig. 279. There are three scales on the front of the stock (the A, D, and K scales), and three on the front of the slide (the B, CI, and C scales). The slide is thinner than the stock and has three scales on the back (the S, L, and T scales), which are ordinarily used by turning the entire slide rule over and pulling the slide to the right.

Advantages

The chief advantages of the Mannheim slide rule are that it is simple, compact, and sturdy. It is smaller than other types of slide rules and yet it has all of the scales necessary for most calculations. These are arranged conveniently and there is not so much chance of confusion in operation as with a slide rule having twice as many scales. The heavy stock is more rigid than those on other types of slide rules and cannot easily get out of adjustment. The runner has only one piece of glass and is not so likely to break when the slide rule is dropped as are the runners on other types. The ruler on the top front edge of the slide rule (divided into sixteenths of an inch) is very useful in making sketches.

Improvements

The only real disadvantage of the Mannheim slide rule appears in comparison with the log-log slide rule which will be discussed later. With the Mannheim, it is not so easy to raise a number to a fractional power or to find the *natural logarithm*

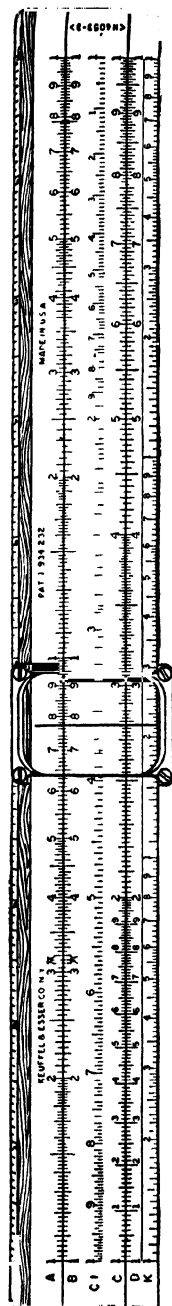


FIG. 279. MANNHEIM SLIDE RULE

of a number. Also, certain specialized calculations take longer, although they can all be done on the Mannheim slide rule.

A scale is a row of marks on the slide rule and each scale is designated by the letter at its end. For instance, notice the A scale on the top of the stock in Fig. 279. There are three general types of scales on the Mannheim slide rule. They are:

1. Logarithmic scales, which are on the front of the slide rule (A, B, CI, C, D, and K).
2. Trigonometric scales, which are on the back of the slide (S and T).
3. The equal-parts scale, which is on the back of the slide (L).

The C scale is representative of all logarithmic scales so it can be used as a model for explanation. The distance from the left end of this scale to the position occupied by a number represents the *mantissa* of the common logarithm of the number. Since this is so, the location of a number on the C scale depends only upon the sequence of numerals in the number and not upon the position of the decimal point in the number. Hence, you can disregard any zeros at the beginning or end of a number. For example, the numbers 0.0278, 27.80, and 27,800 all occupy the same position on the C scale and, therefore, you can think of each as 278 for the purpose of locating it. Then just locate the number as you would on an engineer's scale (decimal system) used for measuring. The runner is set at this number (278) on the C scale in Fig. 279.

The mark at the extreme left end of the C scale is called the *left index*. If the first numeral of the number (disregarding any zeros at the beginning) is 1, the number is located between the left index and the printed numeral 2 which is about three-tenths of the distance to the right end. Within this range you can locate

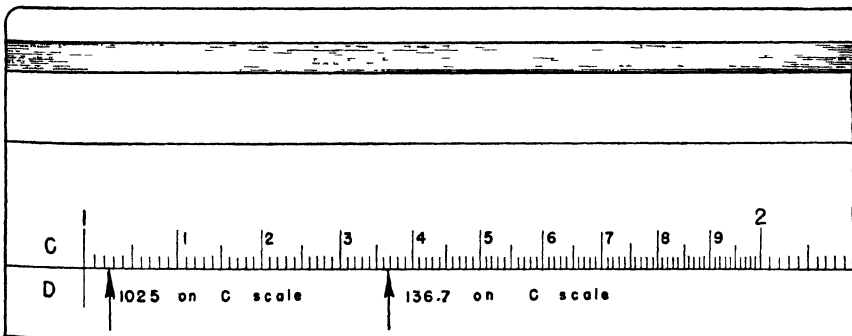


FIG. 280. THE C SCALE BETWEEN THE NUMERALS 1 AND 2

the first three numerals of the number exactly and estimate the fourth. None after the fourth can be represented. Correct locations for the numbers 1.025 or 102.5 and 136.7 are shown in Fig. 280.

*Numbers from
2 to 4*

Between the printed numeral 2 referred to in the paragraph just above and the numeral 4 on the C scale, the marking is a little different. You can always locate the first two numerals of the number exactly, but the third can be located exactly only if it is even. If it is odd, it falls halfway between two marks because each small space represents an increase of two in the third numeral. More than three numerals cannot be represented in this range. The numbers 279 and 0.316 are shown in their proper locations in Fig. 281.

*Numbers
above 4*

To the right of the printed numeral 4 on the C scale, the first two numerals of the number can be located exactly, but the third must usually be estimated. Each small space represents an increase of five in the third numeral. Fig. 282 shows correct locations for the numbers 57.2 and 6570.

*D and CI
scales*

The D scale is an exact duplicate of the C scale; numbers are located on both in the same way. The CI scale differs only in that it is read from right to left instead of left to right in the usual way. It is often called the *reciprocal scale* because the number on it at a given distance from the left end is the reciprocal of the number on the C scale at the same distance from the left end.

*Divided
scales*

The A and B scales are double scales in that each offers a double range of numbers. A given number might be located in either half, and the question of which half to use will be answered in the section where the use of these scales is discussed.

The K scale is a triple scale and a given number might be

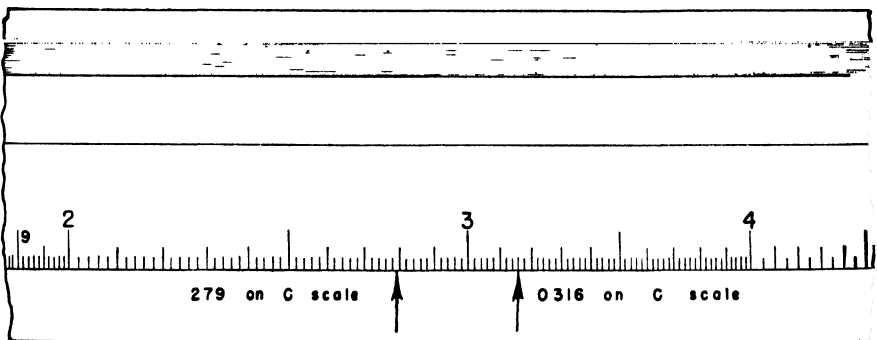


FIG. 281. THE C SCALE BETWEEN THE NUMERALS 2 AND 4

located in any one of its three parts. Which part to use in a particular problem will be considered in the section on the use of the K scale.

The trigonometric scales are on the back of the slide and are the sine scale (marked S) and the tangent scale (marked T). See Fig. 283. On each, the problem is to locate an angle in degrees and minutes. This is easy if you keep in mind that there are 60 min. in one degree. The sine scale ranges from $0^\circ 34'$ to 90° and the tangent scale from $5^\circ 43'$ to 45° . All numerals printed on these scales are to be read in degrees except the numerals 40 and 50 at the left end of the sine scale. These two represent, respectively, $0^\circ 40'$ and $0^\circ 50'$. Each of the first five marks to the right of 70° on the sine scale represents an increase of 2° . Thus the next to the last mark on the sine scale represents 80° . The last mark represents 90° . The last mark on the tangent scale represents 45° . Figure 283 shows part of the tangent scale with correct locations for the angles $16^\circ 25'$ and $21^\circ 33'$.

Trigonometric scales

The equal parts scale is on the back of the slide and is usually marked with the letter L. It is called the *log scale* because it is used in finding the common logarithm of a number. Figure 284 shows part of a log scale with correct locations for the numbers 068 and 123.

Log scale

COMMON OPERATIONS

A great many engineering problems include multiplication of one number by another. On the slide rule, multiplication is ordinarily done with the C and D scales. First you move the slide so that one end (index) of the C scale is over the *first number* on the D scale. Next you set the runner so that the hairline of the runner

How to multiply

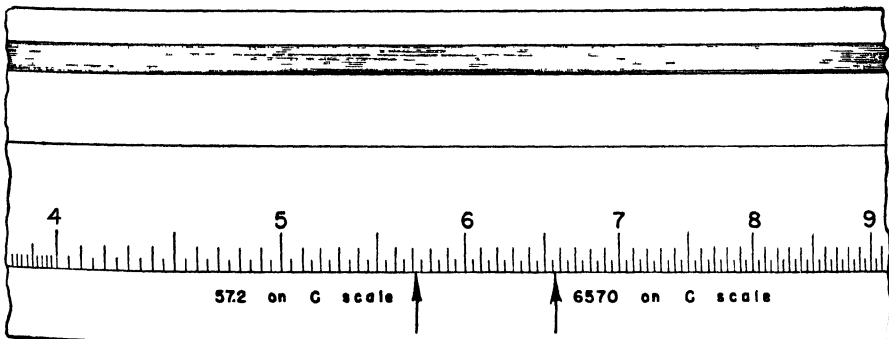


FIG. 282. THE C SCALE BETWEEN THE NUMERALS 4 AND 9

The answer

is over the *second number* on the C scale. Then you read the *answer* on the D scale under the hairline of the runner. Follow a couple of examples on your own slide rule and see how it goes. To multiply 11.6 by 1.25, as in Fig. 285, you set the left index of the C scale over 11.6 on the D scale. Then you set the hairline of the runner to 1.25 on the C scale. This completes the manipulation of the slide rule. The correct setting of the slide and runner is shown in Fig. 285. The answer is read as 14.5 on the D scale under the hairline of the runner.

Decimal points

Of course you do not get the *decimal point* from the slide rule. All it gives you is the sequence of numerals (1, 4, 5) in the answer, but the decimal point can usually be located by exercising judgment as to what would be a reasonable answer. (For instance, you know that 1.45 is too small and 145 is too large for the answer to this problem. Hence it must be 14.5.) If you want a little practice at this point, try the same problem with the numbers in reversed order by multiplying 1.25 by 11.6. Naturally the answer is the same.

Reversing the slide

Now try to multiply 8.5 by 72 in the same way. You cannot do it by the method explained above, because when you have set the left index of the C scale to 8.5 on the D scale, the number 72 on the C scale is far beyond the end of the D scale. When this happens you must solve the problem by moving the slide to the left as in Fig. 286. Set the right index of the C scale to 8.5 on the D scale. Next set the hairline of the runner to 72 on the C scale. Then read the answer as 612 on the D scale under the hairline. (You know that it is 612 because 61.2 is too small and 6120 is too large.)

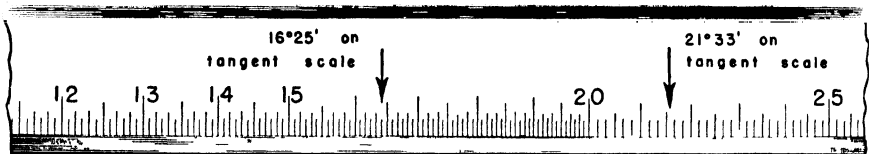


FIG. 283. LOCATING ANGLES ON THE TANGENT SCALE

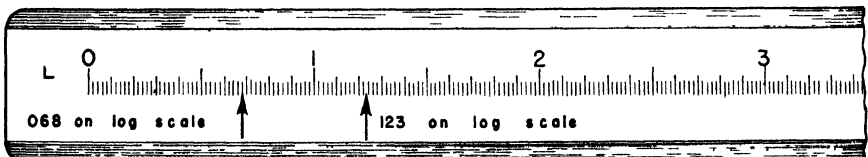


FIG. 284. THE EQUAL PARTS SCALE OR LOG SCALE

At times it is hard to say whether the slide must go to the right *Right or left?* or left until you try it. In such a case do not waste time thinking about it. Try it one way and if it does not work, you can always do the problem by moving the slide the other way. You can multiply any number by any other number with the Mannheim slide rule.

Do not worry if you did not read the answers exactly as given *Accuracy* in the preceding examples. In the first one you might have read the answer as 14.49 or 14.52 rather than 14.50, because you might have made a slight error in setting the slide or runner (or I might have). The difference between 14.49 and 14.52, as a slide-rule answer, is not ordinarily important. *Slide-rule answers are accurate but not exact.* Now try solving the following problems in multiplication. Verify each of the answers given.

- | | | |
|----------------------------------|----------------------------------|-----------------|
| 1. $20.7 \times 1.63 = 33.8$ | 11. $11.04 \times 6.97 = 77$ | <i>Problems</i> |
| 2. $4.82 \times 0.833 = 4.01$ | 12. $0.161 \times 594 = 95.5$ | |
| 3. $0.667 \times 58.4 = 38.9$ | 13. $10.9 \times 0.501 = 5.46$ | |
| 4. $0.707 \times 33 = 23.3$ | 14. $35.6 \times 16.5 = 588$ | |
| 5. $1.008 \times 0.905 = 0.913$ | 15. $6.29 \times 0.437 = 2.75$ | |
| 6. $5280 \times 22.5 = 118,800$ | 16. $0.948 \times 0.876 = 0.831$ | |
| 7. $8.32 \times 2.77 = 23$ | 17. $74.8 \times 3.64 = 272$ | |
| 8. $0.985 \times 6370 = 6280$ | 18. $0.989 \times 31 = 30.7$ | |
| 9. $17.24 \times 5.5 = 94.8$ | 19. $3.03 \times 0.484 = 1.465$ | |
| 10. $0.563 \times 0.316 = 0.178$ | 20. $963 \times 8.94 = 8600$ | |

Division is just the reverse of multiplication on the slide rule. *How to divide* First set the hairline of the runner, on the D scale, to the number into which we wish to divide (dividend). Then move the slide so that the number you want to divide by (divisor) is under the hair-

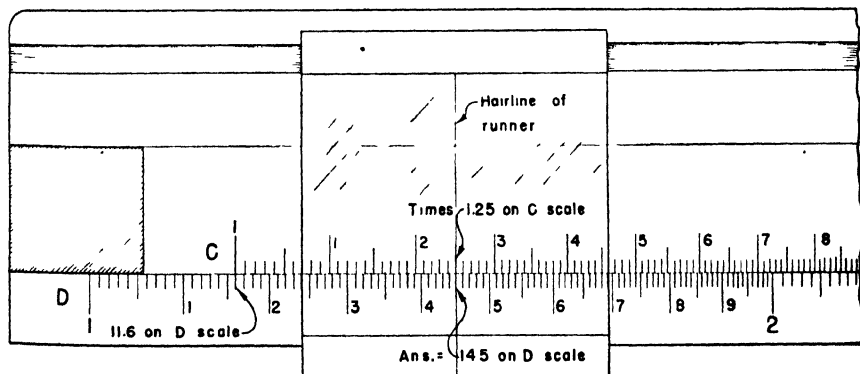


FIG. 285. MULTIPLICATION WITH THE C AND D SCALES

line on the C scale. When you have done this, read the answer (quotient) on the D scale under whichever index of the C scale is between the ends of the D scale. Here is an example to show you the way.

Example

Suppose we wish to divide 756 by 83.5. Figure 287 shows the setting of the slide and runner. First set the hairline of the runner to 756 on the D scale. Next move the slide so that 83.5 on the C scale is under the hairline. Then read the answer as 9.05 on the D scale under the right index of the C scale. In every case, one index of the C scale will be between the ends of the D scale when you are ready to read the answer. Read it on the D scale under this index. Check the answer in each of the following problems.

Problems

- | | |
|-------------------------------|--------------------------------|
| 1. $1.078 \div 5.78 = 0.1863$ | 11. $33.3 \div 9.04 = 3.69$ |
| 2. $36.3 \div 2.07 = 17.54$ | 12. $3.69 \div 57.5 = 0.0642$ |
| 3. $533 \div 29 = 18.4$ | 13. $11.6 \div 5.85 = 1.982$ |
| 4. $72 \div 17 = 4.23$ | 14. $9.58 \div 0.157 = 61$ |
| 5. $4.87 \div 1.05 = 4.63$ | 15. $0.757 \div 8.83 = 0.0858$ |
| 6. $15.22 \div 11.73 = 1.296$ | 16. $376 \div 44.1 = 8.53$ |
| 7. $22.5 \div 9.17 = 2.45$ | 17. $1.503 \div 0.621 = 2.42$ |
| 8. $0.402 \div 6.65 = 0.0605$ | 18. $14.65 \div 7.19 = 2.04$ |
| 9. $0.947 \div 0.413 = 2.3$ | 19. $589 \div 36.7 = 16.05$ |
| 10. $11.5 \div 3.46 = 3.33$ | 20. $9.15 \div 7.16 = 1.275$ |

Multiplication with division

Most technical calculations contain more than two numbers. For instance, you will often find that you need the answer to some fraction like this:

$$\frac{16,300 \times 242}{12,000,000 \times 1.38}$$

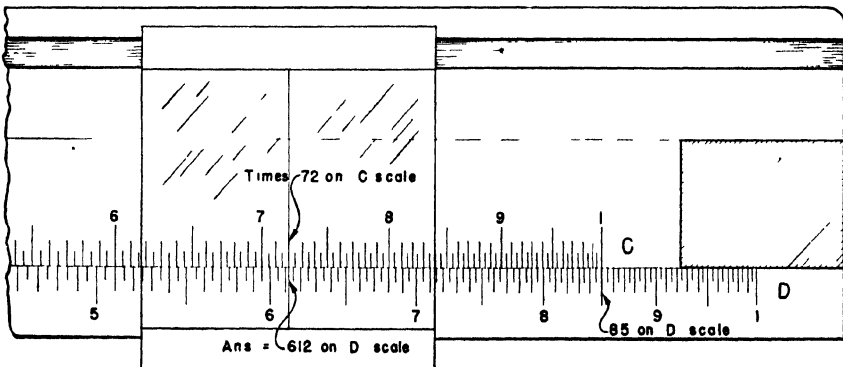


FIG. 286. MULTIPLICATION BY REVERSING THE SLIDE

This problem includes both multiplication and division but you know how to do both. Just *multiply and divide in any order* until you have the final answer. Then write it down. That is all there is to it. In the example given above, one way to start would be to set the hairline of the runner to 16,300 on the D scale. Then divide by 12,000,000 by bringing 12,000,000 (the same as 12) on the C scale under the hairline. Next, multiply by 242 by setting the hairline of the runner to 242 on the C scale. Continuing, divide by 1.38 by sliding 1.38 on the C scale under the hairline. Finally, read the answer as 0.238 on the D scale under the left index of the C scale. You can solve this type of problem quickly and it is not necessary to write down any intermediate results. Locate the *decimal point* in the answer by “rounding off” the numbers and making the same calculation mentally.

Continuing operation

It is not always quite so easy as this example has seemed. Sometimes when you are ready to multiply by one of the numbers in the numerator, this number on the C scale will be beyond the end of the D scale. When this happens you can reverse the slide. Do so by setting the runner to that index of the C scale that is between the ends of the D scale. Then move the other index of the C scale under the hairline of the runner. An example will illustrate. We want the result of:

Reversing the slide

$$\frac{56.3 \times 0.785}{3.75 \times 13.4}$$

The normal procedure is to set the hairline of the runner to 56.3 on the D scale. Next, divide by 3.75 by bringing 3.75 on the C scale under the hairline. At this point, you would like to multiply

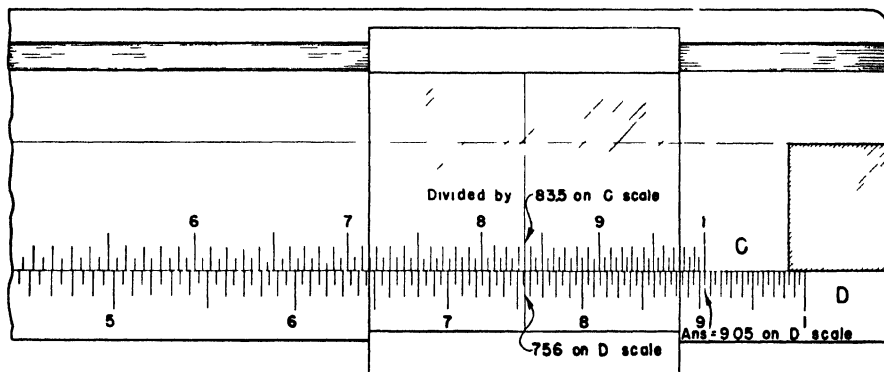


FIG. 287. DIVISION BY USE OF THE C AND D SCALES

by 0.785 by setting the runner to 0.785 on the C scale. But 0.785 on the C scale is beyond the end of the D scale so you cannot move the runner to it. Therefore, move the glass runner to the left index of the C scale, which happens to be above 15 on the D scale. Leave the runner in position while you slide the right index of the C scale under the hairline. Now you can multiply by 0.785 by setting the hairline of the runner to 0.785 on the C scale. Then divide by 13.4 by bringing 13.4 on the C scale under the hairline. This completes the manipulation of the slide rule and you read the answer to the problem on the D scale under the right index of the C scale. It is 0.881.

*Choose of
order*

Another method is even better for this example. First divide 56.3 by 3.75, getting 15. Then divide 15 by 1.34 and multiply by 0.785. By this order or sequence there are fewer motions of the slide and runner since the slide does not have to be reversed. However, in some problems this order would require more operations. You can try and see for yourself that this will be true if 0.785 is replaced by 0.925 in the problem above. The problems in the next list are longer than those you have worked before, but they represent a type that engineers often have to solve. Work them by going all the way through before writing down any result. See if the following answers are correct.

Problems

- | | |
|--|--|
| 1. $\frac{98.3 \times 27.2}{7.95 \times 15.3} = 22$ | 6. $\frac{13.01 \times 7.68 \times 0.437}{3.92 \times 12.65 \times 8.60} = 0.1023$ |
| 2. $\frac{0.753 \times 1.08}{3.85 \times 3.05} = 0.0693$ | 7. $\frac{56.5 \times 0.645 \times 11.82}{7.72 \times 8.4 \times 0.754} = 8.82$ |
| 3. $\frac{433 \times 39}{32.7 \times 683} = 0.757$ | 8. $\frac{0.643 \times 9.48 \times 504}{79 \times 5.30 \times 382} = 0.0192$ |
| 4. $\frac{13.72 \times 5.68}{10.2 \times 2.14} = 3.57$ | 9. $\frac{5.84 \times 2.58 \times 0.925}{44.1 \times 1.518 \times 0.938} = 0.222$ |
| 5. $\frac{6.33 \times 372}{46.7 \times 0.925} = 54.5$ | 10. $\frac{0.375 \times 11.8 \times 2.72}{6.28 \times 38.7 \times 0.163} = 0.304$ |

POWERS AND ROOTS

*Squaring a
number*

The A and D scales are ordinarily used together to square a number or to find its square root. The indexes of the two scales are always aligned since they are both on the frame of the rule. To square a number, set the hairline of the glass runner to the number on the D scale and read the square on the A scale under the hairline. You do not need to use the slide at all. For example, the first step in squaring 43.5 is to set the hairline of the runner to 43.5 on

the D scale. Then you read the square of 43.5 on the A scale under the hairline. It is 1890. Try it yourself for practice. Then try the following problems. Find the square of each number.

1. 18.5 *Ans.* 3422. 6.32 *Ans.* 403. 21.7 *Ans.* 4704. 0.915 *Ans.* 0.8355. 51.2 *Ans.* 26206. 4.27 *Ans.* 18.27. 0.397 *Ans.* 0.1588. 1.085 *Ans.* 1.189. 2.43 *Ans.* 5.910. 0.733 *Ans.* 0.537*Problems*

To find the square root of a number, set the hairline of the runner to the number on the A scale and read the square root on the D scale under the hairline. This raises the question of *which half of the A scale to use* in starting the process. There is a rule that answers this question: *Square root*

In taking the square root use the left half of the A scale for numbers having 1, 3, 5, or any uneven number of digits to the left of the decimal point.

This is a fine rule if you remember it. However, you may forget it and besides it is more complicated when you try to apply it to numbers less than unity. For this reason, it is better to use a little mental arithmetic and not to rely upon the rule. Estimate about what the square root should be and move the runner to that number on the D scale. Then adjust the runner to the original number *on the same half of the A scale*. Finally, the square root will be found on the D scale under the hairline. Try this procedure by obtaining the square root of 23.5. You know that the square root of 23.5 is about 5. Therefore move the runner to 5 on the D scale and then adjust it slightly to 23.5 on the A scale. This position will be in the right-hand half of the A scale and the square root of 23.5 is read as 4.85 on the D scale.

For large numbers and for decimal fractions such as 0.0693, *Decimal fractions* you may find some difficulty in estimating the value of the square root. It is convenient then to point off pairs of digits from the decimal point. For the number 0.0693 the first pair is .06, and since the square root of 6 is between 2 and 3 the square root of 0.0693 begins with the numeral 2. If you set the runner to 0.0693 in the right half of the A scale, you will read 833 on the D scale, but by setting the runner to 0.0693 in the left half of the A scale you read 263 on the D scale. The latter setting is therefore correct, and by mental arithmetic we place the decimal point at 0.263. Now try a little mental arithmetic in finding the square root of each number below. Then check the exact answer on your slide rule.

Problems

| | | | |
|------------------------------|------------|-------------------------------|------------|
| 1. 0.643 ($\sqrt{64} = 8$) | Ans. 0.802 | 6. 1020 ($\sqrt{10} = 3+$) | Ans. 32 |
| 2. 8.17 ($\sqrt{8} = 2+$) | Ans. 2.86 | 7. 34.2 ($\sqrt{34} = 5+$) | Ans. 5.85 |
| 3. 28.2 ($\sqrt{28} = 5+$) | Ans. 5.31 | 8. 0.118 ($\sqrt{11} = 3+$) | Ans. 0.344 |
| 4. 477 ($\sqrt{4} = 2$) | Ans. 21.8 | 9. 23,500 ($\sqrt{2} = 1+$) | Ans. 153.4 |
| 5. 10.2 ($\sqrt{10} = 3+$) | Ans. 3.2 | 10. 0.364 ($\sqrt{36} = 6$) | Ans. 0.603 |

Cubing a number

The D and K scales are used together to find the cube or cube root of a number. When the hairline of the runner is set to a given number on the D scale, the cube of the number can be read on the K scale under the hairline. The slide is not used. The cube of 3.12 is found by setting the hairline of the runner to 3.12 on the D scale and then reading the cube on the K scale under the hairline. The cube of 3.12 is 30.3. Here the decimal point is located in the cube by mental arithmetic. Try this method to find the cube of each of the following numbers.

Problems

| | | | |
|----------|------------|-----------|------------|
| 1. 1.05 | Ans. 1.16 | 6. 0.953 | Ans. 0.866 |
| 2. 0.863 | Ans. 0.643 | 7. 3.72 | Ans. 51.5 |
| 3. 5.78 | Ans. 193 | 8. 6.15 | Ans. 233 |
| 4. 2.27 | Ans. 11.7 | 9. 0.77 | Ans. 0.457 |
| 5. 12.6 | Ans. 2000 | 10. 1.935 | Ans. 7.25 |

Cube root

Finding the cube root of a number is just the reverse of finding the cube. The number is located on the K scale by the hairline of the runner and the cube root is read on the D scale under the hairline. You will notice, however, that *a given number can be located in three different places on the K scale*. You can determine which part of the K scale to use by exercising judgment as to what would be a reasonable answer.

For instance, suppose you want the cube root of 14.7. You know that the cube root of 8 is 2 and the cube root of 27 is 3.

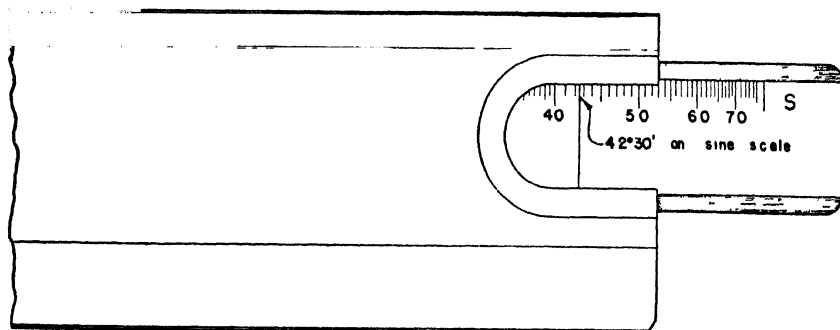


FIG. 288 a. SETTING THE ANGLE ON THE SINE SCALE

Hence, the cube root of 14.7 must be between 2 and 3 or say 2.5. *Example of $\sqrt[3]{14.7}$*
 Therefore, place the runner so that its hairline is at 2.5 on the D scale and then adjust it to 14.7 on the K scale. Read the cube root on the D scale as 2.45. The same method will work for any number although it is easier with large numbers to think in terms of the cube of the approximate answer. For example, we can study the cube root of 93,500. Here you realize that the cube of 40 is 64,000 and the cube of 50 is 125,000. The cube root of 93,500 must, therefore, be between 40 and 50, and you find it to be 45.3.

Special care must be taken with cube roots of decimal fractions *Decimal fractions* such as 0.00474 and 0.0722. Remember that the cube root of a number less than 1 is always greater than the number itself but cannot be greater than 1. You can find the cube root of 0.00474 this way. The cube of 0.1 is 0.001 and the cube of 0.2 is 0.008. Therefore, the cube root of 0.00474 must be between 0.1 and 0.2. Set the runner between 0.1 and 0.2 on the D scale and adjust it to 0.00474 on the K scale. Then read the cube root on the D scale as 0.168. Now try this method for yourself by finding the cube root of each of the following numbers.

- | | | | |
|-----------|------------|------------|------------|
| 1. 83.4 | Ans. 4.37 | 6. 1.87 | Ans. 1.232 |
| 2. 27.9 | Ans. 3.04 | 7. 139,500 | Ans. 51.8 |
| 3. 3.37 | Ans. 1.5 | 8. 0.068 | Ans. 0.408 |
| 4. 31,300 | Ans. 31.5 | 9. 0.0122 | Ans. 0.230 |
| 5. 0.75 | Ans. 0.909 | 10. 0.0007 | Ans. 0.089 |

Problems

TRIGONOMETRIC FUNCTIONS

When you study mechanics and related subjects, you will often need to find the sine or cosine of an angle. On the Mannheim *Sine of an angle* slide rule, the sine is found in the following manner. Turn the en-

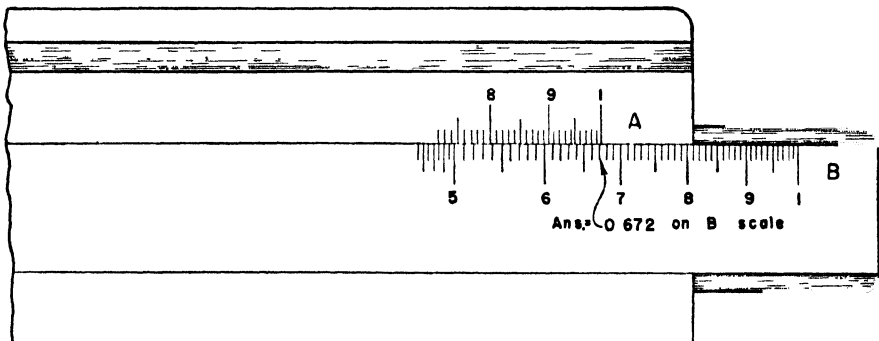


FIG. 288b. READING THE SINE OF THE ANGLE ON THE B SCALE

*Setting
the angle*

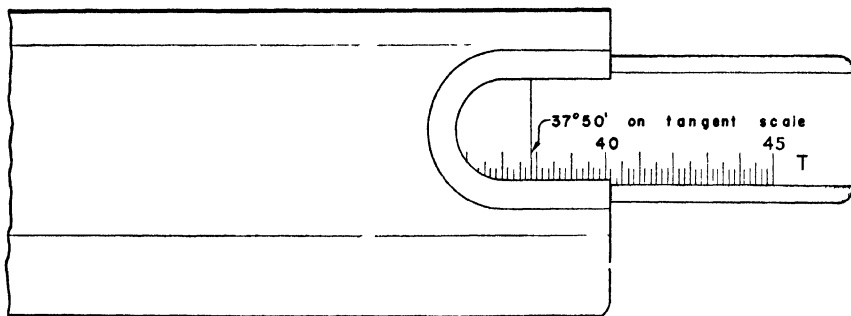
tire slide rule over and pull the slide to the right. Use the sine scale and set the angle for which you want to find the sine under the mark on the celluloid insert. (If there is no mark on the celluloid, use the edge of it instead.) Then turn the slide rule back over and read the sine of the angle on the B scale under the right-hand index of the A scale. As an example we will find the sine of $42^{\circ} 30'$. Turn the slide rule over and set $42^{\circ} 30'$ on the sine scale under the mark on the celluloid insert as in Fig. 288 *a*. Then turn the slide rule back over and read the sine of $42^{\circ} 30'$ as 0.672 on the B scale under the right index of the A scale. The proper setting is shown in Fig. 288 *b*. This procedure is quicker than using a set of tables. To be sure you remember it, find the sine of each angle below.

Problems

- | | | | |
|---------------------|------------|---------------------|------------|
| 1. 35° | Ans. 0.574 | 6. $47^{\circ} 40'$ | Ans. 0.739 |
| 2. $16^{\circ} 30'$ | Ans. 0.284 | 7. $32^{\circ} 30'$ | Ans. 0.537 |
| 3. $22^{\circ} 20'$ | Ans. 0.380 | 8. $61^{\circ} 20'$ | Ans. 0.877 |
| 4. $53^{\circ} 10'$ | Ans. 0.800 | 9. $7^{\circ} 15'$ | Ans. 0.126 |
| 5. 71° | Ans. 0.945 | 10. $3^{\circ} 23'$ | Ans. 0.059 |

Arc sine

To find the *arc sine* of a given number, that is, the angle which has the given number for its sine, just reverse the process of finding the sine. Set the number on the B scale under the right-hand index of the A scale. Then turn the slide rule over and read the angle on the sine scale under the mark on the celluloid insert. The B scale is a double scale so be careful about the decimal point. When you wish to find the sine or the arc sine, think of the left half of the scale as extending from 0.01 to 0.1 and the right half as extending from 0.1 to 1.0. Remember that *the sine of an angle can never be greater than unity*. Find the arc sine of each of the following numbers.

FIG. 289 *a*. SETTING THE ANGLE ON THE TANGENT SCALE

| | | | | |
|-----------|-----------------------|-----------|-----------------------|-----------------|
| 1. 0.0617 | Ans. $3^{\circ} 32'$ | 6. 0.916 | Ans. $66^{\circ} 30'$ | <i>Problems</i> |
| 2. 0.443 | Ans. $26^{\circ} 20'$ | 7. 0.547 | Ans. $33^{\circ} 15'$ | |
| 3. 0.718 | Ans. 46° | 8. 0.633 | Ans. $43^{\circ} 10'$ | |
| 4. 0.0823 | Ans. $4^{\circ} 43'$ | 9. 0.0492 | Ans. $2^{\circ} 49'$ | |
| 5. 0.815 | Ans. $54^{\circ} 50'$ | 10. 0.167 | Ans. $9^{\circ} 37'$ | |

The cosine of an angle cannot be found directly from the slide rule, but you can make use of the fact that the cosine of an angle is equal to the sine of 90° minus the angle. When you want to find the cosine, just subtract the angle from 90° and find the sine of the difference. For example, to find the sine of $63^{\circ} 30'$, subtract $63^{\circ} 30'$ from 90° . This gives $26^{\circ} 30'$. Then find the sine of $26^{\circ} 30'$, which is 0.443. This is the cosine of $63^{\circ} 30'$. The arc cosine of a number can be found by reversing the process. While you have the details in mind, it will be useful to solve the problems below. In problems 1–5, find the cosine of the angle. In the remaining, find the arc cosine of the number.

| | | | | |
|---------------------|-------------|-----------|-----------------------|-----------------|
| 1. $32^{\circ} 50'$ | Ans. 0.840 | 6. 0.384 | Ans. $67^{\circ} 20'$ | <i>Problems</i> |
| 2. $73^{\circ} 10'$ | Ans. 0.290 | 7. 0.0785 | Ans. $85^{\circ} 30'$ | |
| 3. $27^{\circ} 15'$ | Ans. 0.889 | 8. 0.850 | Ans. 32° | |
| 4. $84^{\circ} 45'$ | Ans. 0.0915 | 9. 0.173 | Ans. $80^{\circ} 02'$ | |
| 5. $43^{\circ} 30'$ | Ans. 0.725 | 10. 0.528 | Ans. 58° | |

The tangent of an angle is a commonly used trigonometric function and it is convenient to be able to find it from the slide rule. To illustrate, we will find the tangent of $37^{\circ} 50'$. Turn the slide rule over and set $37^{\circ} 50'$ on the tangent scale under the mark on the celluloid. This is shown in Fig. 289 a. Then turn the slide rule back over and read the tangent of $37^{\circ} 50'$ as 0.777 on the C scale over the right index of the D scale. The correct position of the slide at this point is shown in Fig. 289 b.

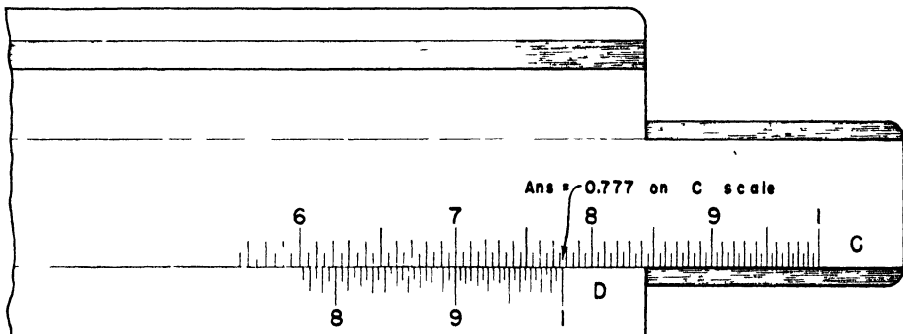


FIG. 289 b. READING THE TANGENT OF THE ANGLE ON THE C SCALE

To generalize, you first turn the slide rule over and pull the slide to the right. Then set the angle on the tangent scale under the mark on the celluloid insert. Next turn the slide rule back over and read the tangent of the angle on the C scale over the right index of the D scale. Think of the C scale as extending from 0.1 at the left end to 1.0 at the right end for this calculation.

Large angles

When the angle is greater than 45° , subtract it from 90° . Set the result on the tangent scale under the mark on the celluloid and read the tangent of the original angle on the D scale under the left index of the C scale. For this purpose, the D scale is to be thought of as extending from 1 at the left end to 10 at the right end.

The *arc tangent* of a given number, that is the angle which has the number for its tangent, is found by reversing the process of finding the tangent. In the first group of problems find the tangent of each angle.

Problems

- | | | | |
|-------------------|------------|--------------------|------------|
| 1. $6^\circ 32'$ | Ans. 0.115 | 6. $52^\circ 40'$ | Ans. 1.311 |
| 2. $41^\circ 40'$ | Ans. 0.890 | 7. $27^\circ 28'$ | Ans. 0.520 |
| 3. $22^\circ 15'$ | Ans. 0.409 | 8. $34^\circ 20'$ | Ans. 0.683 |
| 4. $71^\circ 30'$ | Ans. 2.99 | 9. $67^\circ 15'$ | Ans. 2.38 |
| 5. $33^\circ 10'$ | Ans. 0.654 | 10. $10^\circ 12'$ | Ans. 0.180 |

In the following list, find the arc tangent of each number.

- | | | | |
|----------|---------------------|-----------|---------------------|
| 1. 0.148 | Ans. $8^\circ 25'$ | 6. 0.238 | Ans. $13^\circ 23'$ |
| 2. 2.56 | Ans. $68^\circ 40'$ | 7. 0.905 | Ans. $42^\circ 10'$ |
| 3. 0.456 | Ans. $24^\circ 30'$ | 8. 3.15 | Ans. $72^\circ 22'$ |
| 4. 0.739 | Ans. $36^\circ 26'$ | 9. 0.673 | Ans. $33^\circ 58'$ |
| 5. 1.54 | Ans. 57° | 10. 0.375 | Ans. $20^\circ 35'$ |

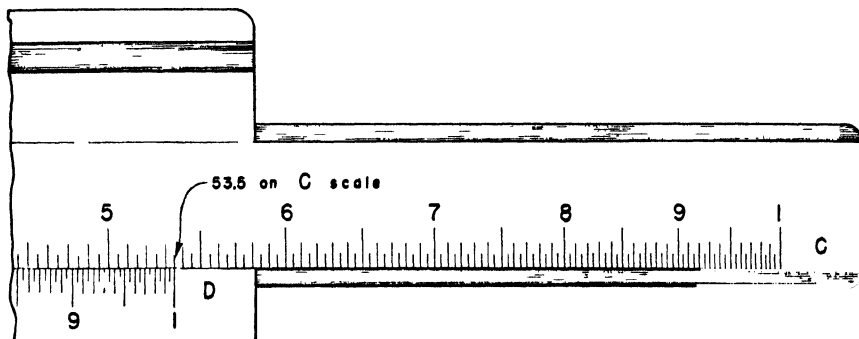


FIG. 290a. TO OBTAIN LOGARITHM SET NUMBER ON C SCALE

USE OF SPECIAL SCALES

Many engineering formulas contain logarithms. Of course, you *Logarithms* can always look up the logarithm of a number in a table of logarithms, but since you use the slide rule for other calculations, you may save time by getting the logarithms also from it. You remember that there are two parts of a common logarithm, the *characteristic* and the *mantissa*. The characteristic is the part of the logarithm to the left of the decimal point and the mantissa is the part to the right of the decimal point. The common logarithm (to base 10) of 350 is 2.545. The characteristic is 2 and the mantissa is 545. The characteristic which is one less than the number of digits to the left of the decimal point of the number itself is determined by inspection. The mantissa is obtained from a set of tables or from the slide rule. It does not depend upon the location of the decimal point, but only upon the sequence of numerals.

To find the mantissa for a given number, you first adjust the *How to find the mantissa* slide on the Mannheim rule so that the given number on the C scale is over the right-hand index of the D scale. Then you turn the slide rule over and read the mantissa of the logarithm of the number on the log scale (marked L) under the mark on the celluloid insert. To illustrate this, the first step in finding the mantissa of the logarithm of 53.5 is to set 53.5 on the C scale over the right index of the D scale as shown in Fig. 290 *a*. Then you turn the slide rule over, as in Fig. 290 *b*, and read the mantissa as 728 on the log scale under the mark on the celluloid. This setting is shown in Fig. 290 *b*. There are two digits to the left of the decimal point in the original number 53.5 so the characteristic of its logarithm is $2 - 1 = 1$. Thus the logarithm of 53.5 is 1.728.

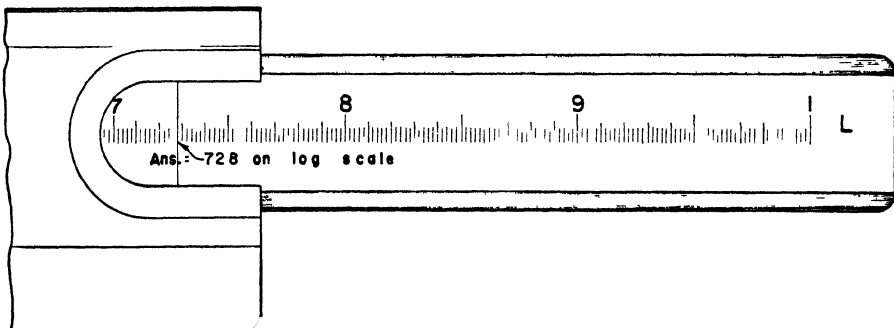


FIG. 290 *b*. READ LOGARITHM ON L SCALE UNDER HAIR LINE

Decimal fraction

There is a special method of writing the characteristic of the logarithm of a decimal fraction. This you probably remember from your study of mathematics. For instance, the logarithm of 0.535 is $9.728 - 10$; the logarithm of 0.0535 is $8.728 - 10$; that of 0.00535 is $7.728 - 10$; and so on. But no matter where the decimal point of a number is located, the mantissa of its logarithm is the same and is found in the same way with the slide rule. In the following problems, find the logarithm of each number.

Problems

- | | | | |
|----------|-----------------|----------|------------|
| 1. 34.7 | Ans. 1.540 | 6. 627 | Ans. 2.797 |
| 2. 1.952 | Ans. 0.291 | 7. 2.74 | Ans. 0.438 |
| 3. 5280 | Ans. 3.723 | 8. 125 | Ans. 2.097 |
| 4. 94.5 | Ans. 1.975 | 9. 22.8 | Ans. 1.358 |
| 5. 0.872 | Ans. 9.941 - 10 | 10. 8.63 | Ans. 0.936 |

Each of the following is the logarithm of a number. See if you can find the numbers which are called the antilogarithms or just antilogs.

- | | | | |
|----------|------------|----------------|-------------|
| 1. 1.579 | Ans. 37.9 | 6. 3.172 | Ans. 1485 |
| 2. 0.483 | Ans. 3.04 | 7. 1.172 | Ans. 14.85 |
| 3. 2.715 | Ans. 519 | 8. 1.950 | Ans. 89.1 |
| 4. 1.617 | Ans. 41.4 | 9. 2.250 | Ans. 178 |
| 5. 0.070 | Ans. 1.023 | 10. 8.316 - 10 | Ans. 0.0207 |

Multiplying with the CI scale

The CI scale is also called the *reciprocal scale* because the number on it at a given distance from the left index is the reciprocal of the number on the C scale at the same distance from the left index. It is just a C scale that reads from right to left instead of from left to right. Used properly, it will enable you to avoid

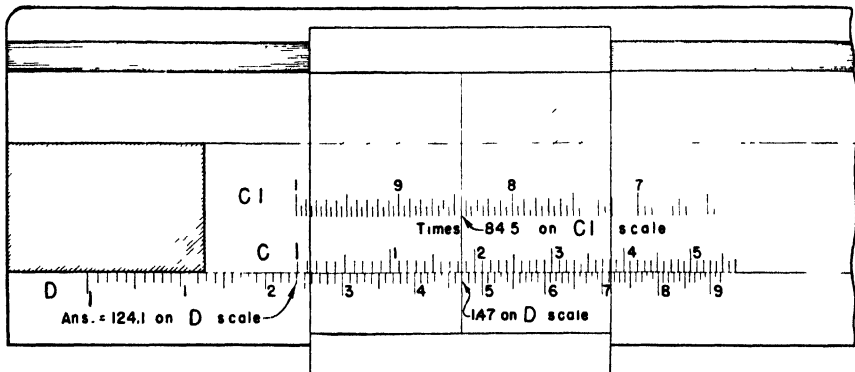


FIG. 291. SETTING THE SLID AND RUNNER FOR MULTIPLICATION WITH THE CI SCALE.

reversing the slide in problems involving both multiplication and division.

To use the CI scale in multiplication, you use the normal procedure for division. Set the hairline of the runner to the first number on the D scale. Then adjust the slide so that the second number on the CI scale is under the hairline. The answer is read on the D scale under whichever index of the CI scale is between the ends of the D scale. This is illustrated in Fig. 291 where we see the correct setting of the slide and runner to multiply 1.47 by 84.5. The answer is 124.1, read on the D scale at the left end of the slide.

To divide one number by another with the CI scale, follow the procedure you have learned for multiplication with the C and D scales. Fig. 292 shows the correct setting of the slide and runner to divide 93.2 by 108.5. The first step in the problem is to place the right-hand index of the CI scale over 93.2 on the D scale. Then set the hairline of the runner to 108.5 on the CI scale. The answer is read as 0.859 on the D scale under the hairline.

Division with the CI scale

If you compare what you have just read with the earlier material on multiplication and division in which you used the C and D scales, you will realize that when you multiply by a number with the CI scale, *you are really dividing by the reciprocal of the number*. This, of course, gives the same result, since $X \times Y = X \div (1/Y)$. Also the process of dividing by a number with the CI scale is equivalent to multiplying by the reciprocal of a number, $X \div Y = X(1/Y)$.

Explanation

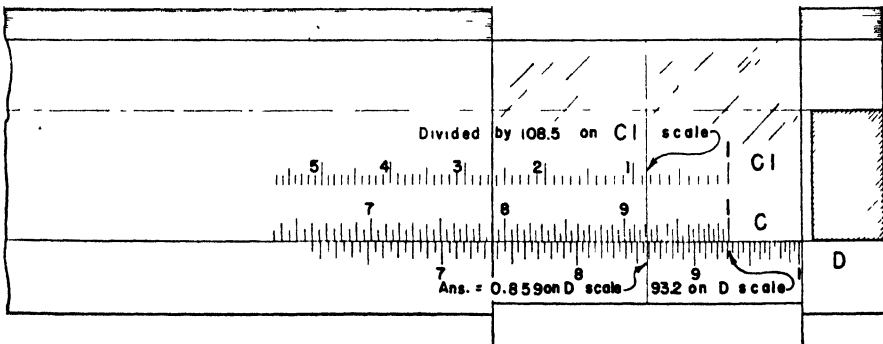


FIG. 292. SETTING THE SLIDE AND RUNNER FOR DIVISION WITH THE CI SCALE

*Other uses for
CI scale*

The CI scale is especially useful in problems involving more than two numbers; for example, problems such as,

$$19.5 \times 27.3 \times 0.408 = ? \quad \text{or} \quad \frac{5.94 \times 65.7}{0.224 \times 343} = ?$$

These answers can be obtained with fewer movements of the slide and runner, using the CI scale, than with only the C and D scales. To demonstrate this we will solve the first one. Multiply 19.5 by 27.3 in the usual way and then multiply by 0.408 with the CI scale to get 217. In the second example, start by dividing 5.94 by 0.224 using the C and D scales. Then divide by 343 with the CI scale and multiply by 65.7 with the CI scale to obtain 5.08. Use of the CI scale in combination with the C and D scales shortens each of these problems by one movement of the slide or runner. For practice, you can turn back and use the CI scale to work the problems given for multiplication and division.

THE DUPLEX-TYPE SLIDE RULE

*Advantages,
disadvantages*

A duplex slide rule is one with scales on both sides of the stock. One form, known as the "Polyphase Duplex Decitrig" (Keuffel & Esser Co.), is shown in Fig. 293. This type of slide rule has been very popular among engineers and is in wide use at the present time. The stock consists of two pieces of wood, held together at the ends by metal plates. The runner has two pieces of glass so that it can be used on each side of the slide rule. The trigonometric scales and the log scale are placed so that they can be used more conveniently than those on the Mannheim slide rule. Also the folded scales (CF, DF, and CIF) make it possible to multiply very quickly by π and are of advantage in certain other problems in multiplication and division. The duplex-type slide rule is not as convenient to carry with you

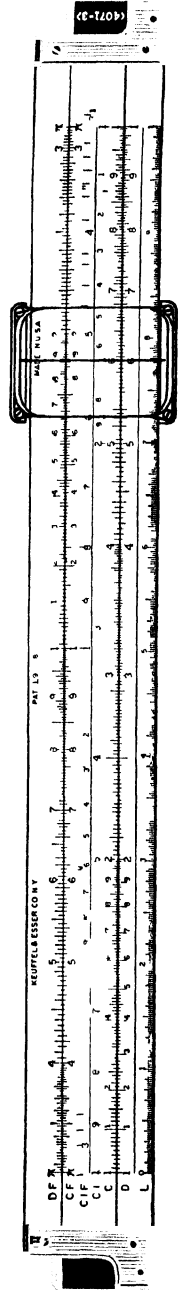


FIG. 293 a. FRONT
OF DUPLEX RULE

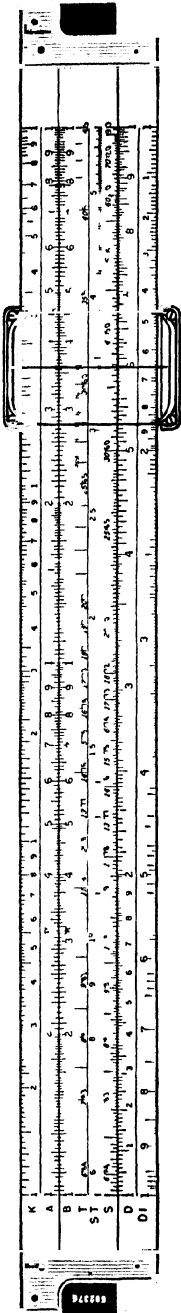


FIG. 293 *b*. BACK OF
DUPLIX RULE

as the Mannheim because it is about 2 in. longer than a Mannheim slide rule which has scales of the same length.

Most of the calculations you have learned for the Mannheim slide rule are done in the same way with the duplex. For instance, the C and D scales are located on the front of the duplex rule (Fig. 293) and are used to multiply and divide just as on the Mannheim slide rule. On the back of the duplex, you notice the A and D scales which you can use together to find the square or square root of a number. The K scale, on the back of the slide rule, is used with the D scale to find the cube or cube root of a number. The CI scale is located on the front of the duplex-type slide rule and can be used with the C and D scales just as on the Mannheim slide rule.

*Similarity to
Mannheim*

The sine scale on the back of the slide is to be used with the D scale to find the sine of an angle. Just line up the ends of the two scales, set the hairline of the runner to the angle on the sine scale, and read the sine of the angle on the D scale under the hairline. This is simpler than the use of the Mannheim rule. The tangent scale, on the back of the slide, is used with the D scale on the same side to find the tangent of an angle. Be sure to line up the ends of the tangent scale and the D scale before you try this operation. If you look closely at the sine and tangent scales, you can see that they are divided into degrees and decimal fractions of a degree rather than degrees and minutes. This represents a modern step in the development of the slide rule, one which is especially useful when you start with the value of the angle in *radians*. Notice also the numbers which lean to the left on the trigonometric scales in Fig. 293. These numbers give you immediately the difference between the angle and 90° , a figure useful in finding the cosine of an angle or the tangent of an angle greater than 45° .

*Sine and
tangent*

Small angles

The *ST scale* is an addition made especially for finding the sine or tangent of a small angle. Use it, after aligning the indices, by setting the hairline of the runner to the angle on the ST scale and reading the sine or tangent on the D scale under the hairline. (For such small angles, the sine and tangent are so nearly equal that you can take them to be the same.) Think of the D scale as extending from 0.01 at the left index to 0.1 at the right index when you use the ST scale.

Logarithms

When the hairline of the runner is set to a given number on the D scale, the mantissa of the common logarithm of the number can be read on the log scale under the hairline. These scales are on the front of the duplex slide rule.

Folded scales

The folded scales, CF and DF, match, respectively, the C and D scales. When the hairline of the runner is set to a given number on the C scale, the product of that number and π is on the CF scale under the hairline. This same relation holds for the D and DF scales. The CIF scale is a reciprocal scale for the CF and DF scales. Since each of the folded scales offers a full numerical range, you can use the two together (and also with CIF) for multiplication and division just as you would the C, D, and CI scales. The only difference is that the index is at the mid-point instead of the end of the slide.

Best of all, the CF and DF scales can be used in conjunction with the C and D scales to make certain calculations work out more easily. You remember in a previous problem when we wanted the result of

$$\frac{56.3 \times 0.785}{3.75 \times 13.4}$$

Using C, D, CF, and DF scales

that we divided 56.3 by 3.75 and then had to reverse the slide in order to multiply by 0.785. The reversal of the slide can be avoided by using the folded scales. You can start the calculation in the same way as before by dividing 56.3 by 3.75 with the C and D scales. Then multiply by 0.785 by setting the hairline of the runner to 0.785 on the CF scale. Next divide by 13.4 by sliding 13.4 on the CF scale under the hairline. Finally, read the result (0.88) on the DF scale above the index at the middle of the CF scale. This method can always be used, but to be perfectly safe, follow this rule. If you move up to the folded scales in the middle of a problem and end the calculation there, read the answer there on the DF scale. Do not try to read it on the D scale.

If you have a rather long problem to solve, such as

Mixing scales

$$\frac{28.5 \times 19.7 \times 0.842}{7.2 \times 1.52 \times 44.8},$$

you may find it desirable to move up to the folded scales for part of the calculation and then back to the C and D scales to complete it. If you do, *the safe procedure is to use the folded scales exactly as many times for division as you do for multiplication.* Otherwise, your answer may be wrong. Now suppose we solve the problem just mentioned. Start by setting the hairline of the runner to 28.5 on the D scale. Then divide by sliding 7.2 on the C scale under the hairline. Next multiply by setting the hairline of the runner to 19.7 on the CF scale. Then divide by bringing 1.52 on the CF scale under the hairline. Now it is safe to return to the lower scales and multiply by setting the hairline of the runner to 0.842 on the C scale. Next divide by sliding 44.8 on the C scale under the hairline. Finally, read the result as 0.963 on the D scale under the right-hand index of the C scale. Rather nice, for you did not have to reverse the slide at all. Now here are problems for you to practice on. Verify each of the following answers.

Example

1. $\frac{63.5 \times 1.06}{9.25 \times 2.27} = 3.2$
2. $\frac{0.707 \times 8.75}{0.313 \times 5.33} = 3.72$
3. $\frac{158 \times 1.973}{0.487 \times 7.85} = 81.4$
4. $\frac{25.8 \times 0.599}{1.17 \times 2.62} = 5.04$
5. $\frac{8.95 \times 6.85}{0.336 \times 1.08} = 168.8$

6. $\frac{43.7 \times 0.85 \times 5.5}{2.05 \times 3.62 \times 7.3} = 3.77$
7. $\frac{11.8 \times 3.50 \times 92}{6.4 \times 3.46 \times 0.67} = 257$
8. $\frac{5.75 \times 1.05 \times 8.93}{72.7 \times 0.042 \times 6.15} = 2.88$
9. $\frac{0.195 \times 2.38 \times 0.916}{47.4 \times 7.4 \times 0.45} = 0.0027$
10. $\frac{87.2 \times 0.519 \times 15.7}{29.7 \times 3.08 \times 0.611} = 12.73$

Problems

THE LOG-LOG SLIDE RULE

The log-log slide rule is a duplex slide rule in that it has scales on both sides of the stock. One form of it, known as the "Log-Log Duplex Trig" is shown in Fig. 294. It represents a development of the polyphase duplex slide rule and differs from it in that it has five log-log scales in addition to those on the polyphase duplex. Otherwise its construction is the same. Its main advantage is that certain *calculations involving exponentials* are made more easily with it than with simpler slide rules. One disadvantage is that there are so many scales on it that you lose time in picking out the proper

Why use the log-log rule

ones to use for the simple calculations such as multiplication, division, and so on. Also, because of its bulk, it is less convenient to carry.

Similarity to other slide rules

Multiplication and division can be performed on the log-log slide rule in the usual manner by using the C and D scales. Also, you can use the A scale for squaring a number and the K scale for cubing a number. Other calculations, such as those which use the trigonometric scales, folded scales, and log scale, are made as with the duplex slide rule.

Log-log scales

The three log-log scales on the front of the slide rule are designated as LL1, LL2, and LL3, respectively. They are so arranged that any power of e (e is a number appearing very often in the more advanced phases of engineering and is approximately equal to 2.718) can be obtained directly, and in addition, any power or any root of any number can be found. To raise e to any positive power, forget the slide and set the hairline of the runner to that power on the D scale. The result can then be read under the hairline on one of the scales LL1, LL2, or LL3. If the power of e is between 0.01 and 0.1, the answer will be found on the LL1 scale; if between 0.1 and 1, on the LL2 scale; and if between 1 and 10, on the LL3 scale.

Any number to any power

You can raise any number to any power by using the log-log scales. Just set one index (whichever one will enable you to complete the problem) of the C scale even with the number on the LL1, LL2, or LL3 scale by use of the hairline. Next move the hairline of the runner to the power on the C scale. Then read the result on the LL1, LL2, or LL3 scale under the hairline. *You can always tell which log-log scale to use by estimating what the result should be.* For example, let us find the value of 1.17 raised to the power 2.38. Begin by setting the left index of the C scale to 1.17 which occurs on the

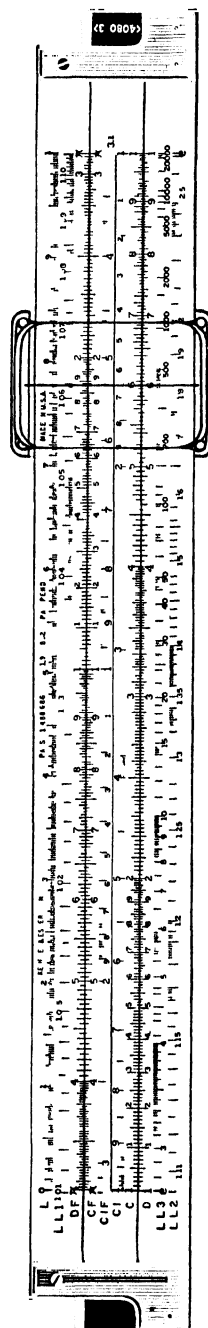


FIG 294 a. FRONT OF LOG-LOG RULE

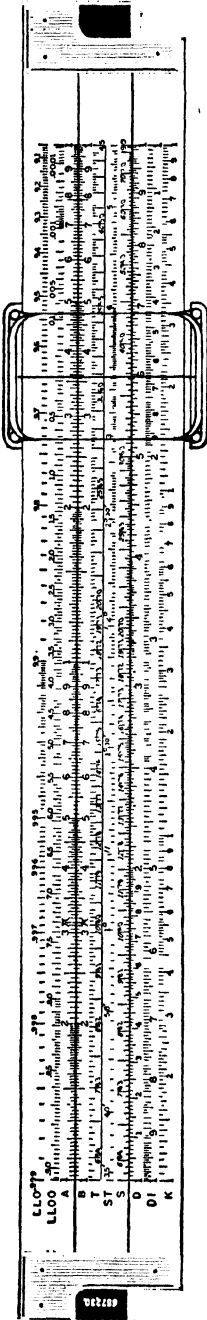


FIG. 294b. BACK OF
LOG-LOG RULE

LL2 scale. Then set the hairline of the runner to 2.38 on the C scale. Last, read the result as 1.45 on the LL2 scale under the hairline. You can tell that the answer must be read on the LL2 scale because you would read 1.038 on the LL1 scale and 42.0 on the LL3 scale. One of these is much too small for the answer and the other is much too large.

In general, when the power is greater than 1, *Rule of thumb* read to the right on the log-log scale with which you start, or read on a log-log scale of higher number. When the power is less than 1, read to the left on the log-log scale with which you start, or read on a log-log scale of lower number.

When the number you want to raise to a *Decimal fractions* power is less than 1, use the A scales with the LL0 and LL00 scales. Set one index of the A scale even with the number on the log-log scale. Then set the hairline of the runner to the power on the A scale. The answer is read on one of the log-log scales under the hairline.

By reversing the process of obtaining a power *Any root* we can obtain any root of any number. For example, we obtain the seventh root of 152 as follows. Set the hairline on 152 which is found on the LL3 scale. Then bring the root number 7 under the hairline on the C scale. The answer is found under the index of the C scale; it happens to appear on LL2 and is 2.05. Evidently *the root number need not be a whole number* but could have been 7.23 or any other number without complication. If you prefer, *you can always think of roots as fractional powers*; for example, the fifth root is the same as the one-fifth power.

$$1. 2^3 = 30.0$$

$$2. (2.14)^{2.14} = 5.11$$

$$3. (0.93)^{0.7} = 0.95$$

$$4. (3.87)^{0.43} = 1.79$$

$$5. \sqrt[5]{77.2} = 2.38$$

$$6. (1.07)^{1.07} = 1.075$$

$$7. (43)^{0.24} = 2.45$$

$$8. (1.33)^{2.1} = 1.82$$

$$9. (0.5)^{1.75} = 0.293$$

$$10. \sqrt[1.38]{4.27} = 3.11$$

Problems

THE CIRCULAR SLIDE RULE

Scale arrangement

A circular slide rule is just what the name suggests. It is circular in outline with each scale laid off on the circumference of a circle as in Fig. 295. While this type of slide rule has not been as popular as the straight ones, it has much to recommend it. Those who use it find it very convenient and feel that it is better than a straight slide rule. The three outer scales are on a separate piece of metal or plastic which can be rotated with respect to the inner part.

Observe endless scales

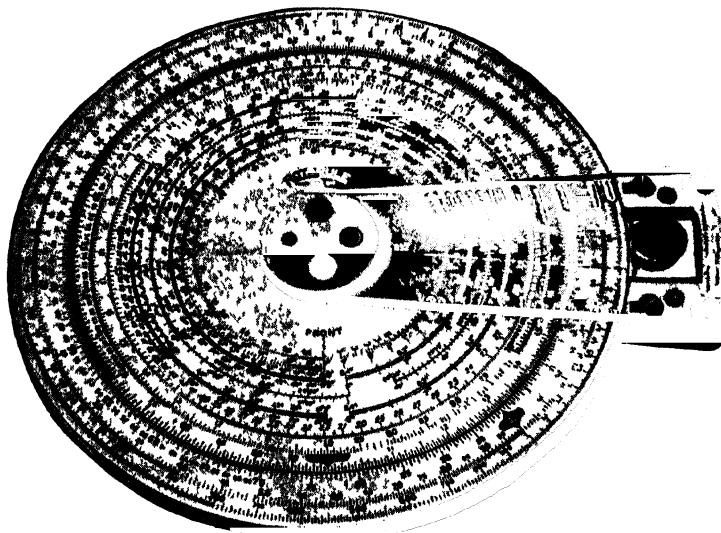


FIG. 295. THE CIRCULAR SLIDE RULE

This outer ring you can think of as corresponding to the stock in the straight slide rule. The inner part corresponds to the slide and the movable arm takes the place of the runner. The rule of Fig. 295 is the Dietzgen Rotarule.

Advantages and criticisms

One great advantage of the circular slide rule is its compactness. With a diameter of 4 in., the circumference of the outer scale is about 12.5 in. Thus you secure a scale long enough for precision while the over-all size of the instrument is small. A further advantage is that the scale begins again where it ends; hence, *the runner never runs off the scale in multiplying*. A disadvantage of the circular slide rule is that the scales are not all of the same length. Those on the inside are shorter than the scales of a conventional 10-in. slide rule and, consequently, they do not give such precise results.

In spite of the different shape, the circular slide rule is really quite similar to the straight ones. The fundamental scales are the same and the fundamental calculations are performed in exactly the same manner. Of course, you do not have to think about the left index and the right index, because each scale occupies a full circle and ends where it starts. Hence, there is only one index. *Similarity to straight slide rules*

In using the slide rule shown in Fig. 295, think of the third scale from the outside as the D scale and the fourth as the C scale. To multiply one number by another, turn the center part of the slide rule to set the index of the C scale to the first number on the D scale. Then set the hairline of the runner to the second number on the C scale and read the answer on the D scale under the hairline. You will see that the procedure is practically the same as that used for a straight slide rule. To divide one number by another, just set the hairline of the runner to the first number on the D scale. Then turn the center part of the rule to bring the second number on the C scale under the hairline and read the result on the D scale under the index of the C scale. *Multiplying and dividing*

The second scale from the outside is the same as an A scale and the outside one is the same as a K scale. Move from the D to the A scale to square a number and from the D to the K scale to cube a number. Reverse the order in taking roots. This is enough description to show the similarity of the circular slide rule to the straight ones. If you know how to use a straight slide rule, you can use a circular slide rule because it is actually simpler. *Special scales*

CONCLUSION

By the time you read this you should be able to use the slide rule. Already you know that it is much faster than working arithmetic problems longhand. As you continue to use it, you will develop more speed and proficiency so that it will become a valuable tool in your work. Use the slide rule for all of your problems and get the feel of it while doing the work you have to do anyway. In a surprisingly short time you will find that you can save at least 90 per cent of the time required to do the work longhand. You will appreciate this in your engineering studies because engineering calculations are often long and tedious. You simply will not have time to multiply and divide longhand. When you are working a problem in mechanics, for instance, you need all of *Practice*
Save time

your time to think about mechanics and will have none to spare for longhand calculations.

*Fatigue and
eyestrain*

There are two other factors closely related to that of saving time. These are fatigue and eyestrain. If using the slide rule makes it possible for you to do your work in fewer hours, then you will undergo less fatigue and less eyestrain. While we are on the subject of eyestrain, we should consider another source of it. Many students use slide rules only 5 in. long. They have to squint to read a short slide rule. Some even use a magnifying glass on such a slide rule with the object of reading more closely which increases the eyestrain. You can get results just as precise with a 10-in. slide rule and your eyes will benefit.

Errors

As a beginner, you may be tempted to check your slide-rule calculations by working the problems longhand. Do so if you wish, but very soon you will realize that you not only work faster with the slide rule but make fewer errors too. After all, the slide rule itself cannot make a mistake; the usual errors come from setting a number wrong or reading one incorrectly. The longer you use your rule, the less chance there will be of your making such errors.

*Tricks and
short cuts*

There are many tricks and special methods that you can learn with the slide rule: methods for solving quadratic equations, making vector calculations, and so on. I have learned many from time to time and forgotten most of them. *Short cuts are worth learning only if you have frequent use for them.* They are not worth learning if you are going to use them only once every year or two because you will forget them in between uses. Concentrate instead on the fundamental operations, which you can use every day and with which you can solve any problem. Be sure to use your slide rule at every opportunity for solving problems. It will soon become a habit.

Light and Electricity

HALLMARKS OF CIVILIZATION

Light and electricity are the products of the electrical engineer. Through their use he has transformed the hard and solitary struggle for existence of early man into a protected life of comfort and convenience, plus many opportunities for communication and social contacts with his fellow men.

Today we live in an age of electricity. The machines in our factories are driven by electric motors and nearly all of the processes in production depend upon electric control or operation. Our streetcars, buses, railway trains, and most of the ships on the ocean are propelled by electricity taken from power lines or produced by power plants carried on the transporting medium. Our communication contacts via telegraph, telephone, or radio are activated by electricity. Our homes, offices, and stores secure their light and power from the electric power plant. *Age of electricity*

The numerous applications of electricity have been developed in the past hundred years. These applications have been stimulated or developed directly from five major inventions which occurred within a period of seventy years. About 1838, Morse developed the first successful American telegraph apparatus. Shortly thereafter, telegraph lines were extended between the leading American communities, and railroad transportation and commerce were stimulated by this first form of long-distance communication. *Morse's telegraph*

In 1876, Alexander Graham Bell completed the first satisfactory telephone instrument. A few years later the system which bears the name of Bell was founded. Since then the extension of communication service for the human voice has grown until today over twenty million telephones are in use in America. In 1881, Thomas Edison built the first successful incandescent electric lamp. The establishment of electric power plants to operate the new incandescent lamps followed rapidly after 1883. The first major use of electric power was for lighting the street, the store, the shop, and the home. *Bell and Edison*

Use of electric power

After about ten years electric power was applied to electric motors and other devices. Today the requirements of electric energy for power far exceed those for lighting. At the present time the United States has about 47 million kilowatts of installed capacity in its power plants. These power plants are generating about 188 billion kilowatt-hours per year. Kilowatts and kilowatt-hours are defined on page 415. Edison's invention had a great stimulative effect in bringing about the electrical age.

Marconi and DeForest

About 1898, Marconi demonstrated that electromagnetic waves (wireless waves) could be used successfully to send messages over the air. In 1907, DeForest added a third electrode (the grid) to Fleming's "electric valve" and called his invention the *audion*. These contributions have given rise to worldwide radio communication.

Electronics

The three-electrode tube has proved to be the most important scientific discovery* of the twentieth century. Strange to say, it took six or more years before science perceived the possibilities of this device or could explain its operation. DeForest's audion made possible the amplification of weak signals which permits communication around the world. Recent modifications of the three-electrode tube have revolutionized control and operation of electric power devices. Today electron tubes and their activating circuits constitute a great branch of electrical engineering known as *electronics*.

ELECTRIC LIGHTING

The progress of man has been marked by the methods developed for producing artificial light. These methods have followed the stages of firebrands, candles, oil lamps, gas lamps, arc lamps, incandescent lamps, and the modern gaseous lamps. The important landmark in this progress was Edison's invention of the electric incandescent lamp. Since this landmark was reached, the science of artificial lighting has been developed and applied at an accelerated pace. The excellently lighted rooms of Fig. 296 are in great contrast to the poorly lighted homes of a half-century ago.

Incandescent lamp

Edison's early lamp used a carbonized bamboo fiber for a filament. The efficiency of this lamp was low and its life was short but the lamp did work and thus paved the way for the electrical age which was to follow. This early lamp was improved by a process which consisted of operating the filament in an atmosphere

of gasoline vapor whereby the carbon in the vapor was deposited on the filament in the form of a smooth hard coating which added to the efficiency and life of the lamp

About 1907, tantalum was tried for incandescent lamp filaments and then a year later the sintered tungsten filament. The latter *Metal filaments* consisted of particles of tungsten in a binding material. After the lamp was lighted, the binding material became brittle and the

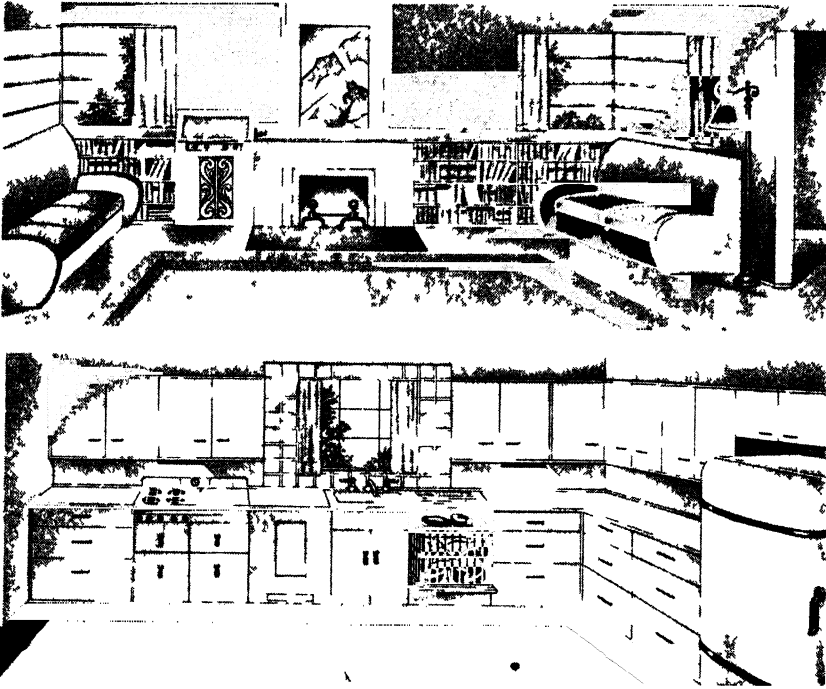


FIG 296 EXCELLENT LIGHTING FOR DAY AND EVENING IS A NECESSARY FEATURE OF MODERN HOMES

sintered tungsten filament was very fragile and broke under the slightest jar. Shortly thereafter, engineering research discovered a heat treatment which made tungsten ductile and permitted it to be drawn into wire. Since that discovery, drawn tungsten wire has been used for incandescent lamps for over a score of years.

Shortly after the discovery of drawn tungsten wire, a research scientist found that the efficiency of the tungsten lamp could be improved by using an inert gas instead of a vacuum in a bulb *Gas-filled lamps*. The addition of the gas (under pressure) reduced the evaporation of the metal tungsten thus permitting a higher operating tempera-

ture. The higher temperature increased the amount of electric energy transformed into light and improved the *efficiency*. The presence of the inert gas also aided the removal of the heat from the filament.

Improvements

One later improvement in incandescent tungsten lamps resulted from the discovery of a method of etching the glass bulb on its inner surface. This etching reduces the apparent brilliancy or brightness of the source, thus giving a softer light and protection for the human eye. In recent years new mechanical methods of coiling a long fine tungsten wire into a concentrated filament have permitted the use of inert gas and increased the operating temperature and efficiency of low-wattage incandescent lamps. The development of complicated machines for producing lamps on a mass-production basis with little human labor has lowered the cost of light bulbs to the point where the ratio of value to cost is exceedingly high.

Gaseous light sources

Later we will explain how *light is produced in a gas as a by-product of ionization*. Light production in gaseous tubes has been used throughout this century. The first general use was in neon lights for advertising signs. These lights consist of long glass tubes filled with neon gas under low pressure or partial vacuum. A high-voltage alternating current applied to electrodes in the ends of the tubes causes ionization of the gas which develops a glow discharge and gives the characteristic reddish-orange color. The color of this source of light limits its application to signal lights and sign lighting.

Mercury and sodium lamps

Mercury-vapor lamps have been used for some industrial lighting applications for many years. They consist of an evacuated glass tube into which a small quantity of mercury is placed. Part of the mercury evaporates forming mercury vapor; this vapor is ionized and gives out light of the characteristic blue color. This light shows up details particularly well but its color makes it objectionable for many applications.

Metallic sodium has been used instead of mercury in some special light sources designed for highway lighting. The sodium lamp is very efficient but is open to the criticism that it requires several minutes to heat the bulb and vaporize the sodium. The yellow color of the sodium light makes the unit objectionable from the psychological point of view but for the penetration of dust and fog it is considered to have some desirable properties.

The fluorescent lamp is the most important development in electric lighting since Edison's invention of the incandescent lamp. This source is a mercury-vapor lamp which utilizes the fluorescent properties of certain materials placed on the inside of the tube. The construction and circuit of a simple fluorescent lamp is shown in Fig. 297. The tube is first evacuated and then a small quantity of mercury is injected. The mercury evaporates, forming mercury vapor under low pressure. The ends of the glass tube contain

Fluorescent lamp

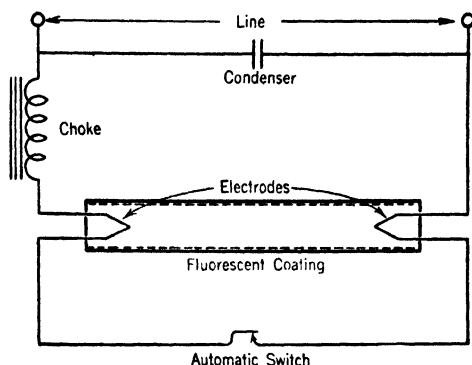


FIG. 297. CIRCUIT FOR A FLUORESCENT LAMP

electrodes which emit electrons when heated. When voltage is applied between the electrodes, *ionization of the mercury vapor* takes place, current flows through the tube, and some light is produced.

A part of this light is the typical bluish mercury-vapor light which is visible to the human eye. Another part of the light produced is ultraviolet light which is invisible and which would be injurious to the human eye. However, this ultraviolet light will not pass through the lead glass used in the tube; hence it cannot injure the eye. The ultraviolet light falls upon a fluorescent coating on the inside of the tube and its energy is transformed into another wave length or color which is visible to the human eye.

Ultraviolet

The color of the transformed light depends upon the particular fluorescent material or combination of materials used. The common colors of light available are red, pink, gold, green, blue, white, and daylight. The efficiency of the fluorescent lamp depends on the color of the light, but in general the efficiency is from two to three times as high as for the corresponding incandescent lamp. In addition to a higher efficiency, the fluorescent source has a lower brightness and gives a better diffusion of light. The disad-

Efficiency

*Circuit of
fluorescent
lamp*

vantages of the fluorescent light are the higher cost for bulbs and the auxiliary equipment required.

The circuit and operation of the fluorescent lamp are interesting. On starting, as shown in Fig. 297, the current flows through a series circuit consisting of a choke, one electrode, automatic switch, second electrode, and line. After a few seconds during which the *electrodes become hot and emit electrons*, the automatic switch opens the circuit between the two electrodes. The opening of this switch places the full line voltage across the tube from electrode to electrode. This voltage is high enough to produce *ionization and resultant current conduction* through the mercury vapor. After conduction starts, *the bombardment of the electrodes by the ions keeps them hot*. The function of the choke is to protect the tube by limiting the flow of current after ionization takes place. As ionization increases, the resistance of the gas column decreases, and the current will rise to damaging values without the limitation produced by the choke. The choke tends to give the fluorescent lamp a low power factor. The low power factor can be improved by the use of the condenser C shunted across the line. Such technical expressions as *power factor* and *condenser* will be defined later. In fact, you can now turn to pages 432 and 433 for these definitions if you wish.

THE PHYSIOLOGY OF LIGHTING

The science of artificial lighting covers the control of light in intensity, direction, and color, and involves parts of the sciences of physiology, psychology, and physics.

The camera

The human eye is analogous to a camera. In the camera (Fig. 298) light reflected from an image passes through a lens

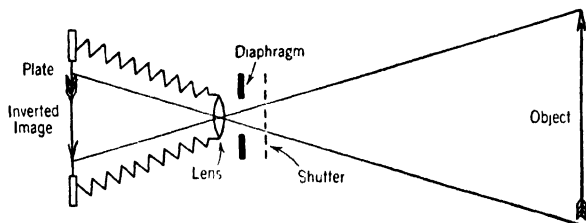


FIG. 298. ACTION OF LIGHT IN A CAMERA

and is focused on a plate or film. The lens must be in the proper position or proper shape to make a sharp focus on the plate. The amount of light admitted to the camera is controlled by the open-

ing of a diaphragm before the lens. The opening to the camera is exposed or closed by a carefully timed shutter.

In a similar manner your eye has a lens called the *crystalline lens* (Fig. 299). The shape of this lens is controlled by muscles which serve to bring the image into focus on the retina. The retina is filled with nerve ends which are sensitive to light in a way analogous to the sensitivity of the chemical surface on the plate or film

The eye

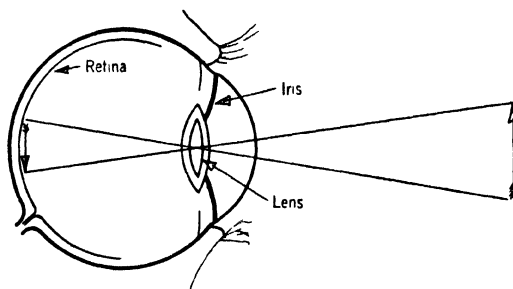


FIG. 299 ACTION OF LIGHT IN THE HUMAN EYE

of the camera. The amount of light admitted to the retina is governed by the iris, which is a curtain in front of the crystalline lens. The iris is controlled by muscles which vary the diameter of the pupil of the eye. Lastly, light is admitted to or shut off from the eye by the eyelids.

In order to obtain a satisfactory picture a sufficient amount of light must be permitted to fall on the film in the camera. Too little light makes a dim picture. In like manner, a certain level of light intensity is necessary on your retina. Too little light gives poor vision and also strains and fatigues the nerve centers in the retina (eyestrain). A camera pointed toward the sun will result in an overexposed film. An eye not shaded from the sun or from a bright lamp in the field of vision will be blinded by the brilliant light. Here the iris of the eye cannot act quickly enough to limit the incoming light. And even if it could act quickly enough, there would be too much *contrast* between the source and the surroundings for any satisfactory image on the retina. Hence the retina is injured and eyestrain results.

Light intensity

If the light is flickering, the muscles controlling the iris will attempt to change the size of the pupil to follow these changes and naturally these muscles will become tired — another case of eyestrain. The surface of the ground and the floors of our homes

Sources of eyestrain

have always been relatively dark. Hence, reflected light of a rather low intensity comes into the eye from angles below the horizontal. The part of the retina receiving this low intensity has become adapted to it and gives good vision. But if you walk about on a ground covered with white snow in bright sunshine, you will suffer great discomfort because an unaccustomed part of the retina of your eyes is receiving a relatively brilliant light.

Contrast

The human eye is uncomfortable when too large a contrast exists in the field of vision. If a light-colored scene occurs under a dark background, the iris of the eye attempts to admit the proper amount of light for the retina to envision both the lighter and the darker part of the view. The result will be that part of the retina receives too much light and another part too little. Too much contrast may be produced by the light reflection from the objects viewed or it may result from a faulty distribution of light.

THE PSYCHOLOGY OF LIGHTING

*Psychology of
seeing*

We humans are very sensitive to the color of light. In general, we prefer a red apple to a yellow one though both may be equally palatable and luscious. The green color of foliage and landscapes seems to have a soothing effect upon human nerves and souls. Blue seems to give a feeling of coldness and sometimes sadness. Red arouses a feeling of warmth and passion. Yellow-orange, the color of a flame, gives rise to a feeling of coziness and comfort.

Daylight color may be defined as the light which is reflected from the sky and enters a north window. This light has more blue than either direct sunlight or the incandescent lamp bulb. This color is very desirable where color matching is important. Hence daylight lamps are used in stores and many factories.

*Mercury-vapor
lamp*

The mercury-vapor lamp gives a light strong in yellow, green, and blue. It is deficient in red and orange, so these colors appear black. People look ghastly and cold under mercury-vapor light, and naturally we would object strenuously to lighting our homes with this light. The advantages of the mercury-vapor lamp are that its color spectrum makes it well adapted for taking photographs and it shows up details on drawings and machines better than daylight or the normal incandescent lamp.

For ornamental lighting at night, for floodlighting of buildings, trees, fountains, or other displays, color may be used to produce artistic and beautiful effects.

Much of the preceding discussion of the principles of lighting may be summarized into seven important suggestions which might be termed the *golden rules of lighting*.

1. Use sufficient light — an illumination suited to the work to be accomplished.
2. Keep bright-light sources out of the field of normal vision.
3. The light source should be large in area so that brightness, which will be defined later in candles per square inch, is less than 0.5.
4. Avoid direct reflections from polished surfaces on table tops, glossy paper, or working surfaces.
5. Avoid too much contrast — light the walls and ceilings as well as the plane of work.
6. Avoid light from any low angle other than normal reflections from floors.
7. Use a color of light suited to its function or to the work being performed.

Golden rules of lighting

UNITS OF LIGHT

The measurement of light makes use of units developed from our early source of the candle. The particular candle is one burning

Candlepower

sperm oil at a specified rate per hour. If you pass a horizontal plane through a lighted candle as shown in Fig. 300, the intensity of light in that plane is assumed to be uniform in all directions. The intensity of light in this horizontal plane is said to be *one candle* (1 c.) or *one candlepower* (1 cp.). The intensity of light refers to direction only, and hence the intensity is 1 c. at all points whether 1 ft., 1 yd., 1 rod, or 1 mile away from the source. The candlepower of a 100-watt incandescent lamp is about 120 in the horizontal plane.

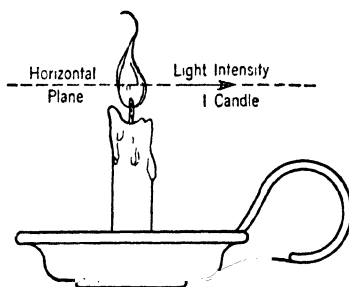


FIG. 300. ILLUSTRATING THE UNIT OF LIGHT INTENSITY

Illumination is the measure of the amount of light falling on a surface and the unit of illumination is called the *foot-candle*. The derivation of the foot-candle unit is illustrated in Fig. 301. Thus if you place a board 1 ft. from a burning candle and at right angles to the direction of the rays, the illumination on the board is said to be *one foot-candle* (1 ft-c.). The direct illumination from the sun at midday under a clear sky may be of the order of 8000 ft-c., whereas the illumination under a full moon is about $\frac{1}{10}$ ft-c. On your study table you should have an illumination of 25 to 50 ft-c.

Illumination

Point source of light

The intensity or candlepower of the light from a candle is not the same in all directions. Thus it will be less at a point directly above the candle than in a horizontal direction from the flame, and at a point underneath the candle it may be zero. In order to

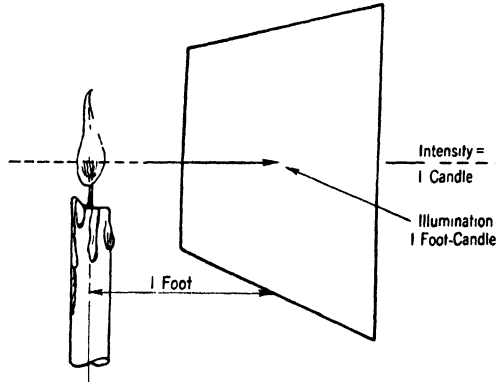


FIG. 301. ILLUSTRATING THE FOOT-CANDLE

define a unit for the quantity of light given forth in all directions or in some given sector, the concept of a point source of light is necessary.

A point source of 1 c. is one which exists at a point and gives forth an intensity of 1 cp. in all directions from the point. If you place a point source of 1 c. at the center of a sphere (Fig. 302) having a radius of 1 ft., the illumination at every point on the inside surface of the sphere will be 1 ft-c.

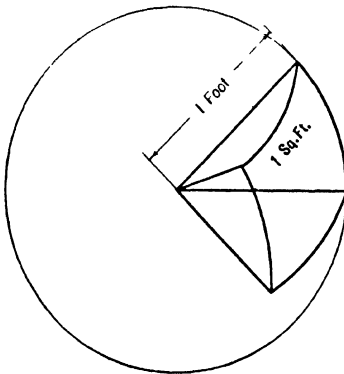


FIG. 302. THE UNIT SOLID ANGLE

Next form on the surface of the sphere a region whose area is 1 sq. ft. This may be done, for instance, by drawing a suitable spherical polygon bounded by arcs of great circles (Fig. 302). Planes drawn from the point source through the sides of this polygon

The lumen

enclose a solid angle. Since the area enclosed is 1 sq. ft., this is called a *unit solid angle*. The amount of light emerging from the point source of 1 c. into the unit solid angle is defined as *one lumen* of light. Three units of illumination are in this concept. A source of 1 c. or 1 cp. produces a uniform illumination of 1 ft-c. at all

points on the inner surface of the sphere and this same source gives forth 1 lumen into the unit solid angle. Since there are 4π solid angles (spherical surface = $4\pi r^2$) around a point, 4π lumens are given forth from the point source of light.

If you construct a sphere having a radius of 2 ft. around the point source of the preceding paragraph, its surface area will be four times that of the one having a radius of 1 ft. (Fig. 303). And

Law of inverse squares

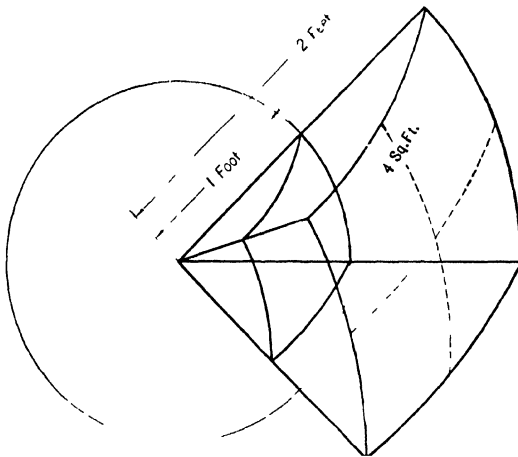


FIG. 303. ILLUSTRATING THE LAW OF INVERSE SQUARES

it will follow that if the boundary lines of the unit solid angle are extended to the larger sphere, the area encompassed will be 4 sq. ft. The lumen of light from the point source will now be spread uniformly over 4 sq. ft. instead of 1 sq. ft., giving an illumination of $\frac{1}{4}$ ft.-c. Since the surface area of a sphere varies as the square of the radius, it follows that the intensity of illumination from a point source varies *inversely as the square of the distance from the source*. The preceding statement, called the *law of the inverse squares*, forms the basis of light measurements and calculations.

PHYSICS OF LIGHT

The distribution of light is controlled by reflection, refraction, transmission, and absorption. Light falling on a mirrored or highly polished surface is reflected so that the angle of reflection is equal to the angle of incidence as shown in part A of Fig. 304. The mirrored surface may be curved so as to direct the incident rays in

Control of light

some desired pattern. An excellent example of this is the parabolic reflector illustrated in Fig. 305. The cross section of this reflector is a parabolic curve which has the property of reflecting all light rays in parallel lines if a point source of light is placed at the focus

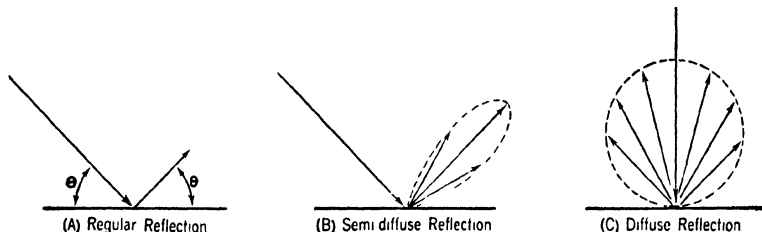


FIG. 304. TYPES OF LIGHT REFLECTIONS

of the parabola. Parabolic reflectors have been used in automobile headlights and in searchlights for years.

*Diffuse
reflection*

If a surface is not mirrored but slightly roughened, the light will be reflected at different angles since all units of the reflecting surface do not lie in the same plane. A perfect matte (roughened) surface will reflect light in all directions. Such reflection is known as "diffuse reflection" and is illustrated in parts B and C of Fig. 304. Porcelain enamel and aluminum paint are examples of diffuse reflectors.

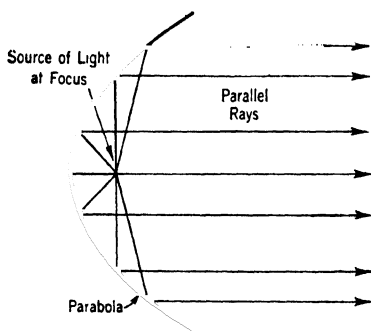


FIG. 305. LIGHT REFLECTION BY PARABOLA

*Refraction
control*

Light rays, passing from one medium to another at an angle other than a right angle are bent or *refracted*. Figure 306 illustrates the action of a prism in bending rays of light. Prisms are molded into glass disks and reflectors for directing and diffusing light. Street-lighting units often use a pris-

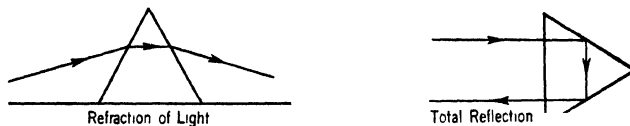


FIG. 306. CONTROL OF LIGHT BY REFRACTION

matic reflector which directs nearly all of the light into the street. That part of the light coming from the source toward the building or trees is directed back (total reflection) into the street where it is

useful. Pressed prismatic glass lenses are also used on automobile headlights to aid in the distribution of the light upon the highway.

Clear smooth glass transmits light with little loss or change of direction. If one side of the glass is ground or etched, this roughened side will spread or diffuse the light. If fine white particles are mixed in the molten glass, it will be milk-white in color and multiple reflections will take place on the inside of the glass. This will give a diffusion of the light transmitted. If glass is colored, some colors of light will be absorbed and the color of the transmitted light may be controlled.

Transmission control

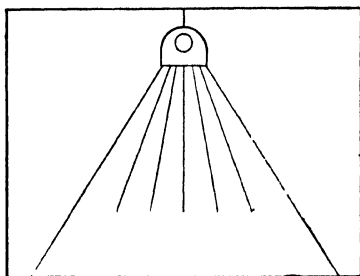


FIG. 307. DIRECT LIGHTING

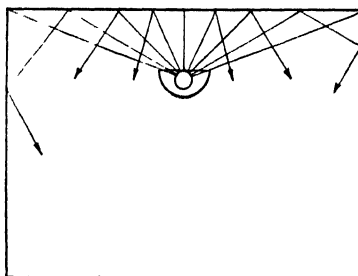


FIG. 308. INDIRECT LIGHTING

Electric lighting is controlled by three different systems. These systems are termed *direct*, *indirect*, and *semi-indirect*. In direct lighting an opaque reflector is used to direct all light upon the plane where it is needed, as illustrated in Fig. 307. This is the most efficient system of lighting but it is open to the objection that the walls and ceiling will appear dark and may offer too much contrast. Generally, the light is not well diffused and shadows are heavy unless fluorescent lighting units are used.

Direct lighting

The indirect system of lighting makes use of an opaque reflector to direct all of the light to the ceiling and upper walls from whence it is reflected to the working plane (Fig. 308). This system gives a soft light with excellent diffusion and distribution and with zero or near zero shadow effect. With sufficient foot-candles of illumination this system meets all of the rules of good lighting suggested on page 399. Yet to many people indirect lighting alone is unsatisfactory for two psychological reasons. The first reason is that they cannot see the source of light, and hence it makes them feel uncomfortable. The second reason is that they say the light is cold because of that inherent human desire for the soft flame-colored rays of light. If some flame-colored ornamental sources of

Indirect lighting

light are added to the indirect system, most persons will be well pleased with the lighting effect.

Semi-indirect lighting

The semi-indirect system of lighting is a combination of the direct and indirect systems. It is effected by using an inverted translucent reflector or bowl as illustrated in Fig. 309. This system gives good diffusion with little shadow, yet it has the desirable direct component of light added to the reflection from the ceiling. An important modification of the semi-indirect system is the I.E.S. unit which is really a portable semi-indirect light source (see

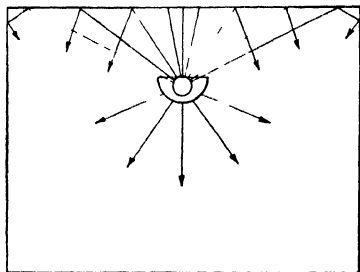


FIG. 309. SEMI-INDIRECT LIGHTING

Fig. 310). The portable feature makes it possible to apply the light locally and get the necessary intensity at a place for studying, reading, writing, or sewing.

I.E.S. lamps

The letters *I.E.S.* signify that the lamp has been built to meet certain specifications drafted several years ago by a committee of the Illuminating Engineering Society. These *I.E.S.* units are made in table-lamp and floor-lamp models. Since the translucent bowl is brought down to the level of the eye (standing) it is necessary to surround it with a shade to protect the eye from the brightness of the bowl. This shade gives the unit an ornamental touch with the possibility of using color. The use of two or more *I.E.S.* units

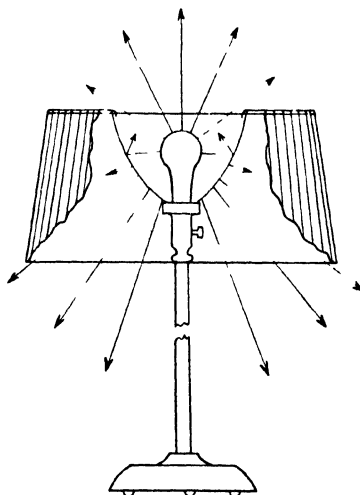


FIG. 310. CONTROL OF LIGHT BY I.E.S. PORTABLE UNIT

Multiple units

in a room permits a localized or nonuniform distribution of light which adds character to a room, whereas the more uniform distribution from an indirect unit tends to give a flat, colorless effect.

APPLICATIONS OF LIGHT

Proper lighting for the study table is important to the student. The method or unit to be used for lighting your study table may

be analyzed by using the golden rules of lighting. First, there should be sufficient light, and that means 25 or more foot-candles on the top of the study table. Some points of the table may have 75 and 50 ft-c. but any parts of the table where you read or write should have 25 ft-c. or more. Second, the source of light must be kept out of the field of your vision. This means that the source and its reflector should be higher than your head when seated and to the left to avoid shadow of your right hand and pencil. A good position for the source is over your left shoulder. Third, a light source of large area is desirable since it gives a better diffusion of light on the table (light coming from many angles). This good diffusion minimizes direct reflection from glossy paper or table top. Fourth, the wall that you face when writing and the ceiling above should have a fair degree of illumination to *avoid strong contrasts* as your eye shifts from table to wall and back again.

*Study table
lighting*

Any color of light such as the regular incandescent or daylight lamp or the white or daylight fluorescent lamp should be satisfactory. However, the color and surface of the table top is important. One or two large blotters should be used because they have a matte finish and will not give a direct harmful reflection. The color of the blotter should be gray, tan, or yellow, but not red, and preferably not too dark or too light (avoid contrast).

Effect of color

The I.E.S. desk lamp will meet the foregoing requirements if it is placed in the proper position on the table — usually the left center of the table. A fluorescent lamp, either table type or one suspended from the ceiling over the left center of the table, will give excellent distribution and little or no glare. Two examples of good lighting of the study table are shown in Fig. 311.

The proper consideration of the applications of electric lighting in the home would require an entire book, and space will permit only a few helpful hints on this subject. The kitchen is a workshop. The tops of the stove, table, and shelves are workbenches and should be well lighted. Some form of shaded or concealed units should light these workbenches. In addition, there should be a central lighting unit at the ceiling to furnish general illumination throughout the room.

*Lighting the
home*

The important consideration in the dining room is the top of the dining table and the food that is placed thereon. This table should be lighted by a fixture suspended from the ceiling. The unit should be one which gives a soft general illumination to the

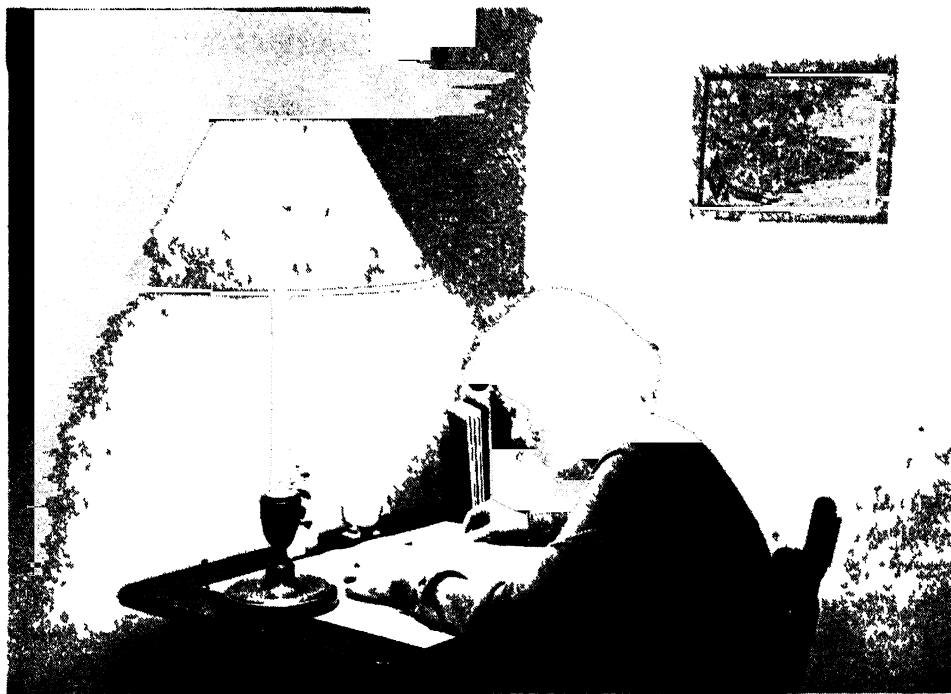


FIG 311 EXAMPLES OF GOOD LIGHTING FOR THE STUDY TABLE

walls, ceiling, and faces of the diners, but a good white illumination on the top of the table.

In the living room a wide range of units and types may be employed. First, there should be some general lighting unit or units for the room which are controlled from a wall switch at two or more points about the room. These general units may be cove lights, ceiling lights, or portables. Localized portable lighting units of the table or floor type should be used at tables, writing desk, radio, reading chairs, davenport, piano, and so on. These portables may be used to spotlight pictures on the wall and in other ways to give color and character to the lighted room. *Lighting combinations*

In the bathroom fluorescent or tubular lights are highly desirable. They may be placed one above and one below the mirror or one on each side of the mirror used for shaving. For the bedroom, lights should be provided on both sides of the mirrors on dressing tables and bureaus. A ceiling unit for general illumination is recommended. Lastly, a unit for giving a soft dim light on the floor is highly desirable to permit movement about the room for anyone awakened in the night or for attending someone who is ill.

The plans for lighting public buildings should be made by illumination engineers. Good designs require lighting experience and judgment plus the employment of the principles of lighting previously considered. Calculations and selections of units require the use of data compiled from many laboratory experiments. The actual calculations for the lighting of large rooms in stores, offices, and factories are not difficult. As an example, let us take the problem of lighting a drafting room 36 ft. by 60 ft. in which we desire to have an illumination of approximately 50 ft-c. on the tables. Such a room can be lighted satisfactorily by an indirect system using *incandescent* lamps or by a direct system using *fluorescent* lamps. *Lighting a drafting room*

For an indirect system the room should be divided into sections small enough to be lighted by one indirect source. Experience shows that such a section should be approximately square, having a range of 10 to 16 ft. on each side. The floor area of 36 by 60 will divide into square sections 12 ft. by 12 ft. Each section has a floor area of 144 sq. ft. It has been shown that when 1 lumen of light is spread uniformly over 1 sq. ft. of area, the illumination is 1 ft-c. Hence to get an illumination of 50 ft-c., 50 lumens would *Design for indirect lighting*

be required for each square foot or $144 \times 50 = 7200$ lumens for each section of the drafting room.

Losses

In the indirect-lighting system a part of the light from the source is lost in the numerous reflections which take place. For a room of the size being considered and for light walls and ceiling, tests have shown that only one-third of the light coming from the source will be effective on the working plane. Therefore, to give 50 ft-c. of *effective illumination*, $\frac{7200}{0.333}$ lumens or 21,600 ft-c. should be produced in each lighting unit.

Bulb size

If you refer to the table below for incandescent lamp bulbs, you will find that a 1000-watt lamp gives 21,000 lumens. This is slightly under the required amount, but the next size, 1500 watts,

OUTPUT OF COMMERCIAL BULBS AND TUBES

| INCANDESCENT 1 AMPS | | FLUORESCENT 1 AMPS | | | | |
|---------------------|----------------------|--------------------|---------------|-----------------|-----------------|--------------|
| <i>Watts</i> | <i>Lumens output</i> | <i>Watts</i> | <i>Length</i> | <i>Diameter</i> | <i>Lumens</i> | |
| | | | | | <i>Daylight</i> | <i>White</i> |
| 40 | 410 | 14 | 15 | 1½" | 390 | 475 |
| 100 | 1600 | 15 | 18 | 1" | 525 | 585 |
| 150 | 2600 | 15 | 18 | 1½" | 450 | 525 |
| 300 | 5850 | 20 | 24 | 1½" | 760 | 860 |
| 500 | 9850 | 30 | 36 | 1" | 1300 | 1450 |
| 750 | 15000 | 40 | 48 | 1½" | 1800 | 2100 |
| 1000 | 21000 | 100 | 60 | 2⅜" | 3700 | 4200 |

would give nearly 50 per cent excess illumination. Hence the 1000-watt lamp will be chosen to light each 12-ft. by 12-ft. floor section. There are $3 \times 5 = 15$ floor sections which also is the total number of indirect units required. The indirect units should preferably be hung at a height of 10 ft. above floor level, but they must be far enough below the ceiling (2 to 4 ft.) to give an even distribution of light over the ceiling. The kilowatt load of this installation will be $(15 \times 1000)/1000$, or 15.0 kilowatts (kw.).

**Design for
fluorescent
lighting**

In considering an installation of fluorescent lights for a drafting room, a number of sizes of bulbs and colors of light are available. In general, the daylight-colored light is preferable for drafting and *the 40-watt fluorescent tube is the most efficient*. This bulb gives 1800 lumens of light and when used in a direct reflector unit 60 per cent of the light will be useful. Thus, $0.6 \times 1800 = 1080$ lumens will be effective from each bulb. Since 50 lumens are required per square foot to produce 50 ft-c., one 40-watt bulb will give that

illumination over $\frac{1080}{4}$, or 21.6 sq. ft. The bulb is 4 ft. long so that one bulb would light an area 4 ft. by 21.6/4, or 5.4 ft. wide.

The calculated dimension indicates the spacing of the fluorescent strip reflectors. Two tubes side by side are desired; hence for a width of 2 ft. \times 5.4 ft. or 10.8 ft., there must be six strips in the 60-ft length of the room. Each strip will require $(\frac{36}{4}) \times 2 = 18$ bulbs. Total bulbs required will be $6 \times 18 = 108$, and, at 40 watts each, we have a load of $(40 \times 108)/1000 = 4.32$ kw.

Fluorescent layout

A comparison of the two systems just analyzed shows a 4.32-kw load for fluorescent lighting and a 15-kw. load for the indirect system using incandescent lamps. A more careful analysis will show that there are some losses in the auxiliaries required with fluorescent bulbs, which lowers their apparent advantage in power consumption, but the power saving will be at least 50 per cent. Also the fluorescent unit will have a higher initial cost than the indirect unit.

Comparison of systems

The calculations just completed give the illumination (foot-candles) when the installation is new. The accumulation of dust on the bulbs, reflectors, and walls, the discoloration of walls with time, and the *decreased efficiency of the lamps with age* will reduce the illumination. This reduction may be as much as 20 to 35 per cent.

Deterioration of illumination

1. The candlepower of a certain 50-watt incandescent lamp is 50 in the horizontal plane. What is the intensity in candles (a) 2 ft. from the bulb? (b) 5 ft? (c) 10 ft? (d) 100 ft?

Problems

Ans. (a) 50, (b) 50.

2. A 100-watt bare lamp is hung above a study table. If the lamp has a candlepower of 150 on the vertical line (through the bulb), what will be the illumination in foot-candles directly under the bulb when it is (a) 1 ft. above the desk? (b) 2 ft. above? (c) 3 ft. above? (d) 6 ft. above?

Hint. Apply law of inverse squares.

Ans. (a) 150; (b) 37.5; (c) 16.67; (d) 4.17 ft-c.

3. The average candlepower of a certain lamp is 25 c. How many lumens will be given forth by the lamp? (Same as point source of 25 c.)

Ans. 314 lumens.

4. A shop for woodworking has floor dimensions of 20 ft. by 40 ft. It is to be lighted by a direct system using eight 300-watt lamps placed in porcelain reflectors. Assume one-third of all the light is useful on the workbenches. Draw a floor plan showing how the lighting units should be placed and calculate the average illumination in foot-candles on top of the workbenches.

Hint. One lumen of light spread uniformly over 1 sq. ft. gives 1 ft-c of illumination. Use the table on p. 408 for lumen output.

Ans. 19.5 ft-c.

5. A single study table is to be lighted by fluorescent tubes placed $2\frac{1}{2}$ ft. above its center line. If the table size is 3 ft. by 4 ft., what size and number of tubes would be needed to produce an average illumination of approximately 30 ft-c? Assume that one-fourth of the light given off by the lamps is effective on the table top.

Ans. Two 20-watt, 24-in. white or daylight fluorescent lamps.

ELECTRON THEORY OF ELECTRICITY

Electrons

Prior to the twentieth century men knew little of the nature of electricity, though some theories of a speculative nature had been advanced. Scientists understood something of the behavior of electricity but admitted that they knew little concerning its true nature. About 1900, J. J. Thomson performed some important experiments and suggested to the scientific world the presence of the fundamental unit of electric charge — the *electron*. A few years later Robert A. Millikan performed many experiments and developed an exact and satisfactory method for determining the charge on the electron. The results of his experiments, announced to the world in 1913, confirmed the work of Thomson and gave scientists a new basis for explaining observed phenomena of chemistry, physics, and electricity.

Proton

Under the new theory the electron becomes the fundamental unit of electric charge and at the same time a particle which is one of the building blocks of matter. The other building block of matter is the *proton* which carries a positive charge, equal in magnitude to the negative charge on the electron, and having a mass about 1830 times that of the electron.

Bohr's atom

In 1913, Bohr offered a theory for atomic structure utilizing the new conception of electrons and protons. Bohr pictured the atom as consisting of a small dense core or nucleus about which one or more electrons revolve (Fig. 312).

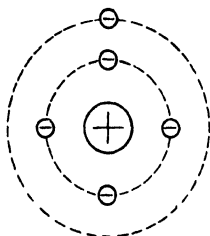


FIG. 312. BOHR ATOM

This structure is analogous to our solar system wherein the sun corresponds to the nucleus and the earth and planets to the revolving electrons. The nucleus contains all the protons and usually a part of the electrons. Chemical elements are determined by the total number of protons and electrons in the atom (always equal in number).

Hydrogen atom

For the simple hydrogen atom a single proton constitutes the nucleus and a single electron serves as the lone planet. You may obtain a conception of the relative magnitude of this hydrogen atomic system by expanding the nucleus to the size of a baseball located at the geographical center of the United States (near Manhattan, Kansas). The revolving electron will then pass through New York City and San Francisco and will be a sphere 300 ft. in

diameter — big enough to fill an average-size stadium or baseball park. Actually the radius of the hydrogen atom is of the infinitesimal size of only one hundred millionth of a centimeter. This new picture of the atom shows it to be a miniature hollow nebulous sphere; its parts form a swarm of whirling specks occupying a small part of space.

In more recent years some other particles and charges such as the *neutron* and the *positron* have been discovered. These and other discoveries have led to later theories defining the structure of the atom. However, for the general purpose of the electrical engineer and for satisfactory explanation of the action taking place in the electron tube, the Bohr atom continues to serve a useful purpose.

Atoms are combined to form the molecules of elements and compounds. Identical atoms may unite to form a molecule of an element and different atoms may unite to form a molecule of a compound. However, in the case of mercury vapor and in argon and neon gas the molecule is a single atom.

The Bohr atom is held together by the natural attraction of the negatively charged electrons and the positively charged protons which constitute its structure. Additional knowledge of the natural *repulsion of like charges* (negative repulsion of negative and positive versus positive) completes the basic theory which may be utilized for explaining many phenomena of electricity. The necessary concepts are relatively simple. Take the example of the electric charge. A neutral or uncharged body is one having an equal number of electrons and protons within its form and on its surface. If a body has an excess of electrons (more than protons), it has a negative charge. Conversely, if it possesses a deficiency of electrons, it is said to be positively charged.

The charge on a glass rod rubbed with silk is positive. This means that the friction of the silk rubbed on glass removes electrons from the glass, leaving it with a positive charge. In the process the silk becomes negatively charged owing to the acquired electrons. It should be noted that the removal of electrons from outer orbits of atoms does not necessarily change the chemical constitution of a substance.

Electric potential depends on the density or concentration of the excess (or deficiency) of the electrons on a body. The difference in potential between two points depends on the difference in the concentration of the electrons at those points. In contrast to

this concept it should be noted that the electric charge on a body depends on the total excess or deficiency of electrons on that body irrespective of the *area* of the body or the relative density of the electrons.

Charge on a body

To illustrate this point, let two neutral spheres, each 1 cm. in diameter, be placed 10 cm. apart and let a billion or so electrons be taken from one and added to the other. Now these spheres will have charges of opposite sign and will attract each other. Also there will be a difference in potential between these spheres determined by the concentration of their charges. If these spheres are shrunk to 1 mm. in diameter, leaving the distance between centers unchanged and without any exchange of electrons, they will still have the same charges and attract each other with the same force. However, the difference of potential between the spheres would now be increased because of the increase of *concentration* of electrons on one sphere and the *deficiency* on the other.

Electric current

An electric current is merely a coordinated flow or transfer of electrons. A random movement of electrons does not constitute an electric current since the result of the motion is not a coordinated action in some direction. The movement of electrons may be continuous and in the same direction as produced by a battery or direct-current generator, or the movement may be in a rapid back-and-forth (alternating) direction as produced by an alternating-current generator.

Electron velocity

The velocity of the individual electrons which constitute a current may vary through wide limits. At one extreme, electrons in a vacuum tube under the influence of a high potential may move with a velocity approaching that of light. Again, they may "drift" at a snail's pace of a fraction of a centimeter per second through a solid conductor.

Magnitude of current

The magnitude of the electric current is determined by the number of electrons which move past a point in a second. Since the charge on the electron is very small, the number of electrons required to equal 1 ampere (amp.) is beyond the power of human conception. The number is 6,300,000,000,000,000,000 (6.3×10^{18}) electrons per second. Moving electrons, instead of protons, constitute the current in an electric circuit because of their small mass, $\frac{1}{1836}$ times the proton.

An electric current will not pass through pure distilled water but if an *acid* base, or *salt* is added to the water, a transfer of electrons

takes place. For example, ordinary table salt (sodium chloride) added to distilled water disassociates into fragments or ions. The sodium atom separates from the chlorine atom and lacks one electron, so it has a positive charge and is called a *positive ion*. The chlorine atom carries an extra electron and is called a negative ion.

If two electrodes are now placed in the solution as in Fig. 313, *Conduction in liquids*

it gives up its extra electron and becomes a neutral chlorine atom. In like manner the positive sodium ion is attracted to the negative electrode where it seizes an electron and becomes a neutral sodium atom which promptly reacts with water to form hydrogen gas and sodium hydroxide. This simple process has served to remove an electron from the negative electrode and to deposit an electron on the positive electrode. This action repeated continuously by millions of ions serving as carriers constitutes an electric current through an electrolyte.

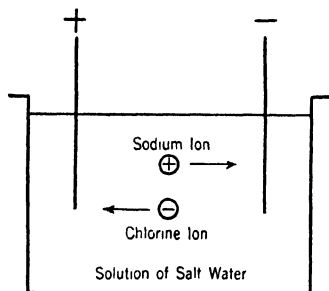


FIG. 313. CURRENT IN AN ELECTROLYTE

In solids there are some "free" electrons. Free electrons are those which are not closely bound to the atoms of which they are a part, and hence are susceptible to being moved along the conductor from atom to atom. This transfer or movement may be imagined as a relay race of electrons and it constitutes an electric current in solids. *Conduction in solids*

Electrical resistance is that property which opposes an electric current or movement of electrons. It depends upon the opposition to a free movement of ions in liquids and gases and upon the resistance to removal of electrons from atoms in solids. *Electrical resistance*

UNITS OF ELECTRICITY AND POWER

Electricity which is used for lighting and power moves in closed circuits or paths. A simple electric circuit is shown in Fig. 314. The flow of *current* in this circuit consists of a movement of electrons. The movement of these electrons is caused by an electrical pressure or *difference of potential* which is created in the battery. The magnitude of the electric current for this circuit is limited by *The electric circuit*

the load or *resistance* which is inserted in the circuit. The three factors which enter into this circuit are related to each other in a simple manner. Thus,

*Law of
flow*

$$(1) \quad \text{Electric current} = \frac{\text{electric difference of potential}}{\text{electric resistance}}.$$

This relationship is known as Ohm's Law and it is the basic equation for all electric circuits. Each of the three quantities used

in Ohm's Law must be represented by some unit of measure. While there are several systems of units, just as there are British units and metric units, for our purposes only the commercial units or practical units will be considered.

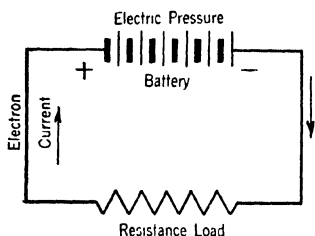


FIG. 314. SIMPLIFIED ELECTRIC CIRCUIT

It would seem logical to measure electric current by the number of electrons passing a point in a given

unit of time, but electrons were unknown at the time our practical system of units was adopted. Hence the practical unit of electric current is the *ampere*, defined as that current which in flowing through a conductor 1 centimeter long placed at right angles to a magnetic field of unit strength will be urged out of the field with a force of $\frac{1}{10}$ dyne. This definition does not give us a simple method of measuring electric current but it does give a basis for determining a standard ampere anywhere in the world. Actually, we measure current by use of indicating meters which are calibrated to read current in amperes.

Volt

Electric pressure or difference of electric potential is measured in *volts*. The volt is defined as the difference in potential produced by cutting magnetic flux at the uniform rate of 10^8 lines per second. Difference in potential in volts is measured by the use of indicating meters calibrated in accord with the definition given above.

Ohm

Electric resistance is measured in *ohms* and is defined as the resistance which produces a difference of potential of 1 volt when a current of 1 ampere flows through it. In practice, resistance in ohms is calculated from measurements of voltage (*volts*) and current (*amperes*) which are substituted in the equation of Ohm's Law.

The common form for Ohm's Law using the units just defined becomes

$$(2) \quad \text{Amperes} = \frac{\text{volts}}{\text{ohms}}, \quad \text{or} \quad I = \frac{E}{R}$$

Ohm's Law

When an electric current moves through a resistance under the influence of an electric difference of potential, energy is expended. The rate at which this energy is expended is electric power which is expressed in *watts* and equals the product of volts times amperes.

Thus, we may write

$$(3) \quad \text{Electric power} = \text{watts} = \text{volts} \times \text{amperes.}$$

The watt is a rather small unit of power, so 1000 watts, or the *kilowatt* (kw.), becomes the more common unit. The common unit of mechanical power is the horsepower which is equivalent to 746 watts, or about $\frac{3}{4}$ kw. Work is done when power is expended over a period of time. Thus watts \times hours = watthours, and 1000 watthours is the *kilowatt-hour* (kw-hr.) which is the accepted unit of electric work or energy. When you buy electricity, you pay for it at so many cents per kilowatt-hour.

An electric current passing through a resistance expends energy and this consumption of energy causes the temperature of the resistance to rise. Thus electric energy is transformed into heat. This action is utilized in all kinds of electric heating devices. The electric hot plate of Fig. 315 is an example. This hot plate is rated at 110 volts and 800 watts. Since watts = volts \times amperes, this unit must have a normal current consumption of



Electric energy

FIG. 315. ELECTRIC HOT PLATE

$$(4) \quad \text{Amperes} = \frac{\text{watts}}{\text{volts}} = \frac{800}{110} = 7.27.$$

Heat may be expressed in a number of units. One common unit is the British thermal unit (B.t.u.) which is the energy required to raise the temperature of 1 lb. of water 1° F. One kilowatt-hour is equivalent to 3415 B.t.u. If 10 lb. of water are placed on the electric hot plate referred to for 15 min., how many degrees will the temperature of the water be raised if all of the energy of the electric plate is transferred to the water without loss? The hot plate transforms 800 watts into heat for 15 min. The watthours equal

Units of heat and electricity

$800 \times \frac{1}{4} = 200$, or 0.2 kw-hr. This energy is equivalent to $0.2 \times 3415 = 683$ B.t.u. But 683 B.t.u. by definition above would raise 1 lb. of water 683° F. Hence, for 10 lb. of water the temperature rise will be $\frac{683}{10} = 68.3^\circ$ F.

Electrical calculations

In using electric circuits it becomes necessary to determine the values of current, resistance, potential, or watts when only two of these quantities are known. Calculations can be made readily through the use of rearrangements of the equation of Ohm's Law. Thus we write

$$(5) \quad \text{Amperes} = \frac{\text{volts}}{\text{ohms}}$$

$$(6) \quad \text{Volts} = \text{amperes} \times \text{ohms}.$$

$$(7) \quad \text{Ohms} = \frac{\text{volts}}{\text{amperes}}$$

$$(8) \quad \text{Watts} = \text{volts} \times \text{amperes}.$$

$$(9) \quad \text{Watts} = \text{amperes} \times \text{amperes} \times \text{ohms}.$$

$$(10) \quad \text{Watts} = \frac{\text{volts} \times \text{volts}}{\text{ohms}} = \frac{(\text{volts})^2}{\text{ohms}}$$

Examples

We may use these equations to solve practical problems. For example, how many amperes pass through a 110-volt, 200-watt lamp? Substituting in equation 8, we obtain

$$\begin{aligned} 200 &= 110 \times \text{amperes} \\ \text{Amperes} &= 1.815. \end{aligned}$$

How many volts are required to cause a current of 5 amp. to flow through a resistance of 10 ohms? Using equation 6, we write

$$\text{Volts} = 5 \times 10 = 50.$$

Resistances in series

Resistances may be connected in *series* or in *parallel*. The series connection is illustrated in Fig. 316 where the current through

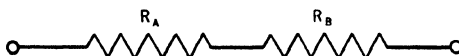


FIG. 316. RESISTANCES IN SERIES

resistances R_A and R_B is the same. The combined resistance of the circuit is the sum of the resistance R_A and the resistance R_B .

Resistances in parallel

The parallel connection is illustrated in Fig. 317 where the current will divide in passing through the parallel paths R_C and R_D . To find the combined resistance of R_C and R_D in parallel (R_p), we first note that the voltage across both R_C and R_D is the same voltage V . Then the current through the resistance R_C is V/R_C and

through R_D it is V/R_D . The combined current is $V/R_C + V/R_D$. Again the combined current should be V divided by the combined resistance R_p , or V/R_p . *Current division*

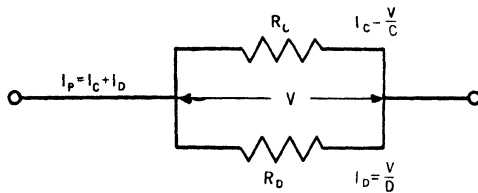


FIG. 317. RESISTANCES IN PARALLEL

Hence $V/R_p = V/R_C + V/R_D$, and dividing both sides of the equation by V gives the general equation for resistance of parallel circuits.

$$(11) \quad \frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots$$

1. A 110-volt flatiron has a resistance of 18.35 ohms. What current does it use? *Problems*
Ans. 6 amp.

2. A current of 3.2 amp. is passed through a resistance of 10 ohms. What voltage drop exists across the resistor? *Ans.* 32 volts.

3. How many watts are consumed in problems 1 and 2?
Ans. 660 watts; 102 watts.

4. A modern heavy-duty flatiron takes 9.1 amp. at 110 volts. What is its rating in watts?
Ans. 1000 watts.

5. An electric water heater is rated at 220 volts and has a resistance of 11 ohms. What is its rated load in kilowatts?
Ans. 4.4 kw.

6. Two resistances of 20 ohms each are connected (a) in series; (b) in parallel. What is the combined resistance?
Ans. (a) 40 ohms; (b) 10 ohms.

7. What is the combined resistance of 20 ohms and 10 ohms connected (a) in series; (b) in parallel?
Ans. (a) 30 ohms; (b) 6.67 ohms.

8. A flatiron having a resistance of 22 ohms and a toaster rated at 660 watts (110 volts) are connected in parallel across a 110-volt supply. Calculate the current through each unit, watts of each unit, total current, and total kilowatts.

| <i>Ans.</i> | <i>Current</i> | <i>Watts</i> |
|-------------|----------------|--------------|
| Flatiron | 5 | 550 |
| Toaster | 6 | 660 |
| Total | 11 | 1,210 kw. |

9. A washing-machine motor requires 250 watts for operation. If the washing machine is used 3 hours per week and electricity costs 5 cents per kw-hr., what does it cost per month (4 weeks) for electricity for the family washing? *Cost of power*

Ans. 15 cents.

10. Assume ironing is performed with a 1000-watt iron used with current on for 4 hours per week. What is the cost of electricity for ironing per month (4 weeks) at 5 cents per kw-hr.?
Ans. 80 cents.

MAGNETISM

*Magnets and
the compass*

The term *magnetism* is associated with anything which has the property of attracting iron. We think of magnetism in connection

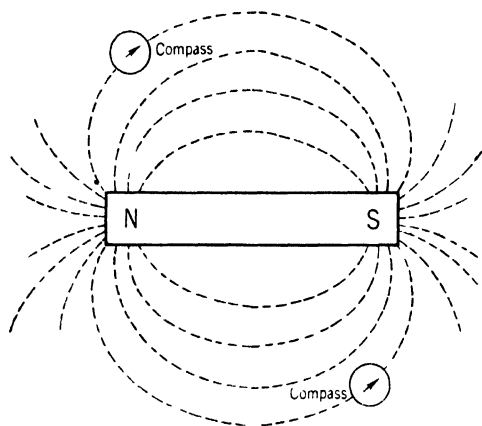


FIG. 318. MAGNETIC FIELD SURROUNDING A PERMANENT BAR MAGNET

with the permanent bar magnet of Fig. 318. Such a magnet will attract iron at regions near its ends which are termed *poles*. If this magnet is suspended freely, one end will swing until it points toward the north magnetic pole of the earth. This north-seeking pole is termed the *north pole* of the magnet, while the other end becomes the *south pole*.

Compasses consisting of little bar magnets have served for centuries to guide ships at sea and explorers in the wilds.

Magnetic field

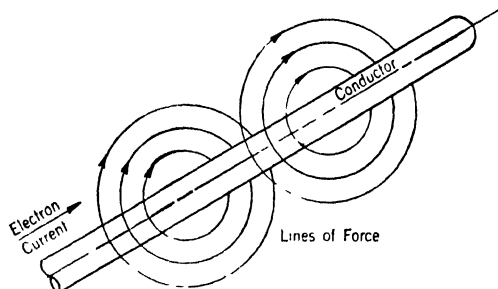
The bar magnet has a magnetic field of stress which exists along the lines shown in Fig. 318. A tiny compass needle may be used to indicate the direction of this magnetic field by the way its north pole is urged. The paths of the field form a closed circuit leading through the bar magnet and out of the north pole through the air and reentering at the south pole. This path constitutes a magnetic circuit and is analogous to the electric circuit previously discussed. Here a magnetic potential or pressure arises in the bar, a magnetic flux flows around the circuit meeting a magnetic resistance or *reluctance* throughout the path. Calculations for this circuit are somewhat complicated and will not be considered here.

*Permanent
magnets*

In order for the bar magnet to retain its magnetism it must be constructed of a hard form of steel. In recent years many alloys of metals have been discovered which give superior magnetic properties — greater permanence, and extremely strong magnets.

For many purposes the usefulness of the permanent magnet is increased by bending it into the form of a horseshoe. This brings both poles close together and reduces the reluctance of the path through air. The electrical engineer uses permanent magnets in

many places. Some of the more common applications are for telephone receivers, loud-speakers, telephone bells, electric meters,



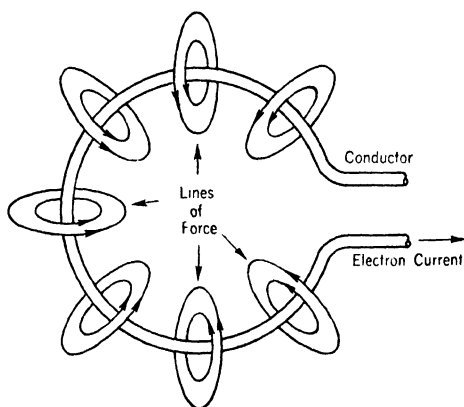
Visualize circles as perpendicular to conductor

FIG. 319. MAGNETIC FIELD SURROUNDING A CONDUCTOR CARRYING AN ELECTRIC CURRENT

and for the fields of magnetos in telephones, gas engines, and speed-indicating devices.

The unit of magnetic flux or flow is the line of force. This unit is called the *maxwell*. Flux density is the number of lines of force or maxwells per unit of area.

An electric current passing through a straight conductor creates a magnetic field in the surrounding medium. The lines of force of this field exist as indicated in Fig. 319. If this straight wire is bent into a circle, the field produced by the conductor on the inside of the circle has a common direction, and hence it is greatly increased in strength. (See Fig. 320.)



Strong field at center

FIG. 320. MAGNETIC FIELD SURROUNDING A TURN OF WIRE CARRYING CURRENT

Winding a long conductor into a circular spool consisting of many turns will further increase the magnetic field along the center of this spool. Such a spool is called a *solenoid* and is illustrated in Fig. 321.

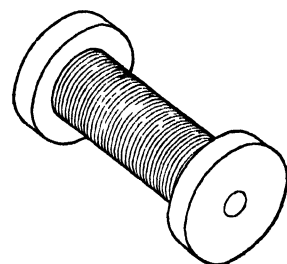


FIG. 321. SOLENOID SPOOL

The magnetism created by the solenoid can be further increased (perhaps 1000 times) by inserting a soft iron core within it.

Solenoids

This iron offers a much lower reluctance than air to the passage of magnetic flux and thus increases the magnetism. The combination of a solenoid on an iron core constitutes an electromagnet and is one of the most important basic parts of electric devices which utilize magnetism. The usefulness of this device arises from (1) the fact that the strength of the magnetism or *pull* of the electromagnet can be controlled by varying the current in the solenoid, and (2) the fact that the magnetism can be reduced quickly to zero or near zero by stopping the current.

Common uses of electromagnets

One of the earliest applications of the electromagnet was in the telegraph sounder, shown in Fig. 322. Here a horseshoe type of electromagnet pulls down an iron armature and gives audible

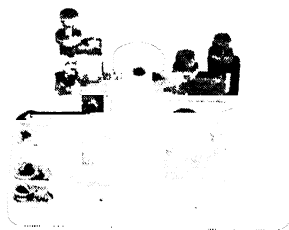


FIG. 322. TELEGRAPH KEY



FIG. 323. DOORBELL

sounds for the dots and dashes of the Morse code. Every school-boy is familiar with the electric bell or buzzer like Fig. 323 which announces the guest at his home. In this device the electromagnet pulls the striking gong against the force of a spring and in doing so breaks the circuit which initiated the pull. The deenergizing of the electromagnet permits the spring to pull the armature back and reestablish a closed circuit so that the operation may be repeated. Electromagnets form a vital part of all types of electric power and communication equipment.

PRODUCING VOLTAGE

Voltage by chemical action

A difference of electric potential (voltage) can be produced (1) by *chemical action*, (2) with *heat*, (3) through *light*, or (4) *mechanically* through magnetic influence. The chemical battery was the earliest source of voltage and power. Today the dry cell is used for operating flashlights and portable radios, whereas the storage battery is an essential part of every automobile and truck. In both of these batteries chemical action takes electrons from the positive electrode and concentrates them on the negative electrode. The

resulting concentrations give rise to the difference of potential at the battery terminals.

When two pieces of dissimilar metal such as iron and copper are joined together at one end with the other ends left free, a difference of potential will exist at the free ends if they are at a different temperature than the united ends. This unit is called a *thermocouple* and its use in conjunction with a voltmeter is one method used for measuring temperatures. *Thermal electricity*

Light falling upon a combination of certain dissimilar materials will produce a difference of potential between them. Such a combination may consist of a thin layer of gray selenium upon a base of iron. When light falls upon the selenium side of this unit it penetrates to the barrier layer and gives up its energy in such a way as to remove electrons from the selenium and carry them into the iron. This action causes the iron to assume a negative potential and the selenium becomes positive. A sensitive meter connected to the two sides, as shown in Fig. 324, will give a deflection which is proportional to the voltage and to the intensity of the light falling on the selenium. Many exposure meters used by photographers operate on this principle. *Voltage from light*

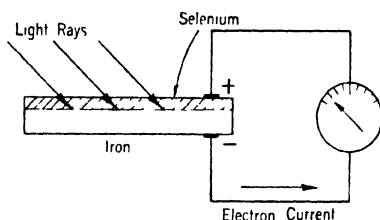


FIG. 324. LIGHT PRODUCING AN ELECTRIC POTENTIAL

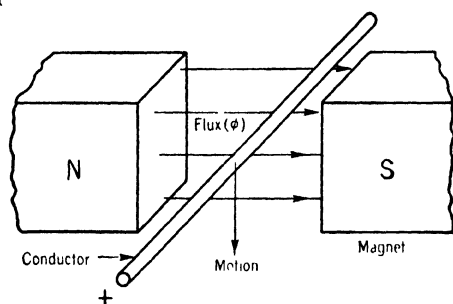


FIG. 325. VOLTAGE INDUCED BY CUTTING FLUX

The common method of producing a voltage for electric power utilizes the *electromagnetic principle*. When a conductor is moved across a magnetic field, a difference of electric potential appears between the ends of the conductor. This action is illustrated in Fig. 325. This phenomenon of a production of a difference of poten- *Voltage by electromagnetic action*

tial may be explained by the electron theory. The conductor is made up of protons and electrons and when it moves it carries the electrons with it. These moving electrons constitute an electric current. It will be explained in an early paragraph how a conductor bearing an electric current is pushed aside when placed in a magnetic field. Thus the electrons in the moving conductor are pushed aside and urged in one direction along that conductor. This transfer of electrons along the conductor produces a difference of concentration of electrons (difference of potential) between its ends.

Voltage

In Fig. 325, the conductor may remain at rest and the field move, or both may move, because as long as a relative motion exists between them a difference of potential or a voltage will be induced. One volt is produced when the conductor cuts the magnetic field at the rate of 100 million (10^8) lines of force or of magnetism per second.

POWER GENERATION

Generators

In order to generate large voltages, it is customary to connect many moving conductors in series; the machine for doing this is called a *generator*. Two conductors may be connected in series and revolved between the two poles of a magnet as shown in Fig. 326.

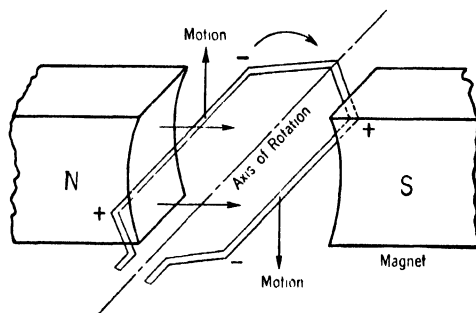


FIG. 326. VOLTAGE INDUCED IN A ROTATING TURN OF WIRE

Here the conductors cut the magnetic-field flux in opposite directions which causes *reversed polarities* of the voltages induced per conductor. This reversed polarity permits the conductors to be connected in series, plus to minus, so that the induced voltages add.

Increasing voltage

The *induced voltage* may be increased manyfold, first, by increasing the number of turns of the single coil of Fig. 326, second, by

using many coils placed around the periphery of a circular cylinder of iron and, third, by connecting all or part of these coils in series. The practical unit embodying these features is called an *armature* and is illustrated in Fig. 327.

In the armature the coils consist of several turns of wire and several similar coils are distributed throughout the circle of rotation. These coils are placed in slots in a laminated iron cylinder, and the ends of the coils are connected to copper commutator bars.

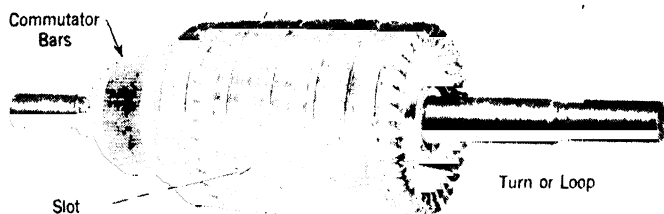


FIG 327. AN ARMAIURE, THE REVOLVING PART OF A MOTOR

The generated voltage and the resulting current is collected from the commutator bars or terminals of the coils by stationary carbon brushes bearing on these bars. The rotation of the armature serves to reverse the connection from coil to brush at the proper time to cause the external voltage to be of the same sign (polarity) and to cause the current flow to be unidirectional (direct current).

It should be noted that if the connection to coils on the armature is not reversed by commutator action, the resulting voltage will change its polarity and the output voltage will be alternating. Thus, inherently, the voltage induced in an electric generator is alternating.

The theory of electric-generator action may be applied to a practical case as follows. Assume that you desire a small *direct-current generator* to be driven by a wind propeller to charge a 12-volt storage-battery plant. For a small plant a current output of about 20 amp. and a potential of 15 volts is desirable. Experiment has shown that a propeller approximately 7 ft. in diameter will develop sufficient power for wind velocities of 12 miles per hour and higher. This propeller will run at about 300 r.p.m. Since this is a low speed for a generator, we will use a higher speed, say 900 r.p.m., and attain this speed by a "step-up" V-belt drive.

Checking an
automobile
generator

To charge the 12-volt batteries we will make a calculation based on the 12-volt generator used on the old 4-cylinder Dodge automobile. Figure 328 gives the dimensions of the generator suggested. From this figure we see that the area under each field pole equals $2.7 \times 3.7 = 10$ sq. in. A reasonable flux density in the air gap between the pole face and the armature is 25,000 lines per square

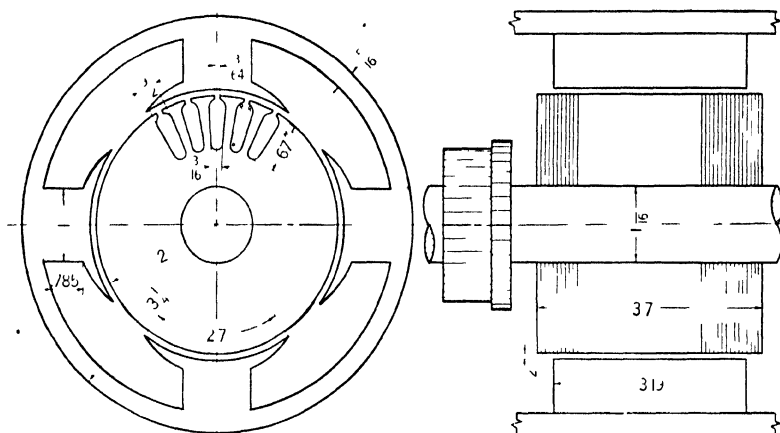


FIG. 328 DIMENSIONS OF A SMALL DIRECT-CURRENT GENERATOR

Flux cut

inch. This flux density would give $10 \times 25,000 = 250,000$ lines or maxwells for the entire face of one magnetic pole. For the four poles on this machine there would be a total flux of $4 \times 250,000 = 1,000,000$ maxwells crossing the air gap and being cut by the conductors on the armature during each revolution.

Conductors

The armature on the Dodge generator has 25 slots and eight conductors were placed in each slot. A conductor may consist of a single wire or two or more wires in parallel. In any case the conductor acts as one wire so far as cutting flux and generating a voltage are concerned. With 8 conductors per slot and 25 slots, there are $8 \times 25 = 200$ conductors on the armature. The voltage output of a generator depends on the way the conductors are connected together at the commutator. For all small machines and many others the wave or series connection, called the *winding*, is used. For this connection there are only two paths in the armature regardless of the number of poles. Thus for a wave winding on the machine under consideration there will be $\frac{200}{2}$ or 100 conductors per path for generating voltage.

A speed of 900 r.p.m. corresponds to $\frac{900}{60}$ or 15 revolutions per

second. Since a single conductor cuts 1,000,000 lines in one revolution, it will cut $15 \times 1,000,000$ or 15,000,000 lines per second. One hundred conductors in series will cut $100 \times 15,000,000 = 1,500,000,000$ lines per second. One volt is produced when 10^8 maxwells are cut per second at a uniform rate. Hence the generated voltage of our direct-current generator will be $1,500,000,000 \div 10^8 = 15$ volts. *Voltage calculation*

The armature of this machine should be wound with 25 coils of four turns or loops each and connected to 25 commutator bars. Each turn consists of 2 conductors. The sides of each coil are placed in slots in the armature separated by the distance between the centers of the field poles. Each conductor will consist of two No. 16 (American Wire Gauge) wires in parallel. Each No. 16 wire will carry 5 amp. continuously without overheating. Two wires in parallel will carry $2 \times 5 = 10$ amp. per path. Since there are two paths in the winding in parallel, the total current output will equal 2×10 or 20 amp., thus fulfilling the specifications for our direct-current, wind-driven generator. *Windings*

1. A conductor cuts the flux under a field pole of a dynamo in $\frac{1}{60}$ sec. If the flux coming from the pole is 1,000,000 maxwells, what average voltage is induced between the ends of the conductor? *Problems*
Ans. 1 volt.

2. A coil of wire has 10 turns and revolves between two poles in a magnetic field of 1,000,000 maxwells ten times per second. What is the average voltage induced in the coil?
Ans. 4 volts.

3. A 10-kw. d-c generator has a speed of 1800 r.p.m. If each path of the winding on the armature has 100 conductors in series and the flux per pole is 1,050,000 maxwells, what is the generated voltage? There are 4 poles. *Ans. 126 volts.*

ELECTRIC MOTORS

A conductor carrying an electric current is surrounded by a *Force in magnetic field* as shown in Fig. 319. The magnetic field passes around the conductor in circular paths. If this conductor is placed in the magnetic field existing between two poles of a permanent magnet or an electromagnet, there will be an interaction between the magnetic field surrounding the conductor and the second magnetic field. This interaction will be such as to *push* the conductor at a right angle to the direction of the main field, thereby developing a *force* as illustrated in Fig. 329.

You will note that the field surrounding the conductor is in the same direction as the main field above the conductor and in oppo-

Why a force

site direction beneath the conductor. The result of these directions of field is to increase the strength of the field above the conductor and to reduce its strength beneath. It is convenient for us to think of the resulting field above the conductor acting like stretched

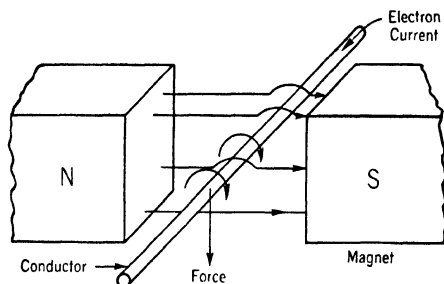


FIG. 329. FORCE ACTING ON A CONDUCTOR CARRYING CURRENT IN A MAGNETIC FIELD

elastic bands displaced upward, which force the conductor downward. This force acting upon a current-bearing conductor placed in a magnetic field is the fundamental principle of all electric motor action. This statement applies to both direct-current and alternating-current motors.

It generally happens that the conductors are partially or completely surrounded by iron so as to increase the field produced by the conductor. Also, the practical motor may hold the conductor stationary and permit the main magnetic field to move. The force or the power developed is the same either way.

Force developed

The force acting upon the current-bearing conductor in a magnetic field is proportional to the strength of the main field, the length of the conductor, and the magnitude of the current in the conductor. This may be expressed by the equation

$$(12) \quad F = \frac{BIL}{10}$$

where F is the force in dynes, B is the flux density in maxwells per square centimeter, L is the length in centimeters, and I is the current in amperes. The ordinary concept of a force as a push or pull is sufficient for an understanding of motor action. However, for a scientific concept of force and rotation you will read a later section of this book on mechanics which covers the effect of forces. An illustration of the effect of a common force (an uppercut to the chin) is shown in Fig. 373, page 471. Since we are familiar with weights and forces in pounds, it is well to remember that 444,823 dynes equal one pound of force.

Dynes per lb.

The length of conductor L may be effectively increased by increasing the number of conductors. Thus to produce an elementary motor, two conductors may be arranged into a loop or coil and

mounted so as to revolve in a magnetic field as shown in Fig. 330. The right conductor is urged downward and the left upward. The *turning forces* will continue as the loop revolves in a clockwise direction until the plane of the loop is vertical. At this point the

Motor action

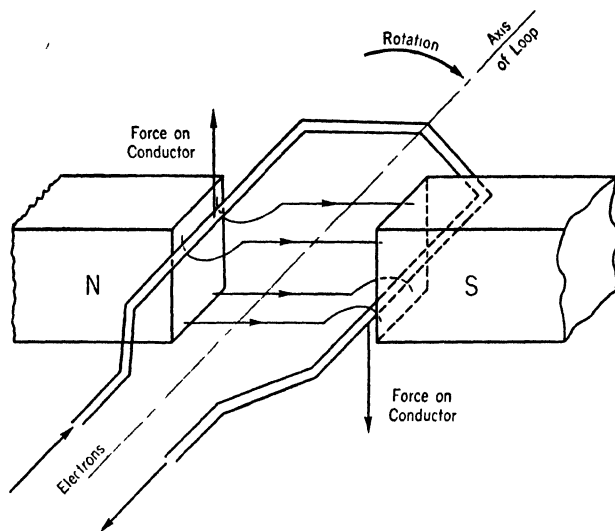


FIG. 330. FORCES ACTING ON A LOOP OF WIRE CARRYING CURRENT IN A MAGNETIC FIELD

motion will cease unless the direction of the current through the loop is reversed.

If the current is reversed and the momentum of the moving coil carries it past the vertical position, the forces now acting upon the coil will continue the clockwise rotation until the plane of the coil reaches the next vertical position, where it will be necessary again for a reversal of current to occur.

Each force acts at a constant radius from the center of rotation. The product of the force F and the radius R is called the torque, T . The total torque produced by a motor is the sum of the individual torques produced by the current flowing through each conductor or coil.

Torque

In practical motors the loop of wire just considered is expanded into a coil of many turns of wire, and several other similar coils are distributed throughout the circle of rotation. The resulting construction is the armature illustrated in Fig. 327. In the motor armature, the current is led to the coils and conductors via stationary brushes bearing on the commutator bars. The rotation of

Motor construction

the armature serves to change the direction of current through the active conductors at the proper time so that the turning effort is always in the same direction.

Motor fields

A motor or generator could be constructed with permanent magnets for producing the magnetic poles. Some small machines are constructed in this manner. However, the more common practice is to use electromagnets consisting of soft iron cores with solenoid coils (Fig. 321). The electromagnet can produce a much stronger field (higher flux density) and it is more economical in the use of material for large machines having four or more poles. The current for magnetizing the field structure is generated by the generator itself or taken from the supply line in the case of the motor.

Automobile starting motor

The active parts of an automobile starting motor are shown in Fig. 331. The motor operates for a few seconds at a time but dur-

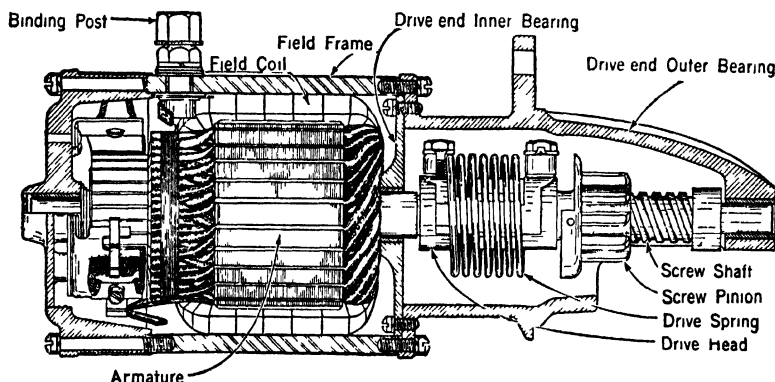


FIG. 331. CROSS SECTION OF AN AUTOMOBILE STARTING MOTOR

ing that time it must produce a *large starting torque* to overcome the inertia and friction of the engine during cold weather when the oil in the crankcase is very "stiff." The voltage to drive this motor may be as low as 4 volts at the motor terminals; hence the current must be large and of the order of 150 to 300 amp. To carry this large current the conductors on the armature are large rectangular copper bars (of low resistance) and only one conductor or bar is placed in an armature slot.

Series winding

In order to obtain a strong magnetic field, the designer connects the solenoids or coils on the field magnets in series with the armature circuit. The complete circuit for the starting motor

operation is shown in Fig. 332. The auto-starting motor is called a *series motor* because the armature and the field coils are connected in series.

The series type of motor develops a large torque on starting because the same large current passes through the armature conductors and the field coils. This large current gives a strong magnetic field acting upon conductors carrying large currents and results in a strong force and torque (torque is force times radius).

Series motors are used wherever heavy torque must be developed for starting an operation. This requirement applies to electric

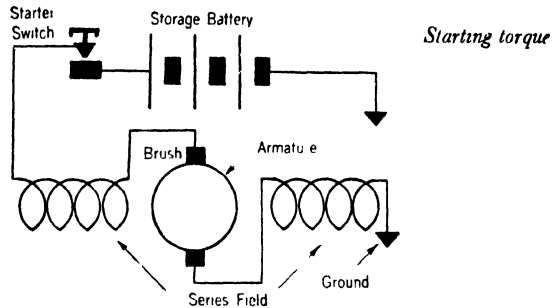


FIG. 332. CIRCUIT FOR AN AUTOMOBILE STARTING MOTOR

railways, buses, elevators, hoists, and most fans. A modern electric railway motor of the series type is illustrated in Fig. 333.

Many other types and connections are used for the millions of electric motors in use today, yet all of them apply the *basic principle of the force acting upon a current-bearing conductor placed in a magnetic field.*

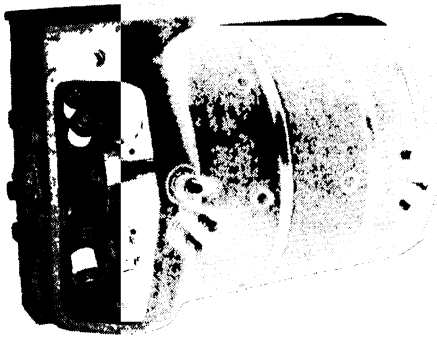


FIG. 333. RAILWAY MOTOR OF SERIES TYPE PRODUCES LARGE STARTING TORQUE

1. A rectangular copper conductor on the armature of an automobile starting motor carries 100 amp. of current. If the conductor is 10 cm. long and the flux density under the field pole is 10,000 maxwells per square centimeter, what is the force on each conductor in dynes? *Problems*
Ans. 1,000,000 dynes or 2.25 lb.

2. A force of 1,000,000 dynes acting on a radius of 3.5 cm. will produce what torque in foot-pounds?
Ans. 0.258 ft-lb.

Hint. Torque is force times radius and there are 444,823 dynes per pound.

3. If there are 14 (out of 20 total) copper conductors within the field flux and the armature makes 850 r.p.m., what horsepower is being developed by the starting motor of problems 1 and 2?
Ans. 0.585 hp.

Hint. One horsepower equals 33,000 ft-lb. per min. Foot-pounds developed is the force in pounds times the distance traveled by any conductor.

4. If the voltage at the starting motor terminals is 5.0 and the current is 200 amp. (2×100 amp. per path), what is the motor's efficiency under the conditions of the problems above? *Ans. 43.6%*

Hint. Efficiency equals output divided by input.

ALTERNATING CURRENT

*Alternating
voltage*

The circuits and electrical equipment that we have considered in this section have been adapted to direct current which flows in one direction. However, the voltage produced by mechanical rotation is fundamentally of a changing or *alternating character*. By referring to the loop of Fig. 326, you will remember that each time the

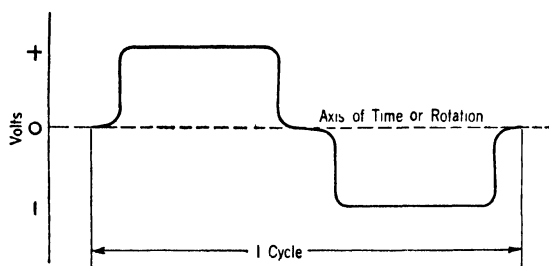


FIG. 334. RECTANGULAR A-C VOLTAGE WAVE

plane of the coil passes the vertical position it cuts the magnetic field in an opposite direction, and hence the polarity of the voltage is reversed. If the generated voltage is brought out from the machine without a reversal of connections, an alternating voltage results.

Voltage wave

If the magnetic flux is distributed uniformly underneath the pole faces, the generated voltage would present a rectangular wave

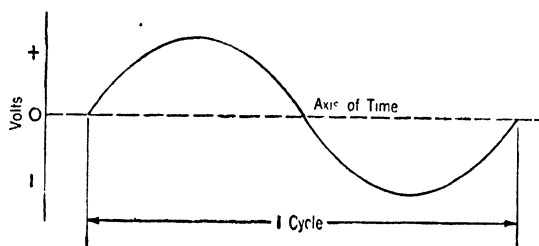


FIG. 335. SINE WAVE

as shown in Fig. 334. Here the height of the curve indicates the magnitude of the voltage and the distance from zero along the horizontal axis represents the degrees of rotation or the time since the armature rotation started. The curve represents the voltage change while the coil or loop turns through a complete circle and

each conductor cuts past two field poles. This period of one revolution is called *one cycle*.

In general, we know that the rectangular voltage wave is objectionable and a smoother form known as a “sine wave” is desired. *Sine wave*

In the sine wave the magnitude of the voltage or current varies with time as the trigonometric sine of an angle changes through a complete revolution from zero to 360 degrees. This wave form, illustrated in Fig. 335, can be attained

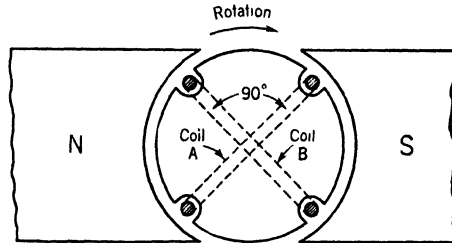


FIG. 336. SIMPLE TWO-PHASE GENERATOR

either by the proper distribution of the field flux or by the proper distribution of the conductors in which the voltage is generated.

Each of two like coils placed at right angles to each other and rotated between the poles of the magnetic field of Fig. 336 will generate the same voltage wave. But their voltage waves will be out of step or out of time because the coils do not pass the same part of the magnetic field at the same time. A picture of these voltage waves is shown in Fig. 337 and the displacement of the *Phase shift*

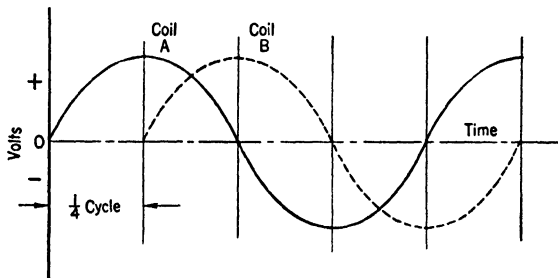


FIG. 337. TWO A-C VOLTAGES 90° OUT OF PHASE

waves is known as “difference in phase” or *phase shift*. For the case assumed, the phase shift will be 90 degrees since the coils are at right angles to each other.

An alternating voltage applied to a resistance will produce an alternating current and the resulting current will be in step or in phase with the voltage as shown in Fig. 338. If the same voltage is applied to a coil of wire wound on an iron core (*choke coil*), the resulting current will lag behind the impressed voltage. (See Fig. 339.) This lag of current is due to a property of choke coils *Alternating current*

which opposes any change of current. By change we mean either increase or decrease of current. This opposition to current change acts like the inertia or momentum of a moving body.

Condenser
capacity

A condenser is a sandwich consisting of two plates or *conductors* separated by an *insulator*. When a voltage is applied to a condenser

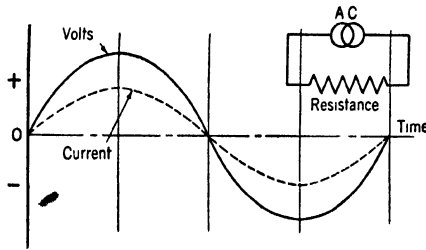


FIG. 338. A-C VOLTAGE AND CURRENT IN PHASE

a charge is built up on the condenser. In producing a charge on the condenser, electrons must be withdrawn from one plate and concentrated on the other. Strangely enough, if an alternating voltage is applied to a condenser, an alternating current will flow in the circuit though no electrons can pass through the insulator of the condenser. This action is a result of the electric storage *capacity* of the condenser.

Thus when the impressed voltage assumes one polarity, one plate of the condenser absorbs a large number of electrons while the other plate gives up an equal number. Then when the polarity of the impressed voltage reverses, the crowded plate gives up all

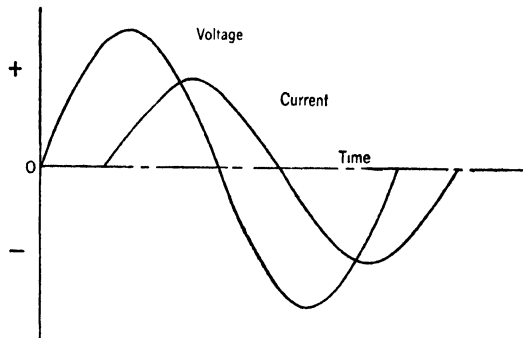


FIG. 339. A-C VOLTAGE AND CURRENT OUT OF PHASE

of its excess electrons plus an equal number more, so that the other plate now assumes or acquires negative charge. By this process a condenser is said to *conduct* an alternating current. Its ability to store electricity and produce this current is its capacity.

Leading
current

The current resulting from an alternating voltage applied to a condenser is out of step or phase with the voltage. The current

through the condenser leads the voltage by one-fourth of a cycle or by 90 degrees.

You are likely to wonder what value should be assigned to a voltage or current which is constantly changing. Your first suggestion might be to take the average of all values as being the solution. However, engineers have long since agreed that the proper value is that of an equivalent direct current or voltage and is called the *effective value*. Since the heating value of a direct current in watts is the current squared times the resistance, the effective value depends upon the square of the instantaneous values. Thus *the effective value is the square root of the average of a large number of the squares of instantaneous values*. Effective values are indicated by a-c voltmeters and ammeters and all alternating voltages and currents should be interpreted as effective values unless otherwise specified.

The shift in phase of the current where an alternating voltage is applied to a circuit containing chokes or condensers introduces complications in calculations of electric power and other factors of the circuit. Ohm's law must be modified to read

$$(13) \quad \text{Amperes} = \frac{\text{volts}}{\text{impedance}} \quad \text{or} \quad I = \frac{E}{Z}$$

Impedance (Z) is the effective sum of the resistance and another opposing factor called reactance. *Reactance* represents the effects of a choke or condenser or both in the circuit.

In the direct-current circuit the product of volts times amperes is equal to the electric power in watts. But in the alternating-current circuit the product of effective volts and effective amperes is called *apparent power* and generally is greater than the true power in watts. This difference between direct current and alternating current is due to the phase shift between voltage and current in the alternating-current circuit. True power in watts is measured by an instrument called a wattmeter.

The ratio between true power in watts and the apparent power in *effective volt-amperes* is called *power factor* (p.f.). The ratio may be expressed thus:

$$(14) \quad \text{Power factor} = \frac{\text{true power}}{\text{apparent power}} = \frac{\text{watts}}{\text{volts} \times \text{amperes}}$$

The maximum value for power factor is 1, and is called unity power factor. Unity power factor exists when the alternating voltage and current are in phase. Unity power factor exists on

loads consisting of incandescent lamps and most electric heating devices. The power factor falls below unity as the alternating voltage and current get out of phase. Hence, for a constant alternating voltage and wattage, the current rises in value as the power factor decreases (equation 14). The increase of current reduces the power which may be obtained from the apparatus because the heating losses in generators, transmission lines and motors vary as the square of the current (I^2R). Low power factor can be corrected through the use of auxiliary devices but only with increase in the cost of electric service. Hence, power factor is a very important consideration in the loads placed on electric power and distribution lines.

Problems

1. A 100-watt, 110-volt incandescent lamp takes 0.91 amp. on a 110-volt alternating-current circuit. (a) What is the impedance Z of the lamp? (b) What is the power factor of the lamp?

Ans. (a) 121 ohms; (b) 1.0.

2. A 100-watt, 110-volt fluorescent lamp and its circuit (without any condenser) takes a current of 1.3 amp. on a 110-volt, a-c circuit. (a) What is the impedance of the circuit? (b) What is the power factor of the circuit?

Ans. (a) 84.7 ohms; (b) 0.7.

3. A 110-volt, $\frac{1}{4}$ -hp., a-c motor on an electric washing machine takes 2.5 amp. and 60 watts when running without load. Under load the same motor takes 4.8 amp. and 375 watts. What is the impedance and the power factor: (a) without load; (b) under load?

Ans. (a) 44 ohms, 0.218 p.f.; (b) 22.9 ohms, 0.71 p.f.

SPECIAL USES FOR A.C. AND D.C.

Importance of voltage trans- formation

Alternating current is almost universally used as the medium of generation and distribution of electric power. Its great value lies in the case of transforming from one voltage to another by means of a simple stationary device called a *transformer*. To illustrate the importance of transforming voltage, let us assume that 5 kw. of power are to be transferred over a line half a mile long. If 1000 volts are used for transmission, the current will be 5000 watts/1000 volts = 5 amperes. For a line consisting of No. 10 copper wire, there will be a resistance of approximately 1 ohm per 1000 ft., or 5.28 ohms for one mile. Under these conditions the voltage drop in the line will be 5×5.28 , or 26.4 volts, which is 2.64 per cent of the voltage of transmission (receiving end).

Next, suppose the same power is to be transmitted at 200 volts. Then the current will be 5000 watts/200 volts = 25 amperes, and the voltage drop would be 25×5.28 , or 132 volts. Here 66 per cent of the voltage would be absorbed in the line.

To avoid such a large drop in line voltage we could use a wire of larger cross section, but that means a greater cost for copper. We could show that for the same power loss in the line *the weight of copper required varies inversely as the square of the voltage*. Hence, for transmitting large blocks of power, high voltages are used. For distribution about a town or city, a voltage from 2200 to 2400 is frequently used, while on the line from Boulder Dam to Los Angeles 287,000 volts has been chosen. Between these extremes a wide range of line voltages is used, determined by the length of line and the amount of power to be transmitted.

Alternating current has two disadvantages. One disadvantage due to the possibility of low power factor has been suggested. The second disadvantage arises from the fact that *alternating-current motors do not have good starting characteristics and also that they usually have a fixed speed*. Hence, if we need a high starting torque or must be able to control the motor speed, direct-current motors are used and some means must be employed for transforming alternating current into direct current. For power purpose this transformation is made by use of a rotary convertor, a motor-generator set, or an electronic rectifier.

The preceding discussion suggests some of the characteristics and problems involved in the use of alternating-current circuits and equipment. A more complete discussion is beyond the scope of this introduction to the problems of electrical engineering.

THE TRANSFORMER

The theory of action of a transformer requires a concept of flux linkages. You understand how one link in a chain connects or links with the next one. Likewise you are familiar with the way several keys may be linked or threaded on a key ring. In a similar way, lines of flux may link or thread through an electric circuit as shown in Fig. 340. These unions or interlinks are called *flux linkages*. In Fig. 340, a single turn of wire is shown linking with one line of force. If there were two lines of force and two turns of wire, there would be 2×2 , or 4 flux linkages. Now you cannot pull a key through a key ring without breaking it, but you can pull lines of force out of their link with an electric circuit.

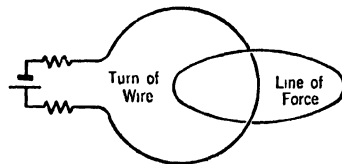


FIG. 340. FLUX LINKAGE

Induced voltage

It was shown in an earlier paragraph that a voltage is induced in a conductor that moves at right angles to a magnetic field. A similar induction of voltage exists when there is a change in the lines of flux linking with or threading through an electric circuit.

If one pole of a permanent magnet be moved into a coil of wire as shown in Fig. 341, there will be a deflection of the voltmeter indicating that a voltage is being induced. Here the voltage is

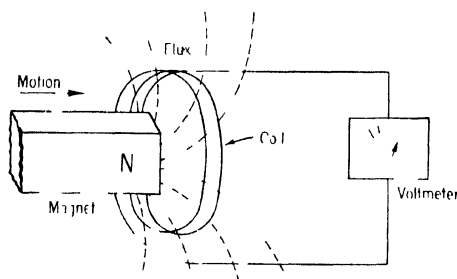


FIG. 341 VOLTAGE PRODUCED BY FLUX CUTTING A COIL OF WIRE

created by a change of the flux linking with the electric circuit. A uniform change of flux linkages at the rate of 10^8 per second will induce 1 volt in the electric circuit. This same factor termed "cutting magnetic flux at the uniform rate of 10^8 maxwells per second" was used when we were discussing the electric generator.

Voltage ratios

The transformer is a simple arrangement of two or more electric circuits *interlinked with a magnetic circuit*. The electric and magnetic

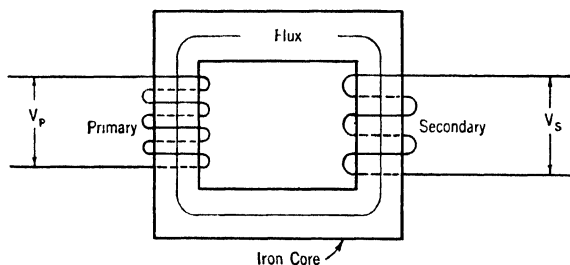


FIG. 342. ELECTRIC AND MAGNETIC CIRCUITS OF A TRANSFORMER

circuits of a simple transformer are illustrated in Fig. 342. An alternating voltage applied to the primary winding on the left will induce an alternating magnetic flux to circulate through the iron core. This alternating (changing) flux will link both the primary and secondary windings and induce an alternating voltage in each one.

The magnitude of the voltage induced at any instant will be equal to the rate of change in flux linkages divided by 10^8 . If the same flux circulates through both windings, the voltage induced in each winding will be proportional to the number of turns on each coil. Hence the voltage ratio between the primary and secondary windings is approximately equal to the ratio of the number of turns on each.

Let us assume that we wish to transform 2300 volts to 115 for distribution in a city. If the magnetic iron core of the transformer has a cross-section of 9 sq. in. and we use a flux density of 70,000 lines per sq. in., the flux will rise to a maximum of $9 \times 70,000$, or 630,000 maxwells. When the current reverses, the flux will reverse its direction to a value of 630,000, thus making a change of $2 \times 630,000$, or 1,260,000 for one-half a cycle. For a full cycle the change is $4 \times 630,000 = 2,520,000$ maxwells. With a frequency of 60 cycles per second the total flux change will be $60 \times 2,520,000 = 151,200,000$. One turn of wire cut or linked by this flux change would have induced in it an average of $151,200,000 \div 10^8 = 1.51$ volts.

*Transformer
calculation*

Since the same voltage will be induced in each turn, the low-voltage winding will have $115/1.51 = 76$ turns of wire, and the high-voltage winding $2300/1.51 = 1522$ turns.

The answer attained in the calculation above is only approximate since it applies to average values of voltage rather than to effective values. For a sine wave of voltage and flux, the number of turns should be divided by the ratio of "effective value to average value" which is 1.11 and is called the form factor. Applying the form factor correction gives us 68.5 turns for the low-voltage winding and 1372 turns for the high-voltage winding.

Form factor

You may be interested in the reactions taking place inside a transformer. An alternating voltage impressed across the primary winding of the transformer in Fig. 342 causes an alternating current to flow in this winding. This current in turn sets up an *alternating flux* through the iron core. This alternating flux induces an alternating voltage in both the primary and secondary windings. In the case of the primary winding this induced voltage is directly opposed to the impressed voltage and acts to limit the current in the primary circuit. If the secondary circuit is open (that is, without load), the induced voltage in the primary will be nearly equal and also opposite in direction to the applied voltage so that very little current

*Transformer
action*

flows in the primary. This primary current is just sufficient to maintain the flux in the iron core and supply the losses. This current is limited by the resistance and the reactance of the primary winding.

*Control of
current needs*

When a load (resistance) is placed on the secondary coil, a current flows having a magnitude determined by the load and the voltage across the secondary coil. The action of this load current upon the primary side is of special interest. The current flowing in the secondary side will also tend to produce a magnetic flux in the iron core, this time in a direction opposite to the flux produced by the primary coil. The effect of this opposing action will be some reduction in the *resulting flux*.

This reduction in flux lowers the induced voltage in both the primary and secondary windings. On the secondary side it lowers the load voltage slightly, while on the primary side it permits more current to flow until the power input in the primary rises to compensate for the secondary load plus the losses in the transformer. Thus the change in magnetic flux causes the transformer to act like a *valve* to permit electric energy to flow through it.

*Transformer
efficiency*

The efficiency of a well-designed transformer is inherently high because the *losses are small*. The principal losses are heat losses due to the current through windings and certain losses in the iron which appear as heat. Naturally there are no friction or air-resistance losses such as occur in rotating machines. Efficiencies of power transformers vary from 90 to 99 per cent.

*Everyday use of
transformers*

The transformer is one of the most common and useful electrical devices known. It has made possible the transmission and distribution of alternating currents at relatively high voltages (up to 287,000) and their use at low safe voltages in the office, factory, store, and home.

Problems

1. The flux in the iron core of a transformer changes from 0 to 1,000,000 in $\frac{1}{240}$ sec. (60 cycles). Assuming that the rate of change is constant, what average voltage will be induced in each turn of wire on transformer coils? *Ans.* 2.4 volts.

2. If 1000 turns of wire are placed in series on one coil (primary) of the core in problem 1 what voltage should be applied? *Ans.* 2400 volts.

3. How many turns should be placed on a secondary coil to deliver 120 volts to a residence? *Ans.* 50 turns.

4. If the input to the primary of a transformer is 5210 watts and the delivered output is 5000 watts, what is the efficiency of the device (efficiency is watts output divided by watts input)? *Ans.* 0.96, or 96%.

ELECTRONICS

In 1883, Thomas Edison discovered that the region surrounding a hot body is a conductor of electricity. Edison placed a metallic plate in a carbon lamp bulb and connected it in series with a sensitive meter, as shown in Fig. 343. He observed that when the

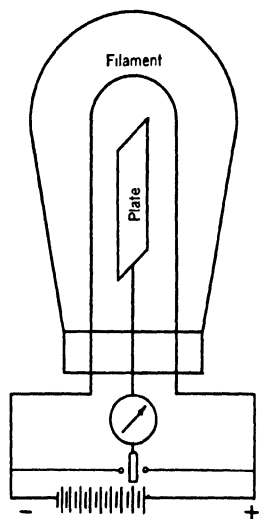


FIG. 343. EDISON EFFECT

plate was connected to the positive side of the lamp filament a current was indicated, but when connected to the negative side there was no deflection of the sensitive meter. This phenomenon came to be called the *Edison effect*. Edison did not attempt to explain this discovery, but other scientists studied the phenomenon a few years later and developed theories and made further discoveries which have culminated in the great modern achievements in the field of electronics.

Electrons are held to the substance of which they are a part by a force. This force or affinity is so large that few if any electrons ever escape under ordinary conditions. The work which must be done in

Electron affinity

removing an electron is called the *work function* of a substance. It is also known as the electron affinity of a substance.

Electron affinity may be overcome and electrons removed from solids in five different ways. These ways are termed: (1) high field emission, (2) secondary emission, (3) photoelectric emission, (4) radioactive disintegration, and (5) thermionic emission. When a metal is cold or at ordinary room temperature it is very difficult to overcome electron affinity. If an extremely high electric field or voltage is produced around a cold metal, it is possible to pull electrons out. The field required is of the order of 1,000,000 volts per centimeter. This extremely high field is difficult to produce with the result that *high field emission* is limited to rare experimental use. If an electron is attracted through space to a plate by a voltage, it usually acquires a velocity sufficient to splash out electrons when it strikes the plate. This splashing-out process is called *secondary emission*.

Electrons from solids

Electron affinity varies with different substances and is low for

some elements like sodium, barium, cesium, and strontium. Under suitable conditions the energy in light may be sufficient to remove a few electrons from cold substances. This form of removal is called *photoelectric emission* and is the basis of action of photoelectric tubes. Radioactive materials such as radium go through a slow process of disintegration during which the highly complex compound or element breaks down into simpler atomic structures. In the course of this process of *radioactive disintegration*, electrons called *beta rays* are ejected into space.

*Brownian
movement*

In 1827, Robert Brown discovered that minute particles of dead matter in suspension in liquids can be seen in a high-powered microscope to perform irregular wiggling motions suggesting "life." This phenomenon came to be known as the *Brownian movement* and was explained fifty years later. These movements are due to the continued bombardment of the inanimate particles by the thermal agitation of the molecules of the liquid. The Brownian movement is proof that constant thermal agitation of molecules exists in all gases, liquids, and solids when above the temperature of absolute zero. This thermal agitation increases with the rise in temperature and serves to explain many physical phenomena including *thermionic emission* of electrons.

Evaporation

The thermal agitation or molecular movement at the surface of liquids causes a bombardment of the surrounding medium. If the medium happens to be a gas, many of the molecules will "jump" out of the liquid into the gas. Some of these ejected molecules return to the liquid while others remain in the gas. As the temperature of the liquid is raised, the *thermal agitation* increases, the number of emitted molecules rises, and the distance which they jump becomes greater. Thus more of the molecules remain in the gas. This is the phenomenon of evaporation of liquids.

*Metallic
vapors*

In solids, and especially in metals, the molecules are very close together and their thermal movement is greatly restricted. However, they do move about positions of equilibrium. As the temperature of the solid is raised, the thermal agitation of its molecules rises and as the substance approaches the melting point, some molecules do leave the metal never to return. This is illustrated by the gradual blackening of the bulb of an incandescent lamp throughout its life. This blackening of the inside of the bulb is due to a deposit of tungsten that has evaporated from the tungsten filament.

In a manner analogous to the preceding explanation, the electrons in a solid may be ejected from that substance due to thermal agitation. The electrons are closely bound to the atoms of which they are a part by electron affinity at ordinary temperatures. But with the rise in temperature the molecules will move more rapidly and through a greater range. The electrons in turn move faster in their orbital motion with the rise in temperature. And with this rise in temperature, a point will be found where a few electrons have a sufficient velocity to escape from the surface. This process may be thought of as a "boiling out" or evaporation of electrons.

Thermionic emission

At the instant of escape, the electron has a certain initial velocity. As the electron (negative charge) emerges from the surface it leaves behind its mate (positive charge) which *attracts the electron back toward the surface*. Then as the electron moves out into space, the force of attraction of the positive charge remaining in the hot solid overcomes its initial velocity and it is pulled back into the solid. Thus, unless some other forces enter into the picture, the electron never gets very far from the heated solid or cathode.

Recapture of electrons

ELECTRON-TUBE OPERATION

Let it be assumed that a filament or cylindrical conductor of tungsten is placed in a vacuum or in any space filled with an inert gas so that oxidization cannot take place. With a rise in the temperature of the filament (produced by an electric current), a point will be found where the thermal motion of the molecules and electrons becomes great enough so that a number of electrons are thrown out into space, as illustrated in part *a* of Fig. 344.

Filament emission

A further rise in temperature of the filament will be accompanied by an increase in the *kinetic energy* of the electrons and an increase in the number and the initial

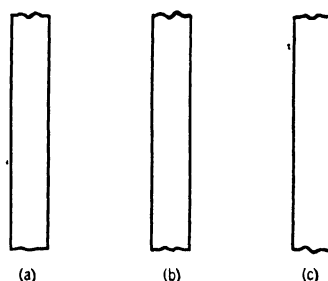
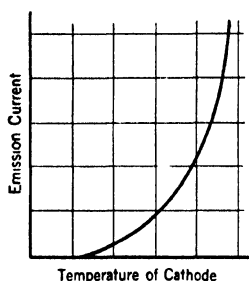


FIG. 344. ELECTRON EMISSION SURROUNDING A HOT FILAMENT

velocity of those emitted. This will result in an atmosphere of electrons around the filament as depicted in part *b* of Fig. 344. A still higher temperature of the filament will result in a greater agitation and velocity of electrons so that the cloud will thicken and move out to the condition of part *c*, Fig. 344.

Electron
activity

It should be understood that Fig. 344 represents an isolated filament and the surrounding space magnified several times. Many electrons emitted from the filament never travel over 0.01 mm.

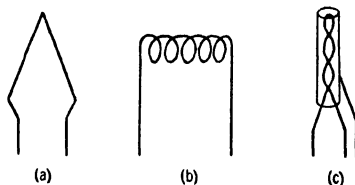


Cathodes

FIG. 345. VARIATION OF EMISSION TEMPERATURE

before returning and very few go farther than 1.5 mm. from the filament. The change of emission with temperature is illustrated by Fig. 345. This shows that the rise in emission is cumulative and becomes very rapid as the melting point of a metal is approached. But, obviously, there is some safe temperature which must be observed in the operation of any emitter.

The electron-emitting element of the electron tube is called the *cathode*. The *hot cathode type* uses the principle of thermal emission described in the preceding discussion. Hot cathodes are constructed in a vast number of shapes, sizes, and materials. Two general types are illustrated schematically in Fig. 346. Parts *a* and *b* illustrate two forms of the directly heated cathodes. Type *a* will be used as a symbol for the cathode in all drawings. Part *c* shows the indirectly heated type where a cylinder or sleeve covered by some emitting material is heated by a tungsten resistance or heater on the inside of the sleeve.



The latter form permits the use of alternating current for heating the cathode, yet serves to insulate the alternating current from the vital circuits of the electron tube and thus prevents any "hum" in the output.

FIG. 346. TYPES OF HOT CATHODES

Cathode
materials

The three active materials used in the construction of hot cathode emitters are tungsten, thoriated tungsten, and oxide coating. Tungsten has a high melting point and is tough and durable so it withstands bombardment of ions at high voltages. It is used for large tubes where a voltage of 4000 and higher is employed. The operating temperature of tungsten cathodes is about 2400° K. (brilliant white heat). The thoriated tungsten consists of tungsten in which thorium oxide is embedded. When heated, the thorium oxide is reduced and the thorium atoms diffuse and come to the surface forming a skin one atom thick. Since thorium has a low

electron affinity, the electron emission is increased about 100 times that for pure tungsten. This material is used in medium-capacity tubes for voltages in the approximate range of 600 to 4000 and is operated at a temperature of about 2100°K . (yellow heat).

Oxide-coated cathodes consist of tungsten or some other metal *Coated cathode* base carrying a coating of barium or strontium oxide. These cathodes operate under a temperature range of 900 to 1200°K . (dull red heat). This low temperature is adapted to the heater type of cathode and requires a low energy input for heating the cathode. The oxide-coated cathode is used for nearly all low-capacity and low-voltage tubes, such as those in receiving sets.

GAS-FILLED TUBES

Many electron tubes contain gas and part of the conduction in *Gas ions* these tubes is by means of *ions* of gas serving as carriers. A positive gas ion is an atom which has lost one electron, and hence carries a positive charge. A negative ion can be an atom which has gained one electron and has a negative charge. A negative ion may also be an electron. In general, negative ions *are* electrons.

Gaseous conduction is a very simple process. A single positive *Gaseous conduction* ion and a negative ion are shown in the tube of Fig. 347. When a difference of potential is applied to the two electrodes as shown,

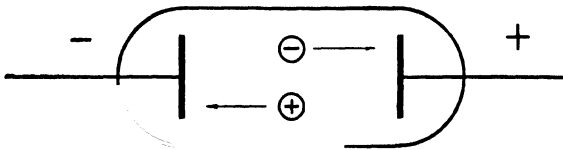


FIG. 347. GASEOUS CONDUCTION WITH IONS

a logical thing occurs. The positive ion is attracted to the negative electrode where it takes on an electron and becomes a neutral atom. The neutral atom then drifts back into space. In the meantime the negative ion (an electron) is attracted to the positive electrode where it gives itself up. The foregoing process has served to remove an electron from the negative plate and to deposit an electron on the positive plate. This is equivalent to a transfer of an electron from one plate to the other, and a movement of an electron is an electric current. Multiply this simple process millions of times per second and gaseous conduction is the result.

Gas ions may be produced in three ways. *Heat* applied to a gas may cause some of the atoms to separate into ions. *X-rays* passed

*Ionization by
collision*

through a gas will cause the formation of ions. The more common process of producing ions where gaseous conduction takes place is by *collision*. In the process of gaseous conduction considered in the preceding paragraph, the positive ion and the electron in moving to the electrodes will travel at an accelerated pace. While en route to the electrodes either one is likely to collide with neutral molecules and atoms of gas which are present. (See Fig. 348.) If any

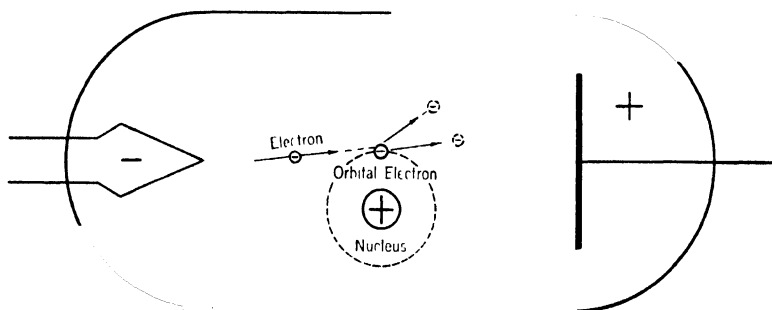


FIG. 348. IONIZATION BY COLLISION

of these collisions is sufficiently violent, an electron in the outer orbit of some atom will be knocked free from it. Thus a new positive and a new negative ion have been formed. Again, multiply these ionizing collisions millions of times and a condition exists where ionization by collision and gaseous conduction may be self-sustaining.

*Light by ion
bombardment*

During the process of ionization by collision, many collisions will occur between ions and molecules wherein the velocities and energies involved are insufficient to overcome electron affinity and produce ionization. Many of these collisions may have sufficient force to remove an electron from its orbit around an atom and set it in an unstable condition. This process of temporary removal from the orbit gives energy to the electron. Shortly after this happening, this electron will give up its unstable condition and return to its original orbit or status. In order to make this return, the energy absorbed from the collision must be released.

The released energy takes the form of a wave motion and may appear as *light visible to the eye*. The color of the light depends on the gas involved. This principle of light production as a by-product of gas ionization is utilized in many electric-lighting units such as neon, mercury-vapor, and sodium lamps.

THE DIODE OR TWO-ELECTRODE TUBE

The early experiment of Edison known as the "Edison effect" showed that electrons would pass from a hot filament or *cathode* to a cold plate whenever the plate was at a higher potential than the filament. This property of a device which permits current to flow in one direction only is called *unilateral conductivity*. The unilateral conductivity of a tube having two electrodes, one hot and the other cold, was utilized by Fleming for the detection of high-frequency radio waves. This device, known as the "Fleming valve," was patented by him in 1905. The Fleming valve was important in the early application of radio telegraphy and it was one of the important developments in the history of electronics.

*One-way
conduction*

A diode is a two-electrode tube containing a cathode and a second electrode called an *anode*. The anode is always the electron-collecting electrode in a tube. In the first tubes built the anode was constructed in the form of a plate or sometimes two little plates connected in parallel and placed on each side of a filament type of cathode. We still use the term *plate* for the anode even though it is often of a cylindrical form as shown in Fig. 349. Anodes may be made of metal or graphite, the latter being common for rectifying tubes.

Diodes

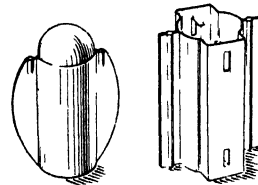


FIG. 349. ANODES

The diode utilizes the principle of unilateral conductivity in its operation. In the high-vacuum tube of Fig. 350, the cathode is

Rectification

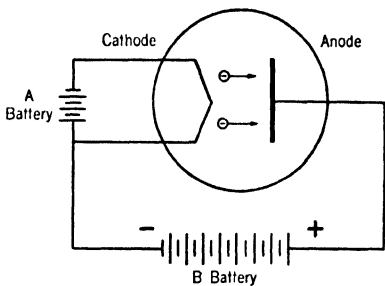


FIG. 350. UNILATERAL CONDUCTIVITY IN A VACUUM

heated to its normal operating temperature so that a cloud of electrons is being emitted. The anode is maintained at a positive potential by the battery *B*. This positive potential attracts the electrons being emitted by the cathode, and a large part of them pass to the anode continuously and then return to the cathode via the battery *B*. If the

connections to the battery *B* are reversed so that the negative terminal of the battery is connected to the anode of the diode, no action will take place in the tube. Now the anode is negatively

*Changing
alternating
current to
direct current*

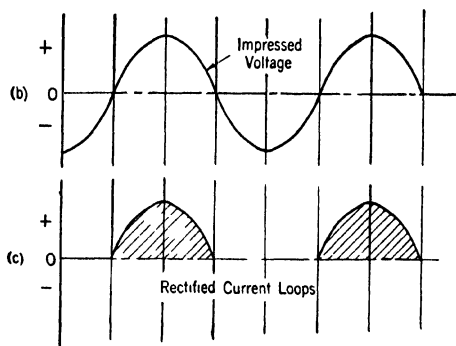
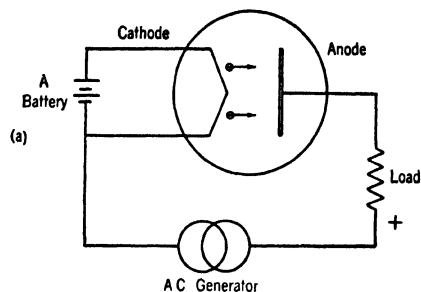


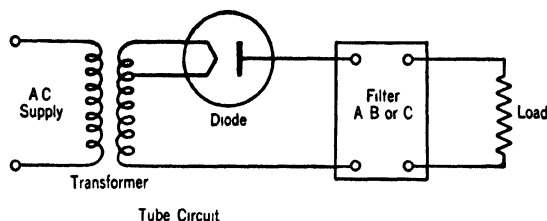
FIG 351 RECTIFICATION OF AN ALTERNATING CURRENT

charged and it will repel the negative electrons; hence the electron movement is blocked.

If we substitute an alternating current having a sine form of voltage wave for the battery as in Fig. 351, we obtain current flowing in only one direction. This is known as "rectification." Thus during the period when the anode of the diode is made positive, electron current flows in the tube, but when it is negative the tube blocks any electron movement. The resulting one-directional current loops are shown in the lower part of Fig. 351.

*Diode for
rectification*

This *rectifying action* of a diode is a very valuable property. The unidirectional pulses of current from a rectifier are satisfactory for charging a storage battery. However, this pulsating current and



SIMPLE RECTIFIER CIRCUIT WITH FILTER

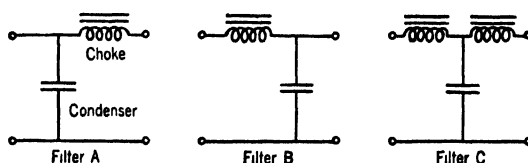


FIG. 352. THREE TYPES OF FILTERS

voltage is not suitable where a steady direct current is desired. The addition of a *filter* will smooth out these pulses.

Filters are made up of combinations of condensers and chokes. *Filters* Some simple filters are shown in Fig. 352. The condenser serves to store and release a charge, while the choke opposes changes of current. The proper combination of condensers and chokes will give the desired degree of smoothness.

The number of electrons attracted to the anode depends upon a number of factors, one of which is the negative space charge. A consideration of the region between the cathode and anode (Fig. 353) shows a cloud of electrons around the cathode with

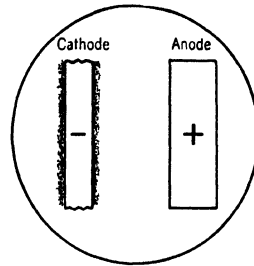


FIG. 353. NEGATIVE SPACE CHARGE

a very high concentration close to the filament. This intense concentration of electrons (negative charges) produces a strong negative charge in the space surrounding the cathode. This *negative space charge repels the electrons* which are ejected from the cathode until they have broken through it; this charge also opposes the attraction of the positively charged anode. Thus the negative space charge is an important factor in controlling the number of electrons passing to the anode.

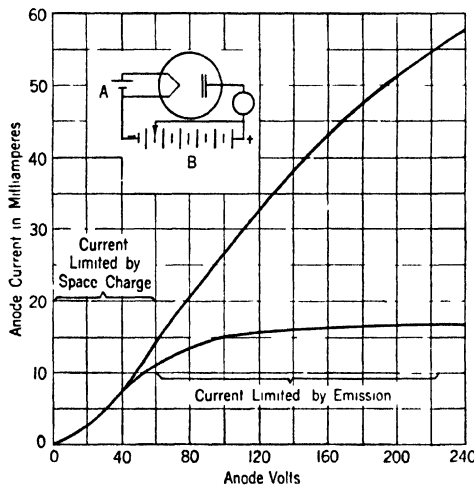


FIG. 354. CHARACTERISTIC OF A VACUUM DIODE

negative space charge limits the electron movement as just explained. Higher positive plate potentials will reach farther back into the electron cloud, thus overcoming the negative space charge. At still higher anode potentials nearly all of the electrons emitted may be drawn to the anode and the curve bends over. This curve

The effect of the anode potential upon the electron current flowing to the anode of a vacuum tube is shown in Fig. 354.

At low anode voltage the

Chart of tube characteristics

represents the important characteristic of the vacuum diode. Anode current is expressed in milliamperes ($\frac{1}{1000}$ amp.). The vacuum diode is used as a detector in radio receivers and as a rectifier for high-voltage applications.

*Gaseous
diodes*

The addition of an inert gas under low pressure to a vacuum diode greatly increases the current flow in the device. This increase is due to two factors. First, the gas becomes ionized and the electron transfer due to *gaseous conduction* is added to the pure electron flow. Second, the presence of the positive gas ions serves to neutralize partially the negative space charge. This action greatly increases the number of *emitted electrons* which are attracted by the anode. Thus the addition of gas reduces the resistance of the tube and lowers the voltage drop from cathode to anode.

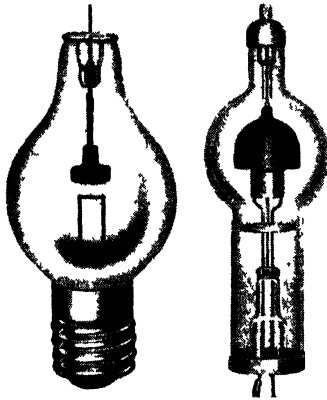


FIG. 355. RECTIFIER TUBES USED FOR CHARGING STORAGE BATTERIES

*Practical uses
of gas tubes*

Two gaseous rectifier diodes are shown in Fig. 355. The one on the left is filled with argon under 2 lb. of pressure. It is used for charging storage batteries and is made in current-carrying capacities varying from $\frac{1}{2}$ to 10 amp. The diode on the right of Fig. 355 contains a small quantity of mercury, and hence operates as a mercury-vapor rectifier. It is used for charging storage batteries and for small direct-current applications. It is built in ratings up to 30 amp.

THE TRIODE OR THREE-ELECTRODE TUBE

*Grid added to
tube*

In 1907, DeForest added a third electrode to the Fleming valve (diode) and called the new device the *audion*. The third electrode, called the *grid*, was placed between the cathode and anode as illustrated in Fig. 356. The value of the grid lay in its ability to control the current between the cathode and the anode. This invention constituted one of the most important developments of the twentieth century.

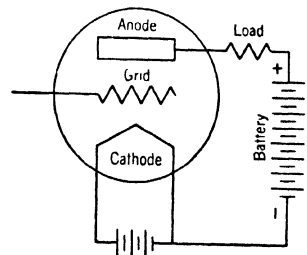


FIG. 356. CIRCUIT OF A TRIODE

It has served to extend the field of communication by wire across continents, and it has made possible radio communication around the world. And now it is revolutionizing the uses of electric power in the industrial world.

The modern three-electrode tube, known as the "triode," uses cathodes and anodes like the diode.

The grid is usually a helically wound coil of fine wire interposed between the cathode and the anode. The grid of a triode is illustrated in Fig. 357.

The grid of the three-electrode vacuum tube functions by a change of the charge residing on the grid. This change in charge (and potential) serves to control the electron stream between the cathode and the anode. Since the grid is very close to the cathode, its changes in potential are more effective than like changes on the anode. When the grid is negative it repels those electrons which pass through the negative space charge, and when it is positive it helps electrons to get through the negative space charge and thus produces a large increase in electron flow.

Grid construction

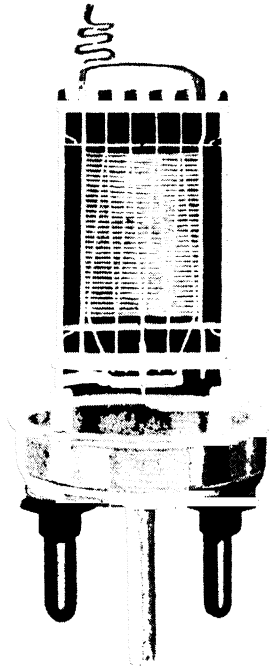


FIG. 357 TRIODE GRID

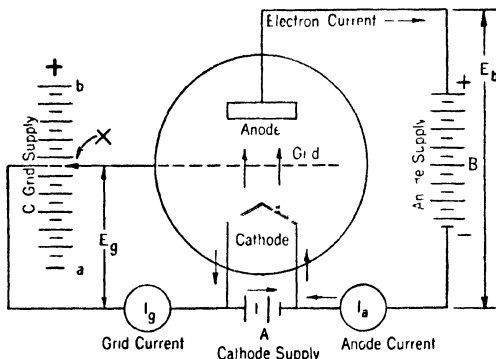


FIG. 358. CIRCUIT FOR VARYING VOLTAGE ON GRID OF A TRIODE

The triode requires 3 sources of potential difference for operation

Grid action

The action of the grid is illustrated by the circuit and curve of Fig. 358 and Fig. 359. The potential supplied to the anode is called the *B supply*, that for the cathode the *A supply*, and that for the grid the *C supply*. As the point *X* is moved from *a* to *b*, the grid goes through a succession of values from strongly negative through zero to strongly positive with respect to the cathode. This variation of grid voltage is indicated by the range of *a* to *b* on the horizontal base of the curve (Fig. 359).

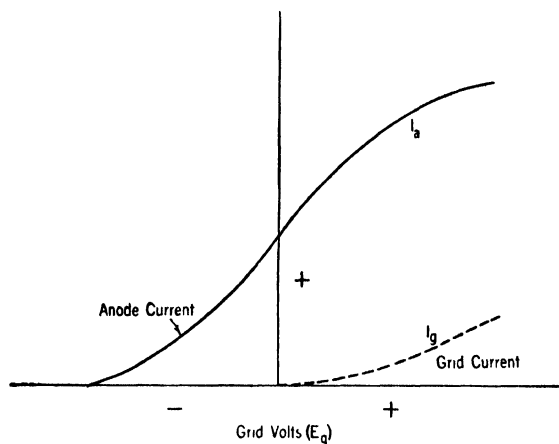


FIG. 359. CHARACTERISTIC OF A VACUUM TRIODE

Cutoff point

With anode-to-cathode potential held constant, the cathode-to-anode electron movement increases as shown by the curve marked I_a . At a certain negative grid voltage, say at *a*, the grid prevents any electrons from reaching the anode, and that status is called the *cutoff point*. Then, as the grid is made less negative, some electrons pass the grid; after the grid becomes positive it helps the electrons pass to the anode or plate. After the grid becomes positive it will attract a few electrons to itself, and this grid current is shown by the dotted curve I_g of Fig. 359.

Uses of triodes

The uses of the vacuum triode in communication may be classified as follows: (1) *amplifier*; (2) *oscillator* or alternating-current generator; (3) *modulator*; and (4) *detector*. The particular class of work performed by the triode does not depend on the tube itself but upon the circuit to which it is connected. Hence, a single tube might be connected in circuits for serving as amplifier, oscillator, modulator, or detector. In practice certain types of tubes are chosen for the different applications, but this selection is not

determined by any difference in their inherent theory of operation.

A simple amplifier circuit using a triode is given in Fig. 360. *Amplifier*
The anode is given a positive, B , supply and the grid is made

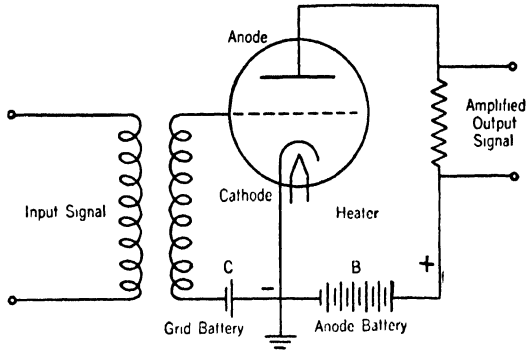
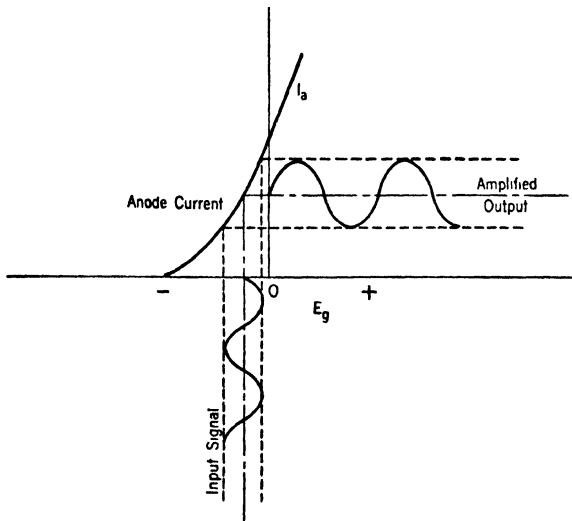


FIG. 360. SIMPLE AMPLIFIER CIRCUIT

normally negative by the battery, C . An alternating-current input voltage (sine wave) of low value is applied across the transformer and between the cathode and grid. This is called the *input* to the tube. The resulting changes in the potential on the grid cause similar



Input signal is a voltage whereas the amplified output is a current

FIG. 361. AMPLIFYING ACTION IN A TRIODE

variations in the current flowing in the anode or output circuit. This action is illustrated in the curves of Fig. 361.

It should be noted that the input signal is a small change in *Volts to*
voltage, whereas the amplified *output* is a *current*. The input is a volt- *amperes*

age change without current and represents no power. The amplified current output may be utilized in different ways. For example, if you desire an amplified voltage to apply to a second tube, the current is passed through a high resistance giving a large alternating voltage (RI) drop. If you desire an amplified power, the current may be passed directly (or via a transformer) into a loudspeaker or other form of load where the current squared times the resistance of the load (I^2R) equals watts or power.

Oscillator

The preceding paragraph explained how a small signal voltage change applied to the grid of a triode would cause an amplification of that signal in the anode circuit. Under normal conditions the amplified signal will have the same frequency and wave form as the input signal. A little consideration will show that if a part of the amplified signal voltage could be applied or "fed back" to the grid with the proper time and phase relationship, the tube would "tickle itself" or stimulate itself so as to give a continuous *self-generated signal* whose frequency would depend upon the characteristics of the tube and of the circuit.

Oscillator circuit

A simple circuit for accomplishing this result is given in Fig. 362. It should be noted that any changes in the anode circuit are fed

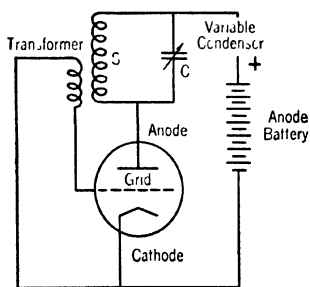


FIG. 362. SIMPLE OSCILLATOR CIRCUIT

back to the grid (input) through the transformer. The frequency at which this tube and circuit will oscillate depends on the capacity of the condenser, C , and the choking action of the transformer winding, S . Any change in the value of the condenser, C , will vary the frequency of the oscillation or of the generated signal. While the power produced by an oscillator circuit is small, the alternating voltage generated may be increased in magnitude by the introduction of a triode amplifier plus a suitable amplifying circuit. Oscillators are used for producing high-frequency currents called *carriers* which will be discussed below.

TRANSMISSION AND RECEPTION

Carriers

Radio communication is a process of sending electromagnetic waves out into space and receiving them at some distant point. If a 60-cycle alternating voltage is applied to a conductor or antenna,

it will set up a changing electric field around the conductor, and this field will move out in waves as shown in Fig. 363. However, for 60 cycles the waves would not carry much energy, and the signal would not have enough strength to travel far.

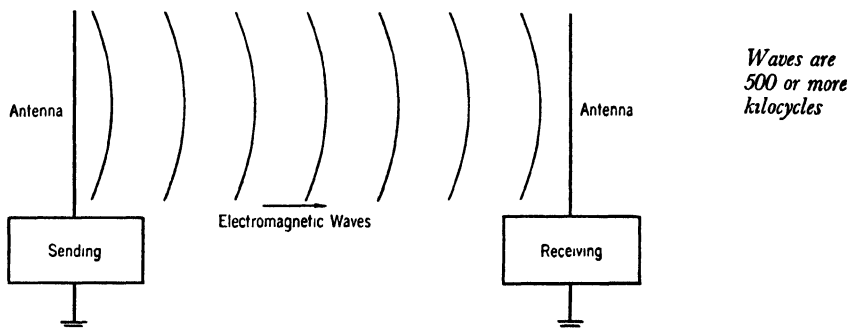


FIG. 363. ILLUSTRATING COMMUNICATION VIA RADIO WAVES

If an alternating current of 500,000 cycles or higher is applied to a conductor of suitable position, length, and so on, electromagnetic waves having relatively large energy are produced, and they will travel far with sufficient signal strength for satisfactory reception. Hence, if it is desired to transmit a signal of 60 cycles via radio waves, it is necessary to use a *high-frequency carrier* and then to impress the 60-cycle signal into the carrier in some manner.

This process of impressing a signal upon a carrier is known as “modulation” and is illustrated graphically in Fig. 364. Part *a* represents the high-frequency carrier, part *b* the low-frequency signal (a sine wave), and *c* the modulated carrier. The modulated carrier has the necessary high frequency for radio transmission, and its amplitude has been modified by the signal being carried.

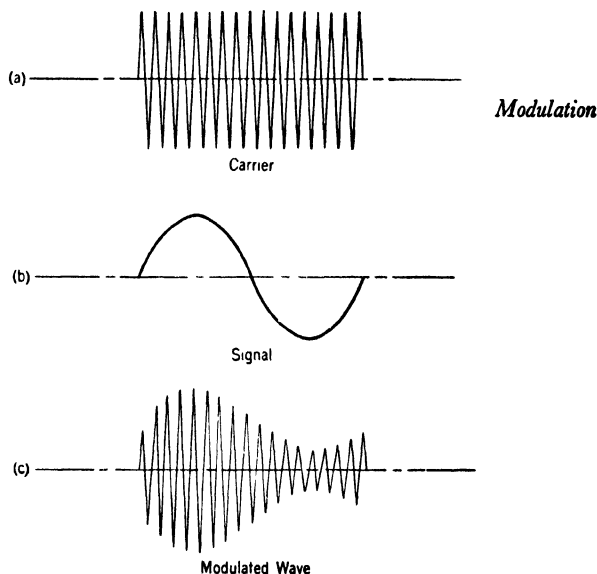


FIG. 364. ILLUSTRATING MODULATION OF A CARRIER BY A SIGNAL WAVE

A simple circuit showing the use of a triode for modulation is given in Fig. 365. The signal is a voice current produced by a microphone. The carrier current is of high frequency produced by an oscillator (not shown). The carrier and the signal are connected in series and fed into the grid-to-cathode circuit. The output current appears in the anode-cathode circuit and is the carrier which has been molded or modulated by the voice signal.

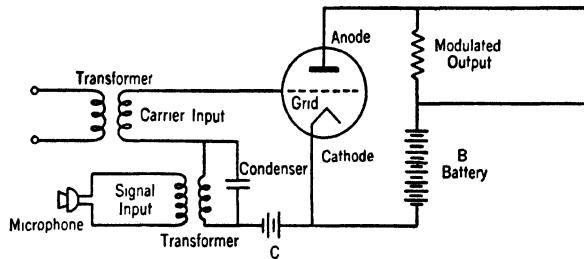


FIG. 365. CIRCUIT FOR MODULATING CARRIER WAVES

Detection

Detection or demodulation is the reverse process of modulation. It involves the detection or separation of a transmitted signal from the modulated carrier. Triodes are seldom used for detection, but they will serve this purpose if desired. A simple detector circuit using a triode is shown in Fig. 366. The modulated carrier is fed into the grid-cathode circuit. This modulated carrier consists of two parts, a high-frequency carrier and a low-frequency signal.

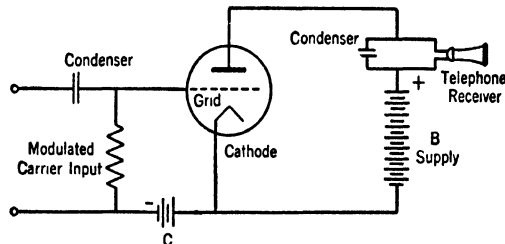


FIG. 366. CIRCUIT FOR DETECTION USING A TRIODE

Filters

When the modulated carrier is amplified in the anode circuit, it is passed through a parallel circuit consisting of a condenser and a telephone receiver. This parallel circuit, when properly designed, acts as a *filter device* in such a way as to pass the high-frequency part of the output through the condenser and to permit the low-frequency signal to pass through the receiver. The receiver delivers a sound that is a close replica of the original sound which produced the voice signal current.

The pentode is a five-electrode tube. It is a triode with two *Pentode* additional grids placed between the regular or control grid and the anode or plate. The pentode does the same things as the triode but performs them better. The first additional grid in the pentode is a *screen grid*. It carries a potential somewhat lower than the

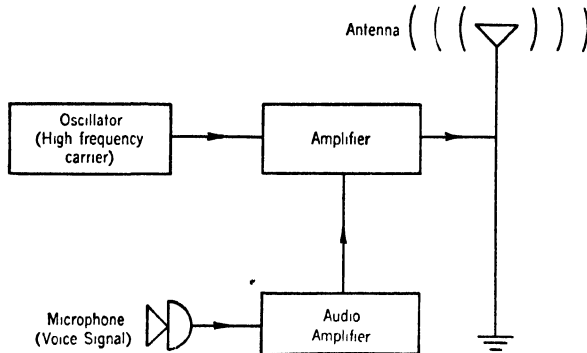


FIG. 367. SCHEMATIC DIAGRAM OF TRANSMISSION VIA RADIO

anode and its function is to set up a constant potential in the space it occupies. This constant space potential assures a constant attraction for the electrons from the cathode, independent of any fluctuations in the anode voltage resulting from changes in load. The second grid added in the pentode is the *suppressor grid*. It is connected to the cathode and ground and it serves to prevent electrons of secondary emission from the anode reaching the screen grid. Pentodes are generally used as amplifiers.

Radio communication is conducted by electromagnetic waves propagated through space from a transmitting station to a receiving station as suggested in Fig. 363. The transmitting or sending station embodies certain electronic units connected together as shown in the block diagram of Fig. 367. A high-frequency carrier current is generated in the oscillator, a voice current or signal is produced by a microphone and amplified in the audio amplifier. The amplified signal and the carrier are fed into an amplifier where both modulation and amplification take place. The modulated output or carrier is fed into an antenna. A very simple circuit using triodes for producing the steps as outlined is given in Fig. 368. The anode battery supply for all tubes is shown at *B*, and the general operation for the component parts of this complete radio transmitter station follows the theory given in the preceding discussion.

*Radio trans-
mitter*

Radio receiving

The receiving station of Fig. 363 must pick up the transmitted radio wave and transform it into sound. The steps for producing the desired result are indicated in the block diagram of Fig. 369.

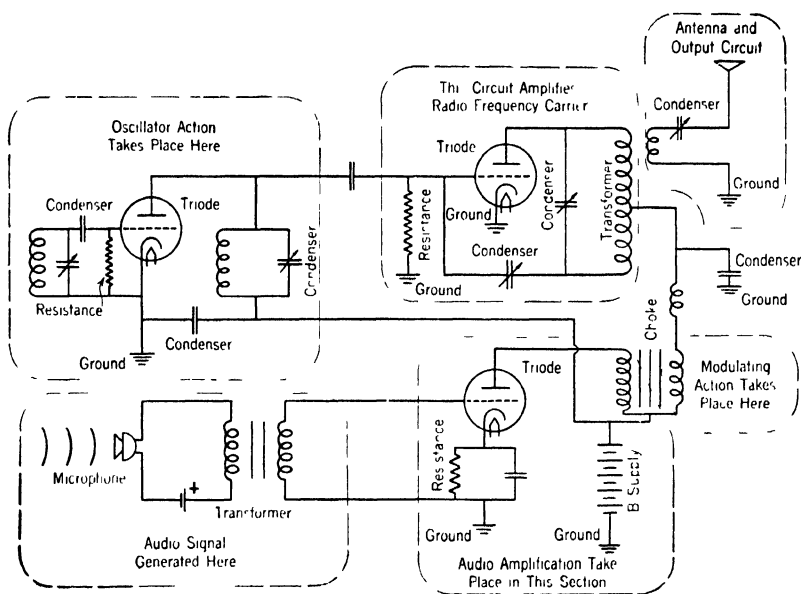


FIG. 368. SIMPLE RADIO TRANSMITTER CIRCUIT

The weak incoming modulated carrier arriving at the receiving antenna is first passed through an amplifier. Then the amplified signal is passed through a detector which extracts the *audio signal* from the modulated carrier. The detected signal is next amplified and then fed into a loud-speaker or receiver which reproduces the

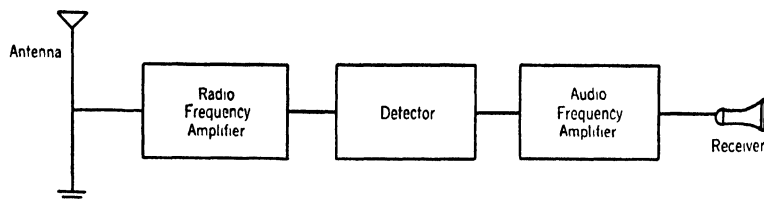


FIG. 369. SCHEMATIC DIAGRAM FOR RADIO RECEPTION

sound originally impressed on the microphone in the transmitting or sending station.

Communication engineering

A simple receiver circuit using two triodes and one diode (detector) is given in Fig. 370. The function of each part of the circuit is given on the circuit diagram, and the action of the tubes and circuits of each part follows the general theory of the preced-

ing discussion. Space does not permit a complete discussion of the theory of action of the transmitter and receiving circuits. A more thorough treatment of mathematics, physics, electronics, and alternating current is necessary for the proper conception of these and

*Science
background
needed*

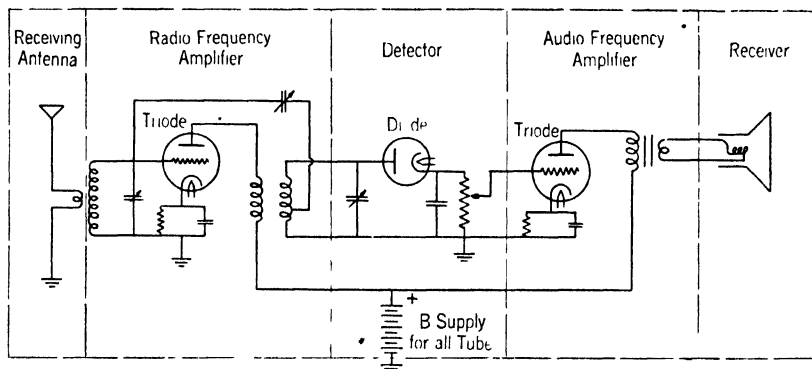


FIG. 370. SIMPLE RADIO RECEIVER CIRCUIT

the more intricate devices and circuits used in radio communication. This knowledge is acquired in the various studies leading to the work of the professional electrical engineer.

PHOTOTUBE

It was suggested previously that electrons may be removed from cold solids by the energy of light rays. Devices which utilize this property of light are called *phototubes*. The construction and simple circuit of a phototube are shown in Fig. 371. The cathode or emitter is made of some light-sensitive substance such as sodium, potassium, barium, or cesium on a metallic subbase and enclosed in a vacuum tube or a tube with inert gas under low pressure. When light impinges upon such a cathode, it gives up its energy to a few electrons in such a way that they are splashed off the surface. These

*Phototube
action*

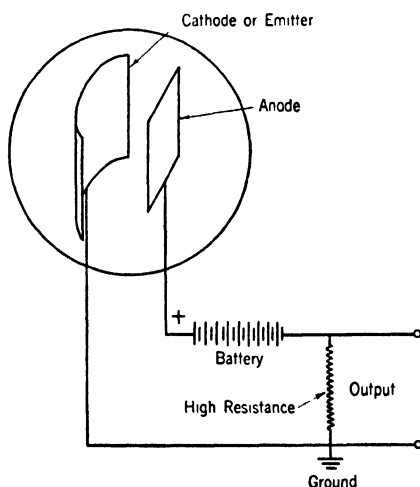


FIG. 371. CIRCUIT FOR A PHOTOTUBE

emitted electrons are then attracted by the positive potential (usually 90 volts) on the anode. After reaching the anode these electrons return via the battery and resistance to the cathode.

Electric eye

The current produced by the phototube is *directly proportional to the intensity of the light* falling on the cathode. Hence the device gives an accurate measurement of illumination. The magnitude of the current is very small, being only a few millionths of an ampere.



FIG. 372. PHOTO-TUBE

This minute current will produce a small drop in voltage across the series resistor R . This voltage change can be impressed across the grid of a triode and thus produce an amplified signal of sufficient strength to operate electromagnets (relays), etc. Through its ability to operate electromagnets, the phototube becomes a very useful control device. A phototube is shown in Fig. 372.

The phototube is often called the *electric eye* because of its ability to transform light changes into electric current. Among the applications of the phototube are (1) measurement of light intensity, (2) analysis of color of fabrics, (3) turning on and off artificial lights with changes in daylight outdoors, (4) counting the passing of people, cars, or packages, and (5) detecting the sound on film for sound motion pictures as will be explained below.

Sound on film

In motion-picture making the voice sounds on the movie set are picked up by microphones and transformed into voice currents. These voice currents control the dark and light area on a small sound track on the film. In reproducing the scene on the screen in the movie theater, a light shines through the narrow sound strip on the film upon a phototube. This light varies (changes) with the frequency of the voice and the phototube changes the light variation back into an electric current variation. This variation is amplified and passed into the loud-speakers behind the projection screens. Thus phototubes play a vital part in the dual transformation chain of *sound — electricity — light — shadow on film — thence shadow on film — light — electricity — sound*. Many other electron tubes play a vital part in the magic chain just cited.

SPECIAL ELECTRONIC DEVICES

Industrial tubes

In addition to the electron tubes that we have discussed, there exists a large number of special tubes and devices which are exceed-

ingly useful and very interesting in theory and application. The *thyatron* is a three-electrode tube containing gas and is probably the most valuable of all control devices for commercial and industrial applications of electricity. The *ignitron* is a three-electrode mercury-vapor tube which has wide application in the control of electric welding and for controlling electric-power rectification. The *magnetron* is capable of generating ultrahigh frequencies needed for radio aircraft detection through distance, darkness, smoke, and fog.

The *cathode-ray tube* is capable of producing pictures of alternating currents (cathode-ray oscillograph) and of reproducing pictures transmitted via television. X-ray tubes produce penetrating rays which permit man to see through opaque materials; they can also be used for the treatment of cancerous growth. Newly developed electron microscopes magnify fine particles 50,000 times and offer man the opportunity of new studies into the unknowns of nature. These and many other devices and applications make the study of electronics an interesting excursion into the latest developments in the scientific world. *Oscillograph*

Mechanics

THE EFFECTS OF FORCES

*How forces are
applied*

This section takes up a study technically called *analytic mechanics*. Of course, you need not be too concerned with this technical name unless perchance you wish to impress your friends with the extent of your vocabulary. No matter what you call it, this subject is a study of forces, the kind that makes an airplane fly, the action which a bump in the road produces on your automobile, the blow which gave you that black eye. Forces may be applied to an object, gently at first, then with gradually increasing intensity, they may act constantly without change, or they may be applied with a bump. In this study, we shall think of forces as unchanging, although the forces may change the instant after we analyze them. While we work a problem, we think of an object as being in a suspended state of change, if change is taking place, much as the diver who is held momentarily in mid-air in the movies.

When an engineer sets out to design an airplane, a machine, or the steel framework of an office building, he must first *estimate the magnitudes of the forces* on each part of the structure or machine. How big is the force on the bolts which attach the engine to the body (fuselage) of the plane? Before the size of the bolt can be decided on, this force must be known. What forces act on the framework of an office building because of the action of the wind, the weight of the building, or the snow on the roof? What is the maximum force on the bearing of the connecting rod in your automobile?

*Why materials
fail in service*

These forces must be known, or approximately known, else the engineer is working in the dark. If he is too much in the dark, or if he makes a serious error, a bridge may fall, a balcony collapse, or a connecting rod may fly through the crankcase of your car. Now do not get the notion that these questions are going to be answered completely in this book. That is too big an order. However, we can learn some of the principles which control the design of structures, airplanes, and machines, even if the actual design must be studied later.

HOW A FORCE IS REPRESENTED

As an illustration of a force, imagine a boxer delivering an uppercut to the chin. Since this is not your chin, you can make a scientific and objective analysis of the force. First, you note that this force will be directed sharply upward, that is, you see that the force has *direction*. Then you notice that it landed squarely on the chin, that is, you find that a force has a *point of application*. Finally, you may imagine the force to be a gentle tap or a heavy blow. In other words, you observe that a force has *magnitude*.

Definition of force

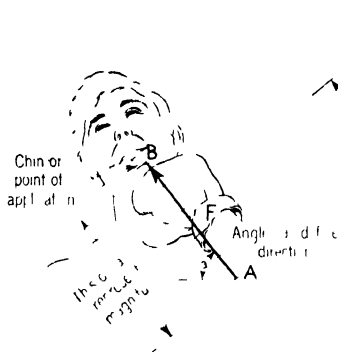


FIG. 373. FORCE OF A FRIENDLY UPPER CUT

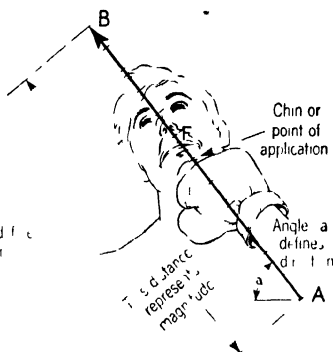


FIG. 374. FORCE OF PUGILIST'S UPPER CUT

Asking ourselves how we can represent something that has magnitude, point of application, and direction, we answer immediately, "By an arrow." Thus, in Fig. 373, the arrow AB represents a force completely. Its length is proportional to the magnitude of the force. The angle a , which the line of the arrow makes with the horizontal, determines the direction of the force. The point of application is designated, and may be any point on the line of the arrow, depending of course upon the position of the arrow. That is, the point of the arrow is not necessarily the point of application. Hence, we are actually concerned here only with the location of the direction line of the arrow, called the *line of action of the force*. As we see in Fig. 374, the arrow AB passes beyond the point of application, but it still is an accurate representation of the force.

The arrow in Fig. 373 is 5 units long. Each unit of length represents a predetermined number of pounds of force. If a unit of length represents 15 lb., the force in Fig. 373 is $15 \times 5 = 75$ lb. Similarly, the force arrow in Fig. 374, being 12 units long, represents $15 \times 12 = 180$ lb. to the same scale.

WHAT CAUSES A FORCE?

Piston pressure

Forces are always caused by the action of one body on another. The explosion of the fuel in the engine of your car results in a large pressure or force on the top of the piston exerted by the burned gases. This gas pressure exerting a force on the piston causes the piston to move downward and makes the engine run. (See Fig. 375 and its caption.) Observe that the force described in this paragraph exists because two bodies (the piston and the hot gases) are in actual contact. The location of the point of application of the force may be determined from the area over which the force acts, which is the top of the piston in this instance.

Force of gravity

Another force which is always present is the *force of gravity*, commonly called the weight of the body. This force exists, not because of the physical contact of the earth and the body, but because of the attraction of the earth and the body for each other.

Center of gravity

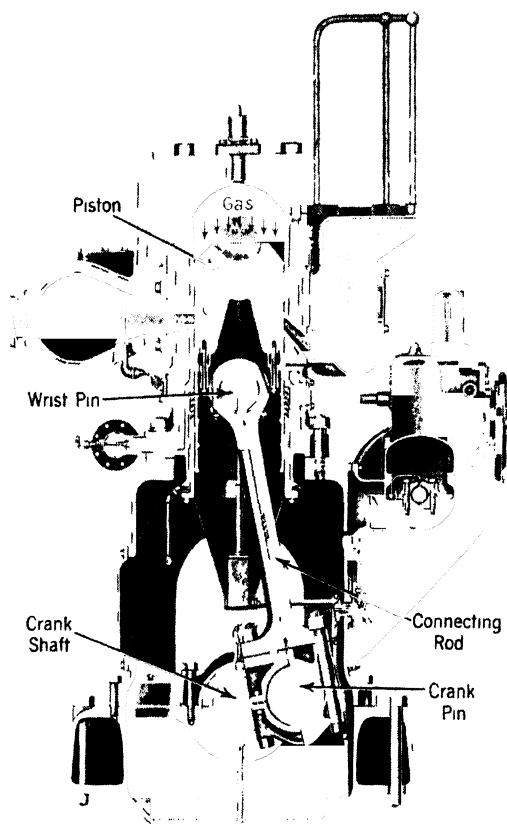
What is the location of the arrow that represents the force of gravity (weight)? The arrow which represents a force, by the way, is technically known as a "vector," and we might as well call a vector a *vector*.^{*} A force may act in any one of all directions, but one force, a vector quantity, can act only in one direction. But back to the question. The vector of the force of gravity passes through the *center of gravity*. Those readers who have some familiarity with the airplane may know of the center of gravity as the *c.g.* The location of the center of gravity or *c.g.* in an airplane is an important factor in its design. If the weight and control surfaces are not properly located with reference to the *c.g.*, the airplane will be unstable and dangerous, if not impossible, to fly. Automobiles are now safer machines, for one reason, because the *c.g.* is lower in modern automobiles than in older ones.

LOCATING THE CENTER OF GRAVITY

Balance and symmetry

The center of gravity may also be termed the *center of balance*. If you balance a stick on the edge of a knife, the *c.g.* of the stick will be exactly above the knife edge. Another criterion, and the one which we often rely upon, is that a plane of symmetry of a body will contain the center of gravity.

^{*} Force is thus a vector quantity, whereas you are more accustomed to thinking of so-called scalar quantities. Scalars are measured by simple numbers and have no property of direction. The area of a room is a scalar quantity.



Burning gas produces the downward force on the piston

Connecting rod crank shaft connection produces the rotation

Courtesy Fairbanks Morse and Company

FIG. 375. SECTION OF A DIESEL ENGINE

While this engine is not an automotive engine, the two possess many points of similarity. Common to both types are the crankshaft, crank pin, connecting rod, wrist pin, piston, and as mentioned in the text, the pressure or force of the gas on the piston during the power stroke. The fundamental difference between the automotive and Diesel engines is that the fuel in the automotive engine is ignited by an electric spark across the gap of the spark plug, while in the Diesel engine (as above), the fuel is ignited because the incoming gases are so highly compressed that they become hot enough to ignite the fuel without the use of a spark. As the crankshaft rotates, the center of the crank pin moves in the circle shown. This movement causes the piston to reciprocate or move up and down. A movement of the piston from top to bottom or *vice versa* is called a *stroke*.

The pressure of the gases on the piston results in a whole series of forces: on the wrist pin, the connecting rod, the crank pin, the crankshaft; and then more forces for members not shown, as, for example, on the shafts and gears of the transmission and differential, on the axles, and finally, frictional forces of the tires on the road, which make it possible for the car to move. It is interesting to observe that without friction it would be impossible to start the car when it is on a level plane.

Engine forces

Symmetry

The connecting rod of Fig. 376 has two planes of symmetry. Since their line of intersection (see page 193) passes lengthwise down the center of the rod, it is only necessary to balance the rod in one other position, either *a* or *b*, as shown in Fig. 376. The c.g. is located at the point where the vertical plane through the knife edge intersects the center line. Now, as stated before, the force of gravity always acts through this point, and it does not matter in what position the connecting rod is placed, as may be seen in Fig. 377.

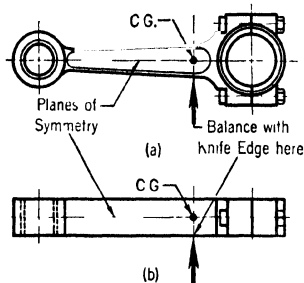


FIG. 376. C.G. OF A CONNECTING ROD

Center of balance

Suppose we drill a hole through the connecting rod at the c.g. in Fig. 376 *a* and support the rod on a smooth shaft in this hole.

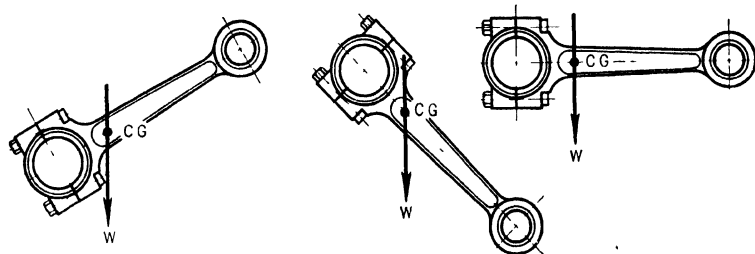


FIG. 377. WEIGHT ACTS THROUGH C.G.

No matter what the position of the connecting rod, the force of gravity acts vertically downward through the center of gravity.

We may then rotate the rod into any position and it will remain in balance as placed (Fig. 378).

That is, the c.g. is the center of balance.

Locating the c.g.

Knowing that all planes of symmetry of homogeneous bodies contain the c.g., we conclude that the c.g. of a sphere, cube, cylinder, or of any body symmetrical in relation to three planes, is at the geometric center of the body. We can use this knowledge in locating the force-of-gravity vectors of many structural and machine parts.

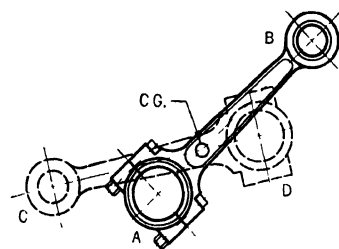


FIG. 378. CENTER OF BALANCE

If it is supported on an axis through the c.g., the connecting rod remains in any position in which it is placed. It will stay in the position *AB*, or in any other position *CD*.

PRINCIPLE OF THE LEVER

Archimedes (287–212 B.C.), who was one of the earliest and most famous engineers, once said something like this, “Give me a place to stand and a suitable pivot point and I can move the world.” He planned on using the principle of the lever, if anyone had called him on his boast. While we still see no practical way of actually moving the earth with a mechanical lever, the principle involved has been of incalculable assistance in technological developments.

Moving the earth

One of the simplest applications of the lever is a pair of balances, Fig. 379. In its elementary form, there will be a pivot at the point O

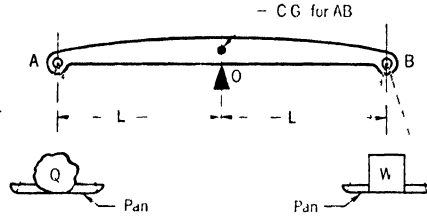


FIG. 379 SIMPLE WEIGHT SCALE

which is directly under the c.g. of the arm AB when the points A and B are on a horizontal line. An object Q in one pan may now be weighed by placing weights, W , in the other pan until the whole system balances with A and B in a horizontal plane. If a weight of $W = 1$ lb. balances Q , then Q weighs 1 lb.

However, suppose the pivot is moved off center as shown in Fig. 380, and, to simplify the discussion, suppose the arm AB and the pans weigh so little in comparison with the objects in the pans

Off-center balance

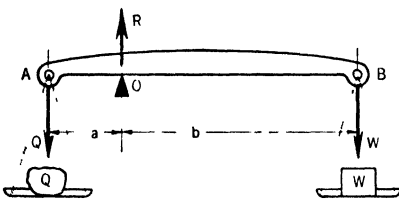


FIG. 380. UNSYMMETRICAL SCALE

that we may neglect their influences altogether. For a condition of balance under this arrangement, the weight W is not the same as the weight of Q , but, with knowledge of the dimensions a and b , the weight of Q may be computed.

The principle that Archimedes discovered was

(1) Weight $W \times$ distance $b =$ weight $Q \times$ distance a .

In technical terms, we say the same thing in this way,

Moment of W about $O =$ moment of Q about O .

Since W and Q are symbols for the forces of gravity and since a and b are distances from the pivot O to lines along which these forces act, we see that the *moment of a force* about a particular point

Moment of a force

is the magnitude of the force times the distance from that point to the line of action of the force. This distance, which is called the *moment arm*, is always measured in a direction perpendicular to the vector representing the force.

Definition

$$(2) \quad \text{Moment} = \text{force} \times \text{moment arm.}$$

EXAMPLE.

Let $a = 5$ in., $b = 10$ in., and $W = 2$ lb. How much does Q weigh (Fig. 380)?

Using equation 1, we have

$$(3) \quad Wb = Qa, \text{ or } 2 \times 10 = Q \times 5,$$

from which we obtain

$$(4) \quad Q = \frac{2 \times 10}{5} = 4 \text{ lb.}$$

which is the weight of the body Q .

In equation 3, we have $Wb = Qa$. Rearranging this equation into the form $W/Q = a/b$, we may generalize to obtain the following statement.

Archimedes' principle

The forces on a lever are inversely proportional to their distances from the pivot point, $W:Q = a:b$.

Remember that the distance to a force is measured perpendicularly to the force vector.

EQUILIBRIUM

There is more to Fig. 380 than meets the eye at first. We ought to observe that the system of forces is in *equilibrium*. When a set of forces is in equilibrium, the body on which the forces act is at rest or it has a constant motion.

Loads and reactions

Consider a body, in particular, the balance arm AB . Think of AB as isolated all by itself, free, so to speak, and decide what forces were acting on it when it was a part of the balance. There are the forces Q and W , of course, which act vertically downward. Now if this arm is to be in equilibrium, there must also be found a force or forces acting upward to counteract

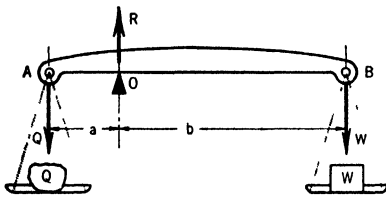


FIG. 380. REPEATED

these downward forces. The equalizing force in this instance is at the pivot O ; that is, the pivot O pushes up on the arm with the

same force that the arm pushes down on the pivot, and the downward forces are Q and W . To be in equilibrium, the forces on the arm AB must add up to zero. Since all the forces in this case act in a vertical direction, we may write, for equilibrium,

$$(5) \quad \text{Sum of the vertical forces} = 0.$$

This equation expresses *one* of the conditions of equilibrium. This condition of equilibrium may be used to determine the *reaction* at O . By *reaction*, we mean the force R at O caused by other forces on the arm. To sum forces, we must account for their directions. In this instance, we can let the upward direction be positive; then, the downward direction is negative. Thus, applying equation 5 and summing the forces in Fig. 380 when $W = 2$ lb. and $Q = 4$ lb., we find

Directions and signs

$$R - Q - W = R - 4 - 2 = 0,$$

from which $R = 6$ lb.

Another condition of equilibrium may be stated.

Moment equilibrium

$$(6) \quad \text{Sum of the moments of all the forces about any point} = 0.$$

Although the rule is good for any point, you will find certain points more convenient than others. In the balance problem, point O is best. Let us take moments about O and equate the sum of all moments to zero in accordance with equation 6. To do this, we must choose positive and negative directions.

A moment has a direction in a clockwise sense or in a counterclockwise sense. Thus, in Fig. 380, the force W tends to turn the arm about the pivot O in a clockwise sense, and the force Q tends to turn it about O in a counterclockwise sense. Suppose we consider clockwise moments to be negative. Then the sum of the moments about O is

Sign of moments

$$(7) \quad Wb - Qa = 0.$$

The force R has no moment about O because its moment arm about this point is zero. Equation 7 is an application of equation 6. But watch carefully as we use a little algebra. Transpose the term Qa to the right-hand side of equation 7. Then you will find

$$(8) \quad Wb = Qa.$$

The result, you notice, is the same as equation 1. You should not be surprised, since all correct methods of solving a problem must give the same answer. The only difference between equa-

Comparison

tions 1 and 6 in this application is simply in the point of view. The point of view in equation 6 is more useful generally. Now you

may use the foregoing principles and solve the following problems.

Problems

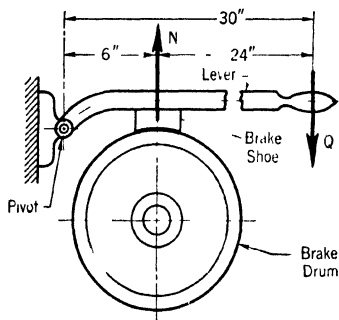


FIG. 381. FORCE ON A BRAKE SHOE, PROBLEM 1

1. A force $Q = 100$ lb. is applied to a 30-in. brake lever as shown in Fig. 381. What is the total force of the brake shoe on the drum? (This force is equal and opposite in direction to the force N of the drum on the shoe, as shown.) *Ans.* 500 lb.

2. A crowbar is used to lift the edge of a crated airplane engine (Fig. 382). It is pivoted on a block of wood at O and the worker exerts a force of $Q = 150$ lb. as shown. What force, F , is applied to the

crate? (This force is equal and opposite to the force F of the crate on the crowbar.) Also, what is the reaction R at the pivot?

Ans. $F = 1650$ lb., $R = 1800$ lb.

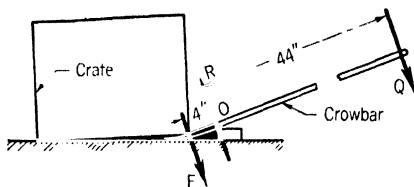


FIG. 382. LEVER ACTION, PROBLEM 2

3. A simple winch is composed of a cylinder, around which a rope winds, and a crank which the operator uses for turning the cylinder (Fig. 383). Winches are frequently found at water wells. A bucket of water, the total weight of which is $W = 20$ lb., is lifted by a winch (Fig. 383) in which the operator applies a force Q at a distance of 10 in. from the axis O of the winch. Find the value of Q when friction is neglected.

Ans. 10 lb.

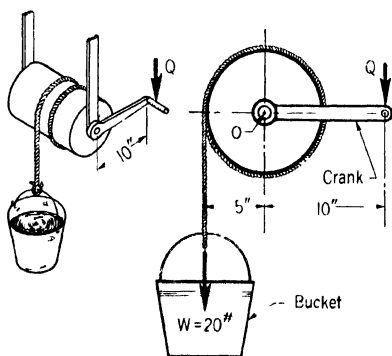


FIG. 383. LEVERAGE OF A WINCH, PROBLEM 3

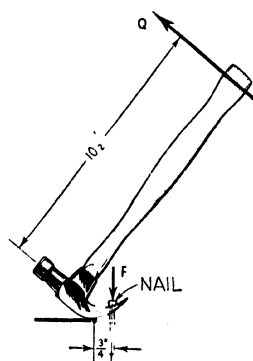


FIG. 384. PULLING FORCE, PROBLEM 4

4. A carpenter had to exert a force of $Q = 50$ lb. on a claw hammer in order to pull a nail when the conditions were as shown in Fig. 384. What force F was necessary to move the nail?

Ans. 700 lb.

Radial
leverage

Nonparallel
forces

COMPONENTS OF FORCES

Suppose you wished to move a loaded industrial truck along a horizontal floor. No doubt you would grab the handle and start pulling. The direction of the force exerted on the truck would be indicated by the inclination of the handle. Assume that the handle makes an angle of b degrees with the horizontal as suggested in Fig. 385. In this event, one effect of the force Q is to tend to lift

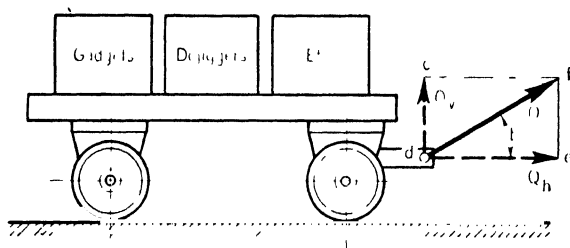


FIG. 385. FORCE COMPONENTS

the front end of the truck off the floor (which is not what you want), and the other effect is the horizontal pull which will cause the truck to move. (Suppose for the moment that there is no friction.) These two effects are measured by what we call *components of the force Q* .

The component of Q in a horizontal plane will be called Q_h , and in a vertical plane, Q_v . As seen from Fig. 385, the horizontal component is the leg de of the right triangle def . The vertical component is the leg dc of the right triangle dcf , which is equal in magnitude to the leg ef of triangle dcf . With these observations and a bit of trigonometry, we find the value of the components to be

$$(9) \quad Q_h = Q \cos b \quad \text{and} \quad Q_i = Q \sin b.$$

Components of a force are other forces which may replace the given force without changing the action of the body on which they act. That is, we may *remove* Q and use Q_h and Q_v with the same effect on the truck as when Q acted alone. Thus, when you think of components, imagine the original force as removed. There are other components of a force, but we shall specialize on the two described above because they are the most useful.

Perhaps you are wondering why we should bother with components when the force itself is known? Well, suppose you tug on the truck and find it so heavily loaded that you need assistance.

(Friction is involved now.) We will assume that a rope is tied to the truck and your assistant pulls on the rope as in Fig. 386 with a force of A lb. (at a degrees with the horizontal) while you pull with B lb. (at b degrees). What total pull results, that is, technically speaking, what is the resultant pull? Let us take some real

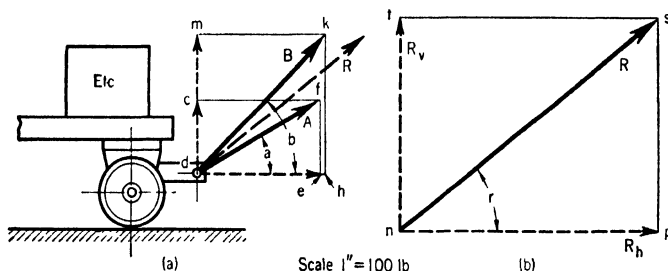


FIG. 386. COMBINING FORCES INTO A RESULTANT

numbers. Your assistant pulls with a force $A = 80$ lb., but you, being more energetic, exert a pull $B = 100$ lb.; let $a = 30^\circ$ and $b = 45^\circ$. What is the resultant pull? First, find the horizontal components of A and B .

$$(10) \quad A_h = de = A \cos a = 80 \times \cos 30^\circ = 80 \times 0.866 = 69.3 \text{ lb.}$$

$$(11) \quad B_h = dh = B \cos b = 100 \times \cos 45^\circ = 100 \times 0.707 = 70.7 \text{ lb.}$$

*Adding
collinear forces*

Since we know that simple addition works for forces acting in the same direction, we can add A_h and B_h and find

$$(12) \quad R_h = A_h + B_h = 69.3 + 70.7 = 140 \text{ lb.,}$$

which is the net *horizontal* force you both exert when pulling together. Now we can find and add the vertical components.

$$(13) \quad A_v = dc = A \sin a = 80 \times \sin 30^\circ = 80 \times 0.5 = 40 \text{ lb.}$$

$$(14) \quad B_v = dm = B \sin b = 100 \times \sin 45^\circ = 100 \times 0.707 = 70.7 \text{ lb.}$$

$$(15) \quad R_v = A_v + B_v = 40 + 70.7 = 110.7 \text{ lb.}$$

Next consider R_h and R_v . These two forces must bear the same relation to some resultant force R as, say, A_h and A_v bear to the force A . This analogy suggests the arrangement of forces in Fig. 386 *b*. The line np is proportional in length to $R_h = 140$ lb., and nt is proportional to $R_v = 110.7$ lb. In thus laying out the forces, we are drawing the vectors to scale.

VECTORS AND TRIGONOMETRY

*Resultant
vector to scale*

The engineer chooses a scale for the forces. In this case, let 1 in. represent 100 lb. of force, so that np is $\frac{140}{100} = 1.40$ in. long,

and nt is $\frac{101.7}{100} = 1.11$ in. long. (Use a ruler and check these figures.) By completing the rectangle $ntsp$ and drawing the diagonal ns , you may determine the magnitude and direction of R . Lay a ruler along ns and you will find that it measures about 1.78 in. If 1 in. represents 100 lb., then 1.78 in. represents $1.78 \times 100 = 178$ lb., which is the magnitude of R . A protractor is used to measure the angle r , which is found to be about 38° . (Note: The vector for R is drawn too heavy for accurate graphical work. In graphical solutions, use a hard pencil with a fine point. Graphical accuracy is usually satisfactory for the solution of engineering problems, the range of error for careful work being one or two per cent.)

To find the resultant R by an algebraic solution, we note that it is the hypotenuse of a right triangle, nps . Its magnitude is therefore the square root of the sum of the squares of the legs (see page 267); thus

*Resultant
obtained
algebraically*

$$(16) \quad R = \sqrt{np^2 + ps^2} = \sqrt{R_h^2 + R_v^2}.$$

Using the numbers of this example, we may write

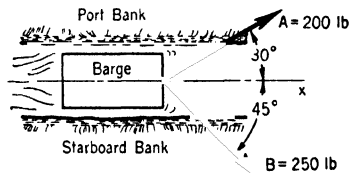
$$R = \sqrt{140^2 + 110.7^2} = 178.2 \text{ lb.}$$

To find the angle r , we notice that

$$(17) \quad \tan r = \frac{ps}{np} = \frac{R_v}{R_h}.$$

Therefore, since $\tan r = \frac{110.7}{140} = 0.79$, the angle r obtained from the table of tangents on page 566 is 38.3° . This angle defines the direction of R . The actual location of the vector for R , or of any resultant force vector, is such that the resultant's line of action passes through the intersection of its components. Because the original components A and B intersect at d , the location of the vector R is as shown by the broken line in Fig. 386 *a*.

1. Two horses are pulling a barge along a canal, Fig. 387. At a certain instant, horse A on the port bank is exerting a pull of $A = 200$ lb., and horse B on the starboard bank is pulling with a force $B = 250$ lb. Determine the magnitude, direction, and location of the resultant force. *Hint.* Watch out for positive and negative components. Observe too that if the horses continue to pull as stated, the barge will soon strike the bank of the canal. Which bank would it strike? *Ans.* $R = 358$ lb., $r = 12.4^\circ$ to the starboard of the x axis.



Problems

FIG. 387. RESULTANT PULL,
PROBLEM 1

2. In a tugging contest, four men pull in the directions and with the magnitudes shown in Fig. 388. Determine the resultant force. (Watch out for negative components.) Which side (A and B vs. C and D) wins this contest?

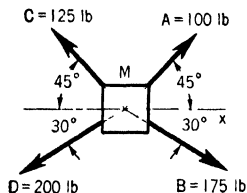


FIG. 388. RESULTANT OF FOUR FORCES, PROBLEM 2

Ans. $R = 48.5$ lb. pointing downward and leftward at an angle of 36° below the x axis.

BRIDGE TRUSS

Now let us consider a problem of equilibrium. Since all parts of a bridge truss are at rest, the forces to be studied in this structure are in equilibrium. Whenever a body is in equilibrium, the resultant force on the body is zero. If the resultant force is zero, its components must also be zero. Hence, the sum of all the horizontal components and the sum of all the vertical components of the forces acting on a body in equilibrium must each add up to zero. We may therefore write these important equations:

(18) Sum of horizontal components = 0.

(In mathematical symbols, $\Sigma H = 0$.)

(19) Sum of vertical components = 0.

(In mathematical symbols, $\Sigma V = 0$.)

Suppose we have a bridge truss loaded as shown in Fig. 389. In order to design the members of the truss, we must determine the load on each member, AB , AC , BC , etc. To do this, we consider

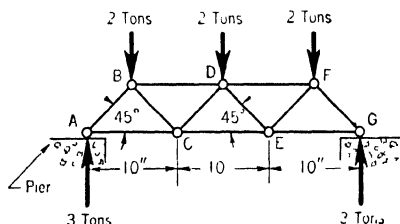


FIG. 389. TRUSS REACTIONS

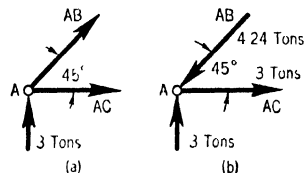


FIG. 390. FORCES ON PIN A

first the pin at A (all members are assumed to be joined by pins). It is acted upon by the upward force or pier reaction of 3 tons (caused by the three 2-ton loads), and by unknown forces in the members AC and AB . In Fig. 390 *a*, the pin A is pictured with the forces AC and AB acting away from the pin, but we may find that we are wrong about these directions. Anyway, we shall proceed with the summing of components, vertically first. Let the upward direction be positive.

Vertical sum of force components = $3 + AB \sin 45^\circ = 0$.

*Summing
components*

Solving for AB from this equation, we find

$$AB = -\frac{3}{0.707} = -4.24 \text{ tons.}$$

Here the minus sign tells us that we have chosen the wrong direction for AB and that it should act toward the pin. We then redraw the figure with the corrected direction for AB as shown in Fig. 390 *b*. Evidently the diagonal end post AB is a compression member that pushes on the pin A . Now using the revised drawing, Fig. 390 (*b*), we sum the horizontal force components and find

$$\text{Horizontal sum} = AC - AB \cos 45^\circ = AC - 4.24 \times 0.707 = 0,$$

from which

$$AC = 4.24 \times 0.707 = 3 \text{ tons.}$$

The positive answer indicates that AC is shown acting in the proper direction in Fig. 390. Next we consider the pin B , Fig. 391 *a*, and again the unknown forces, BD and BC at this point are shown

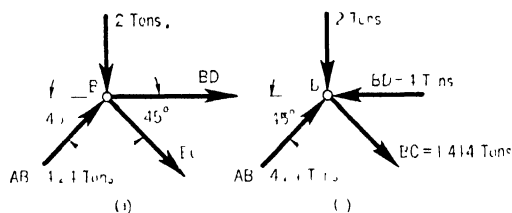


FIG. 391. FORCES ON PIN B

acting *away* from the pin. Of course, AB acts toward pin B because the member AB has been found to be in compression and because a compression member pushes outward at both ends (as a compressed spring acts). Summing vertical force components first, we find

*Compression
member*

$$\text{Vertical sum} = 4.24 \times \cos 45^\circ - 2 - BC \cos 45^\circ = 0,$$

$$BC = \frac{4.24 \times 0.707 - 2}{0.707} = 1.414 \text{ tons.}$$

The positive sign shows that the vector BC is pointed correctly in Fig. 391 *a*. We therefore use the same figure to sum the horizontal force components.

$$\text{Horizontal sum} = BD + BC \cos 45^\circ + AB \cos 45^\circ = 0,$$

$$BD = -1.414 \times 0.707 - 4.24 \times 0.707 = -4 \text{ tons.}$$

*Tension
member*

The negative sign shows that the direction of BD should be reversed, as redrawn in Fig. 391 *b*. When the force in a member points away from a pin, the member is being stretched (in tension) since it is pulling on the pin. If BC , for example, pulls on the pin B , it also pulls on the pin C . To find the forces in all the members of the truss, the foregoing procedure is repeated for each pin. Figures 392 and 393 show typical trusses.



From "Analytic Mechanics," Chambers and Favres

FIG. 392. STEEL TRUSS HIGHWAY BRIDGE

Observe that the members of this structure are so arranged that they always form triangles. A moment's reflection will convince you that the triangular form is the most rigid. Compare pin-connected links in triangular form with pin-connected links in, say, a rectangular form.

Forces in kips

The forces discussed here have been given in tons. American bridge engineers always state all forces in *kips*, a *kip* being 1000 lb. You can therefore multiply all forces in tons given above by 2.0 and they will then be given in *kips*.

Problems

1. Using forces as found above and the pin C in Fig. 389, find the forces CD and CE .
Ans. $CD = 1.414$ tons, compression; $CE = 5$ tons, tension.

2. Using forces as found above, the results of problem 1, and the pin D in Fig. 389, find the forces DF and DE .

Ans. $DF = 4$ tons, compression; $DE = 1.414$ tons, compression.

3. Using the pin G in Fig. 389, find the forces in FG and EG .

Ans. $FG = 4.24$ tons, compression; $EG = 3$ tons, tension.

4. Using the pin F in Fig. 389 and the results of problem 3, find the forces in EF and DF .

Ans. $EF = 2.83$ kips, tension; $DF = 8$ kips, compression.

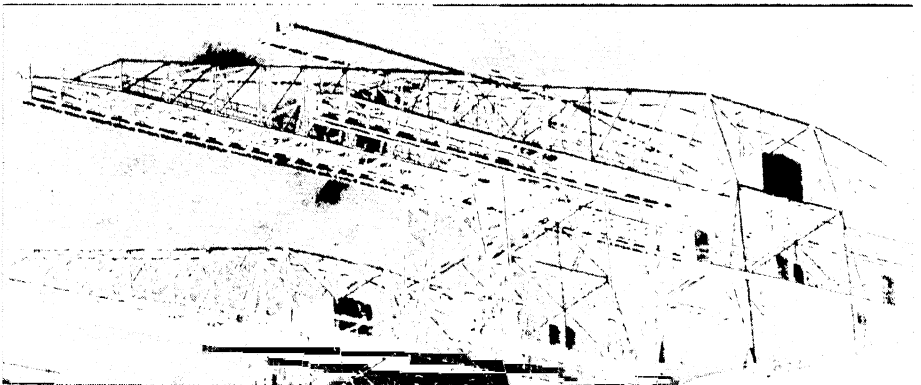
FRICTION

Friction is a contradiction. On the one hand, it serves a very *Using friction* useful purpose in many instances. As a matter of fact, while you might be able to stand up, you could not walk if it were not for friction. Friction holds one foot in place while the weight of the body is shifted to the other foot. On the other hand, in bearings and between pistons and cylinders, for example, we make every effort to reduce friction to a minimum, at least, every effort within reason considering the value of the machine.

If there is rubbing between two surfaces, the frictional force F *Coefficient of friction* is computed from two quantities — from what we call the *normal force*, N , and the *coefficient of friction*, f . The relation between these quantities may be expressed mathematically as

$$(20) \quad F = fN.$$

Here we see that the coefficient of friction f is that number which when multiplied by the normal force N gives the frictional force F . The normal force is that part of the reaction between two surfaces in contact which is normal or perpendicular to the surfaces. The frictional force always acts in a direction to resist motion.



Courtesy Westinghouse Electric and Mfg. Company

FIG. 393. TRAVELING HAMMERHEAD CRANES

This structure with four such crane trusses was erected for use in building a large dam. How would you like the job of finding the force on each member of this trusswork?

An application of these ideas will help to clarify them. Suppose we wish to tip a heavy object (a machine or a crate, for example), so that grapples may be inserted under the object with the intent of picking it up by a crane and moving it. We may drive a wedge under one side of the object as suggested by Fig. 394 *a*. To simplify the picture, we shall *assume* that the greased top surface of the wedge is *frictionless* and then make a free body of it as shown in Fig. 394 *b*.

Friction of a wedge

One-half of the weight $W = 2000$ lb. is taken as acting vertically downward on the wedge ($W/2 = 1000$ lb.). Let the coefficient of friction be $f = 0.3$ and find the value of the force Q to move the wedge. The normal force acts upward on the wedge and perpendicular to the surfaces (floor and wedge) in contact. The wedge is to move toward the left; so the frictional force on it acts toward the right.

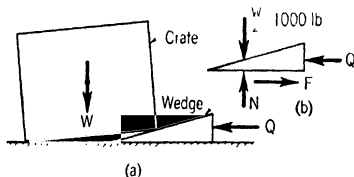


FIG. 394. WEDGE USED TO TIP A CRATE

Our free body diagram of the wedge with a frictionless top surface is now complete (Fig. 394 *b*). We sum vertical forces to obtain

$$\text{Vertical sum} = V - W/2 = N - 1000 = 0,$$

from which $N = 1000$ lb. Since $f = 0.3$, we next find

$$F = fN = (0.3)(1000) = 300 \text{ lb.}$$

Then,

$$\text{Horizontal sum} = F - Q = 300 - Q = 0,$$

from which $Q = 300$ lb. (While the assumption of a frictionless top surface makes this an incomplete picture of the action of the wedge, it does show how to compute the frictional force.)

Friction coefficients

Typical values of the coefficient of friction, which are seen to depend on the kind of materials in contact, are as follows (dry surfaces):

| | | | |
|-------------------------|-----|------------------------------|-----|
| Wood on wood | 0.3 | Leather on iron | 0.3 |
| Wood on metal | 0.4 | Leather on wood | 0.3 |
| Iron on iron | 0.2 | Rubber on concrete | 0.6 |

Problems

1. A boy on a tricycle is coasting downhill and decides to stop. Using his foot on the sidewalk as a brake, he presses down with a force of 30 lb. If the coefficient of friction between the leather sole and sidewalk is 0.4, what is the braking (frictional) force?

Ans. 12 lb.

2. Another youngster, more mindful of the cost of leather than the one in problem 1, uses a stick for a brake and, by virtue of the leverage he obtains, he produces a normal force of 50 lb. with a coefficient of friction of 0.3. What is the braking force?

Ans. 15 lb.

VELOCITY AND ACCELERATION

You are familiar with the concept of *velocity* or *speed*. You can *Velocity* imagine a car going 60 m.p.h., or a plane going 300 m.p.h., and you know that speed is the distance traversed in some unit of time. Instead of using the familiar unit of miles per hour, we find it more convenient in this study to use the number of feet traversed in 1 sec. (that is, ft. per sec.). Thus we may write,

$$(21) \quad \text{Average speed (or velocity)} = \frac{\text{distance in feet}}{\text{time in seconds}}$$

Suppose that the distance between two towns is 490,000 ft. and *Computing* that you drive this distance in 7000 sec. Then your average speed *speed* is $\frac{490,000}{7000} = 70$ ft. per sec. Of course you do not drive constantly at this rate. You find yourself speeding up and slowing down for one reason or another. When the speed of your car is changing, the car is said to be accelerating. Specifically, the acceleration is the rate at which the speed (or velocity) changes. In equation form, we may write

$$(22) \quad \text{Average acceleration} = a = \frac{\text{change in speed}}{\text{time}} = \frac{v_2 - v_1}{t},$$

where v_2 represents the final speed, v_1 represents the initial speed, and t represents the time elapsed during the change.

For example, a car which changes speed from 30 ft. per sec. to *Computing* 50 ft. per sec. in 4 sec. has an average acceleration of *acceleration*

$$a = \frac{(50 - 30) \text{ ft. per sec.}}{4 \text{ sec.}} = 5 \text{ ft. per sec.}^2.$$

Since acceleration is a change in velocity per unit of time, and since the velocity here is in feet per second, the acceleration is in (ft. per sec.) per sec., or *ft. per sec.*², as given. If the initial speed is zero, that is, if a body starts from rest, equation 22 becomes

$$(23) \quad a = \frac{v}{t}, \quad \text{or} \quad v = at.$$

It is common knowledge that a falling body gathers speed. Until the time of Galileo (1564–1642), most people thought that heavy bodies would fall faster than light ones.

*Galileo's
falling bodies*

By dropping various sizes of stones from the famous Leaning Tower of Pisa, Galileo showed that a small light stone gains speed as rapidly as a big heavy one. Some people at the time thought he was being tricky and were in favor of executing him as a practitioner of black magic. However, he managed to squeak through before the Inquisition and made many more valuable contributions to science. Anyway, it is a fact that a rock and a feather fall at the same rate when both are in a *vacuum*, that is, when there is no air resistance. Moreover, they fall with a *constant acceleration*, the velocity steadily increasing. This particular constant acceleration is called the *acceleration of gravity* and is represented by the letter g . The value of g is different for different points on the earth's surface, but its average value is about $g = 32.2$ ft. per sec.².

Problems

1. A train travels 1000 miles in 30 hours. What is the average speed of the trip?
Ans. $33\frac{1}{3}$ m.p.h.
2. An airplane travels 1000 miles in 3 hours. What is its average speed?
Ans. $333\frac{1}{3}$ m.p.h.
3. An automobile goes 170 miles in 4 hours. What is its average speed in feet per second? There are 5280 ft. in 1 mile.
Ans. 62.3 ft. per sec.
4. An automobile starts from rest and attains a speed of 28 m.p.h. in 7 sec. What is the average acceleration?
Ans. 4 m.p.h. per sec.
5. An automobile going 60 ft. per sec. is brought to a stop in $\frac{1}{2}$ min. What is the average acceleration?
Ans. 2 ft. per sec.².
6. An automobile starting from rest has an acceleration of 1.5 ft. per sec.² for 20 sec. What is its final speed?
Ans. 30 ft. per sec.
7. A stone is dropped from a cliff and falls for 5 sec. If air resistance is neglected, with what speed does it strike the ground?
Ans. 161 ft. per sec.

WHAT CAUSES ACCELERATION?

Newton's laws

Sir Isaac Newton (1642–1727) first formulated the laws of motion, but not, as the story goes, because he watched an apple fall from a tree. We have already seen the application of one of his laws, to wit, *action and reaction are equal and opposite*, when we recognized, for example, that the force of the lever on the brake drum (Fig. 381) was equal to but directed opposite to the force of the drum on the lever.

*Force and
acceleration*

Another of his laws is: *the acceleration of a body is proportional to the force producing the acceleration*. Thus, we see that a *force* causes acceleration. This simple law leads into all sorts of ramifications

which we cannot explore in this short study. However, we can learn the basic meaning of the statement.

Since a freely falling body has a known acceleration of g ft. per sec.² and since we know that the force of gravity (weight) is the force which produces this acceleration, we may set up a simple proportion involving these known values of W and g and any other force R producing an acceleration a . Thus we may write,

$$(24) \quad \frac{R}{W} = \frac{a}{g}, \quad \text{or} \quad R = \frac{W}{g} a.$$

This equation may be used to find the resultant force R required to produce a certain acceleration a of a rigid body which weighs W lb. The force R and the acceleration a in equation 24 are *always* in the same direction. Suppose you bring a 3220-lb. car to a stop with an acceleration of 5 ft. per sec.². The resultant force on the car must be

$$R = \frac{W}{g} a = \frac{3220}{32.2} \times 5 = 500 \text{ lb.}$$

Force of acceleration

This force acts between the tires and the road. Thanks to the principle of the lever (and a hydraulic principle not explained here), you do not have to exert a 500-lb. force on the brake pedal. A foot pressure of about 50 lb. will do the job.

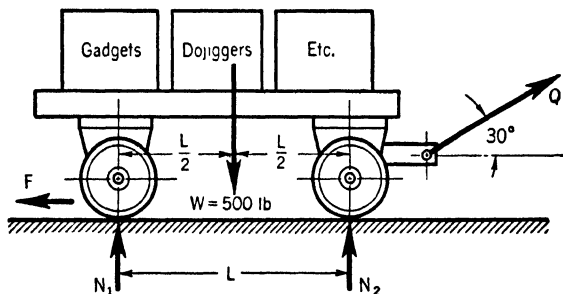


FIG. 395. ACCELERATION OF A TRUCK

We have previously learned something about finding the resultant, and now we see that the resultant of all forces acting on a body is a key to the computation of the acceleration of the body. Reconsider the industrial truck of Fig. 385, redrawn in Fig. 395 where *all* forces acting are represented. Let the pull be $Q = 100$ lb. at 30° with the horizontal. The force $F = 50$ lb. represents the total frictional resistance, a force parallel to and opposite to the

Resultant and acceleration

direction of motion. Obviously, this truck, being on a level floor, will move neither up nor down; therefore, the acceleration in the vertical direction is zero and the sum of the vertical forces is zero (by equation 24, where $a = 0$). Consequently, the resultant force is necessarily horizontal, so that the resultant is equal to the horizontal sum. Thus, summing horizontal components, we find

$$\begin{aligned}\text{Horizontal sum} &= Q \cos 30^\circ - F = 100 \cos 30^\circ - 50 \\ H = R &= 86.6 - 50 = 36.6 \text{ lb.}\end{aligned}$$

Now using equation 24, we get

$$R = \frac{W}{g} a \quad \text{or} \quad 36.6 = \frac{500}{32.2} a,$$

from which $a = 2.36$ ft. per sec.², the acceleration of the truck.

*Equilibrium
again*

Naturally, after you get the truck up to your walking speed, you will not need to continue to pull with a force of $Q = 100$ lb., because the force you must then exert is just enough to maintain a steady speed and does not include the effort that would be necessary to cause acceleration. While it is moving at constant speed, it is in equilibrium. By the way, other forces remaining the same, what would be the pull Q to maintain constant speed? In this case, the horizontal sum of the forces is equal to zero. Hence, $Q \cos 30^\circ - F = Q \cos 30^\circ - 50 = 0$. From which, the value of Q is 57.7 lb.

Problems

1. In Fig. 395, suppose $Q = 80$ lb. and $F = 50$ lb. Determine the acceleration and the speed attained after 3 sec.

Ans. $a = 1.24$ ft. per sec.², $v = 3.72$ ft. per sec.

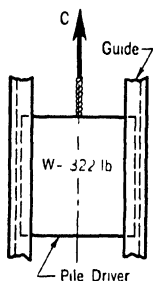


FIG. 396. LIFTING FORCE, PROBLEM 4

2. The driver of a 3220-lb. automobile allows it to coast to a stop from a speed of 50 ft. per sec. If it takes 150 sec., what is the average resisting force?

Ans. $33\frac{1}{3}$ lb.

3. A train with a total weight of 3220 tons is brought from rest to a speed of 48 ft. per sec. in 60 sec. The total resistance to motion is 30,000 lb. What is the drawbar pull in pounds? (The drawbar pull is the pull of the locomotive on the train.)

Ans. 190,000 lb.

4. A pile driver (Fig. 396) weighing 322 lb. is raised with an acceleration of 9.6 ft. per sec.². The total frictional resisting force in the guides is 200 lb. What is the force in the cable C ? (Do not overlook the force of gravity in the vertical sum.)

Ans. 618 lb.

5. A 3000-lb. automobile starts from rest down a 4% grade. (A 4% grade is one that rises 4 ft. in 100 ft. on the horizontal, Fig. 397.) The total constant resistance

to motion (brakes and engine) is $F = 100$ lb. What is the acceleration? Is the car speeding up or slowing down? What is its speed after 1 min.?

Hint. The resultant force acts parallel to the incline; therefore, sum forces in this direction. *Ans.* $a = 0.215$ ft. per sec.², speeding up, $v = 12.9$ ft. per sec.

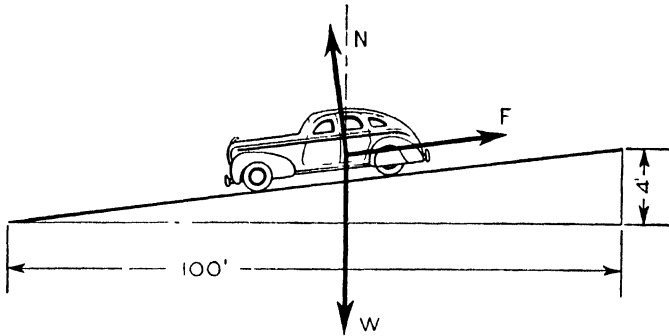


FIG. 397. ACCELERATION OF CAR, PROBLEMS 5 AND 6

6. A 3000-lb. automobile starts moving down a 4% grade (see problem 5) at a speed of 40 ft. per sec., when the total constant resistance to motion is $F = 150$ lb. What is the acceleration? Is the car speeding up or slowing down? What is its speed after 1 min.? How long before it stops?

Ans. $a = 0.322$ ft. per sec.², slowing down, $v = 20.7$ ft. per sec., $t = 2.07$ min.

MECHANICAL WORK

There are different kinds of work. Some people call it work to write or even to study a textbook. Be that as it may, the kind of work we shall now consider is called *mechanical work* and is done by a force. Remember the industrial truck? Well, here it is again

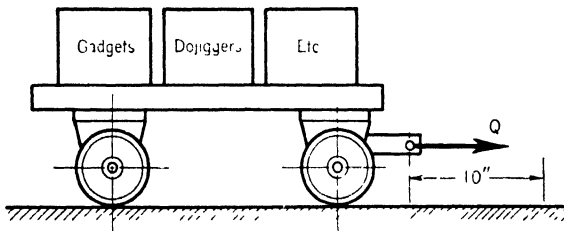


FIG. 398. WORK OF A HORIZONTAL PULL

in Fig. 398. Suppose that you are moving it with a constant horizontal force Q . The work that you do is the product of the force Q times the distance the truck moves. If you pull the truck 10 ft. with a constant force of 50 lb., the work is $10 \times 50 = 500$ ft.-lb. Thus, in its simplest form, we may state

(25) Work = Force times distance moved in the direction of the force.

For example, the work of an automobile engine develops as the pressure of the gases acting on the top of the piston forces the piston to move down. The force exerted by the gas then acts through the distance measured by the stroke of the piston. Because this gas force is variable, the analysis of the work of this engine is a little too complicated for our present discussion.

*Work of a
force*

If you do not pull on the truck in a *horizontal* direction, then the work is the *horizontal* component of your pull times the distance the truck moves, but only when the truck moves in a *horizontal* direction. In other words, *only that force component which acts along the line of motion does work*. Thus, in Fig. 399, the horizontal

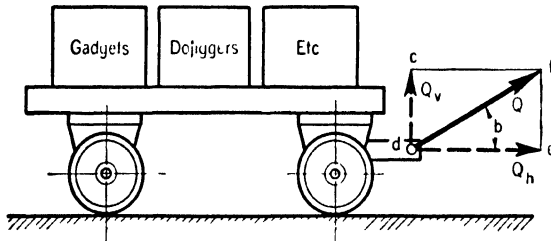


FIG. 399. WORK OF AN INCLINED FORCE

component of Q is Q_h , so that the work done in moving the truck 10 ft. is $Q_h \times 10$ ft.-lb. The vertical component Q_v of Q does no work because its point of application does not move *in the direction of* Q_v . Neither would there be any work done by a sideways component of the pull, unless the truck veered off in that direction. We may now make our definition of the work of a force more complete.

Definition

$$(26) \left\{ \begin{array}{l} \text{Work} \\ \text{of any} \\ \text{force} \end{array} \right\} = \left\{ \begin{array}{l} \text{component of the} \\ \text{force in the} \\ \text{direction of motion} \end{array} \right\} \times \left\{ \begin{array}{l} \text{distance through which} \\ \text{the point of application} \\ \text{of the force moves.} \end{array} \right\}$$

Thus, if you pull the truck 10 ft. with a force of $Q = 50$ lb. at an angle $b = 30^\circ$ with the horizontal, as shown in Fig. 399, the work is computed as

$$\text{Work} = Q \cos b \times 10 = 50 \cos 30^\circ \times 10 = 433 \text{ ft.-lb.}$$

You pull just as hard as before, but you do less work, even though you do not feel less tired.* Carrying the idea to its limit, you may

* This does not prove that the minimum effort to move the truck from one place to another is done when the pull is horizontal. The effect of the frictional force will cause the optimum pull to incline a bit above the horizontal. However, the computations above are correct for the conditions stated.

huff and puff until you collapse exhausted in trying to move a stalled automobile, but if *it does not move*, you have done no work in the sense of this discussion.

WORK AND KINETIC ENERGY

There are many forms of energy; electrical, heat, work as just defined, potential, chemical, and others. Kinetic energy is that energy of a body which exists because of the body's motion. Suppose an airplane of weight W is moving with a speed of v ft. per sec.; then, its kinetic energy is

$$(27) \quad KE = \text{kinetic energy} = \frac{Wv^2}{2g}.$$

Of course g is the acceleration of gravity. Thus, if a plane weighs $W = 10,000$ lb. and is moving at 300 ft. per sec. (these units *must* always be used in equation 27 when g is used in ft. per sec.²), its kinetic energy is

$$KE = \frac{Wv^2}{2g} = \frac{10,000 \times 300^2}{2 \times 32.2} = 13,980,000 \text{ ft-lb.}$$

There are many laws which Congress and the state legislatures have not passed. One of them is the *Law of the Conservation of Energy*. This law says that energy is indestructible and that it cannot be created. It may disappear in one form, but, if so, it reappears in some other form. We cannot prove this law. You would be surprised at how many scientific principles cannot be *proved*. You rightly ask, "Then how do we know that the law is true?" and I answer, "Because we know of no instance where the law has been violated and because we have abundant scientific verification." There may be circumstances beyond our experience (out in celestial space, for example) where this law does not hold. Nevertheless, on this insignificant planet, it does yeoman service for the engineer and scientist and has yet to be found wrong.

If we apply this law to a system where there are just two kinds of energy (work and kinetic energy), the only conclusion possible is that

$$\text{Net work done on a body} = \text{change of kinetic energy.}$$

This statement means that energy which "disappears" as work reappears as kinetic energy. Let $Wv_1^2/2g$ be the original kinetic energy and $Wv_2^2/2g$ be the final kinetic energy of some body

after a net amount of work equal to $R \times s$ has been done. Then we may write

$$(28) \quad R \times s = \frac{Wv_2^2}{2g} - \frac{Wv_1^2}{2g}.$$

In this equation R is the resultant force on the body and s is the distance moved by the point of application of the force in the direction of the force. You have probably noted from previous paragraphs that *a body moving in a straight path always moves in the direction of the resultant force.*

Resultant motion

We can now make use of that truck again (Fig. 400). On page 480, we found that for $Q = 100$ lb., $F = 50$ lb., $W = 500$ lb., and $b = 30^\circ$, the resultant $R = 36.6$ lb. Since the resultant force

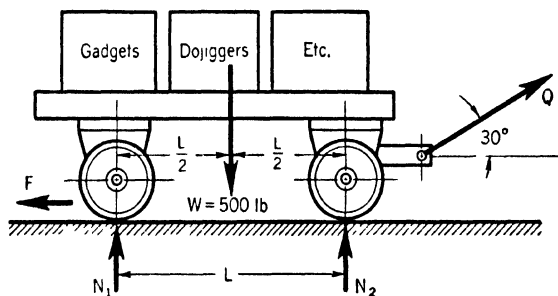


FIG. 400. NET WORK

(which acts horizontally in the direction of motion) times the distance traveled gives the *net work*, that is, the work considering *all forces* which are acting, we can compute the net work for a movement of the truck of, say, 20 ft. Assume that the truck starts from rest, so that $v_1 = 0$ and $KE_1 = 0$. Then we may write

Work and acceleration

$$(29) \quad R \times s = \frac{Wv_2^2}{2g} = 36.6 \times 20 = \frac{500 v_2^2}{2 \times 32.2}.$$

From this relationship we find $v_2 = 9.71$ ft. per sec., the speed which the truck will attain in a distance of 20 ft. if you pull with $Q = 100$ lb. as specified. This is pretty fast moving, even if you are in a tremendous hurry, since 9.71 ft. per sec. is better than 6.5 m.p.h. (Recall that 4 m.p.h. is brisk walking.) What would probably happen is that before the truck had moved 20 ft., you would let up on your pull, not work so hard, and be satisfied with a speed of 2.5 to 3 m.p.h. Why not compute the distance s to get up a speed of 3 m.p.h. or 4.4 ft. per sec.? You ought to find it to be 4.12 ft.

1. The truck of Fig. 400 is moved 20 ft. while the frictional resisting force $F = 50$ lb. What is the work done to overcome friction? *Ans.* 1000 ft-lb.

2. The conditions described in problem 1 are assumed and a pull $Q = 100$ lb. at $b = 30^\circ$ with the horizontal is exerted. What work W_Q is done by the force Q ? What work W_W is done by the weight W ? Using the result from problem 1, determine the net work W_n and compare with the net work obtained from equation 28, p. 484; that is, compare with $R \times s = 732$.

Ans. $W_Q = 1732$ ft-lb., $W_W = 0$, $W_n = 732$ ft-lb.

3. The train described in problem 3, page 480 (weight = 3220 tons, total resistance to motion = 30,000 lb.) is brought from rest to a speed of 48 ft. per sec. by a drawbar pull of 190,000 lb. How far does the train travel in the meanwhile? (Be sure that R = sum of forces in the direction of motion.) *Ans.* 1440 ft.

4. In Fig. 400, $Q = 80$ lb. and $F = 50$ lb. The truck is brought to a speed of 3.72 ft. per sec. from rest. How far does the truck move during this period. (See problem 1, page 480.) *Ans.* 5.58 ft.

5. The driver of a 3220-lb. automobile allows it to coast to a stop from a speed of 50 ft. per sec. on a level road. The constant resisting force (= resultant force) = $33\frac{1}{3}$ lb. How far does the car go? (See problem 2, page 480.) *Ans.* 3750 ft.

6. The pile driver of Fig. 401 is released and falls through a distance of 10 ft. to strike a pile. The constant resistance is 100 lb. and the force $C = 0$. With what velocity does it strike the pile?

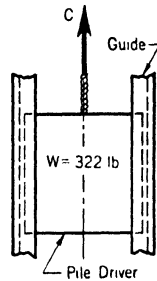


FIG. 401. FORCE OF PILE DRIVER, PROBLEM 6

Ans. 21 ft. per sec.

7. A 3000-lb. automobile is moving down a 4% grade at a speed of 40 ft. per sec. when the total constant resistance to motion is $F = 150$ lb. What is its speed after it has moved 2000 ft.? (See problem 6, page 481.) *Ans.* 17.7 ft. per sec.

8. A 20-lb. projectile from an antiaircraft gun is to be hurled vertically to an altitude of 25,000 ft. What must be its muzzle velocity, if air resistance is neglected? (Note that the resultant force on the projectile after it leaves the gun is the force of gravity.) *Ans.* 1270 ft. per sec.

9. Suppose that Galileo had dropped a 10-lb. stone from the top of the Leaning Tower of Pisa, 179 ft. above the ground. With what velocity would it have struck the ground? (Air resistance neglected, the resultant force is the weight of the stone.) *Ans.* 107.1 ft. per sec.

MOMENTUM

Another law, one that solves many problems which could not be solved by any other principle, is the *Law of the Conservation of Momentum*. If your automobile collides with another car moving in the same straight path, the only thing which is unchanging for that instant before you lose consciousness is momentum; that is,

Conservation of momentum

the momentum of your car plus the momentum of the other car before collision is equal to the sum of the momentums of the two wrecked cars the instant after impact. In other words, *any system of bodies has a constant total momentum as long as no external force acts on the system.*

*Loss of
momentum*

In the instance of the automobile crash, immediately after impact occurs the cars begin to slide about on the road. Since the frictional forces during this sliding constitute an *external* force, momentum is not conserved during this period immediately after the crash, but it is reduced finally to zero when the cars come to rest. On the other hand, the force *between* the cars at the instant of impact is not an external force. It is internal to the system of bodies (the two cars), the force on one car being equal and opposite to the force on the other car.

*R
m*

*Definition of
momentum*

Assuming that you are now interested in knowing what momentum is, it will not be difficult to inform you.

$$(30) \quad \text{Momentum} = \text{mass} \times \text{velocity}.$$

Simple enough. Mass, as far as the units we are using are concerned, is the weight W divided by the acceleration of gravity g ; that is,

$$(31) \quad \text{Mass} = \frac{W}{g}.$$

Therefore, by substitution we may write

$$(32) \quad \text{Momentum} = \frac{W}{g} v.$$

Suppose a body A has an initial velocity of v_{A1} ft. per sec., a final velocity of v_{A2} ft. per sec., and a weight of W_A lb. Then the change of momentum of the body is the second momentum minus the first, or

$$(33) \quad \frac{W_A}{g} v_{A2} - \frac{W_A}{g} v_{A1} = \frac{W_A}{g} (v_{A2} - v_{A1}).$$

Similarly for a body B , the change of momentum is

$$(34) \quad \frac{W_B}{g} v_{B2} - \frac{W_B}{g} v_{B1} = \frac{W_B}{g} (v_{B2} - v_{B1}).$$

*Impact and
momentum*

If these two bodies collide, the decrease (or increase) of momentum of one body will be equal to the increase (or decrease) of momentum of the other body. In short, the algebraic sum of the

above changes is equal to zero, according to the Law of the Conservation of Momentum.

$$\frac{W_A}{g} (v_{A2} - v_{A1}) + \frac{W_B}{g} (v_{B2} - v_{B1}) = 0$$

or

$$(35) \quad \frac{W_A}{g} v_{A1} + \frac{W_B}{g} v_{B1} = \frac{W_A}{g} v_{A2} + \frac{W_B}{g} v_{B2}.$$

This equation is true for bodies moving in the same straight line, either in the same or in opposite directions. Equation 35 is easy to remember because it says that *the momentum of the bodies before impact is equal to the momentum of the bodies after impact.*

Let a 6440-lb. artillery gun *A* fire a 64.4-lb. shell *B*. The muzzle velocity of the shell is 2500 ft. per sec. What is the initial velocity of recoil of the gun? *Recoil of a gun*

Analyzing this problem, we note that the gun *A* and the shell *B* have an initial velocity of zero, $v_{A1} = 0$, $v_{B1} = 0$. Applying equation 35, we get

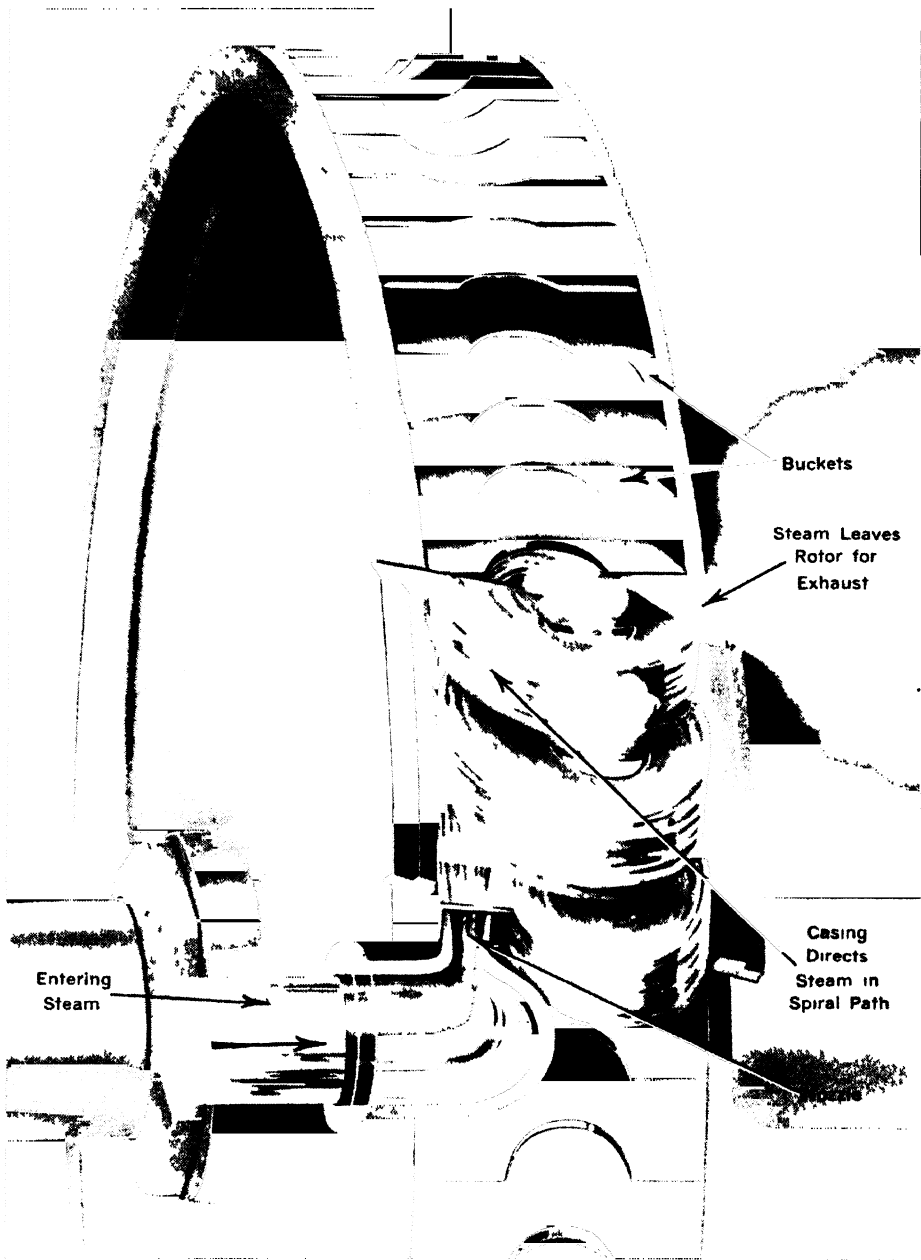
$$0 = \frac{6440}{g} v_{A2} + \frac{64.4}{g} \times 2500$$

or

$$v_{A2} = - 25 \text{ ft. per sec.}$$

The negative sign in this answer simply means that the direction of the velocity of the gun is opposite to the direction of the velocity of the shell, as we already know. The velocity of 25 ft. per sec. is the velocity of the gun the instant after firing and before the recoil mechanism begins to act.

This same principle is used for rocket propulsion. You are familiar with the "sky rockets" which afford fun on July 4th. The burning of the powder charge causes gases to rush out of the back end of the rocket, the gases thus changing their momentum from nothing to something. The change of momentum of the gases results in a change of momentum of the rocket, and away it goes. Observe that, as in the case of the gun and shell, both the rocket and gases are initially at rest and their momentum is therefore zero. Since momentum, like force, is a vector quantity, when the gases gain momentum in, say, a downward direction, the rocket must gain momentum in an upward direction, provided some external force does not hold it in place. These mutual gains of momentum in opposite directions are essential if the sum of the momentums is to remain constant (and zero in this case). This *Rocket propulsion*



Courtesy Terry Steam Turbine Company

FIG. 402. CHANGE OF MOMENTUM PRODUCES IMPULSE OF TURBINE ROTOR

This illustration shows the rotor of a re-entry type of turbine. The steam leaves the nozzle and passes into a bucket. The bucket changes the direction of travel of the steam and directs it into a similar channel in a casing. The casing, which is not shown in this picture, is the cover of the rotor. A channel in the casing

principle is also responsible for the "bazooka" gun which has proved of great use as an antitank weapon. It also explains the basic idea that accounts for the operation of a turbine. (See Fig. 402.)

A recent contribution of this principle to engineering development is the *jet propulsion* of airplanes. In the operation of this type of plane, the air is first compressed in a rotary compressor. Then fuel is mixed with the air and burned. The heat released by the fuel adds to the energy of the air and increases its volume and temperature. Part of the hot air passes through a turbine which drives the compressor, the remainder rushes through a nozzle, leaving the nozzle as a high-velocity jet, pointed toward the rear. This high velocity corresponds to a large momentum. Since the momentum of the air entering the compressor is less than the momentum of the jet leaving it, the increase in momentum results in a force which moves the plane forward. The processes described are continuous.

1. A gun weighing 160,000 lb. fires a 900-lb. projectile whose muzzle velocity is 1400 ft. per sec. What is the initial velocity of recoil? *Problems*
Ans. 7.85 ft. per sec.

2. A 10-lb. box of sand *A*, at rest and hanging from a long rope, is struck by a 0.2-lb. bullet *B* which has a velocity of 2000 ft. per sec. in a horizontal direction. The bullet imbeds itself in the sand. What is the velocity of the sand box immediately after impact? (Note that the momentum of the bullet before impact is equal to the momentum of the bullet and box after impact.)

Ans. 39.2 ft. per sec. in the same direction as the bullet.

3. A 10-lb. box of sand *A*, moving toward the left on a horizontal plane with a velocity of 5 ft. per sec., is struck by a 0.2-lb. bullet *B* which is moving toward the right with a velocity of 2000 ft. per sec. The bullet imbeds itself in the sand. What is the velocity of the sand box immediately after impact and before friction of the box on the plane begins to reduce the momentum?

Hint. Assume rightward direction as positive. Then $v_{A1} = -5$ ft. per sec. and $v_{B1} = +2000$ ft. per sec. in equation 35.

Ans. + 34.3 ft. per sec. toward the right.

redirects the steam back into a bucket on the rotor, where a path similar to the first pass of the steam is repeated. In its journey through the turbine, the steam travels along a spiral-like path. Because the direction of motion and the speed of the steam are changed, there is a change in its momentum. A change of momentum is equal to an impulse, where impulse is defined as force *times* time. Thus, the change of momentum results in a force acting on the rotor, a force which causes the rotor to turn. If the shaft of this rotor is connected to another machine which the turbine drives, useful work will be done by this force in driving the other machine. This type of turbine is manufactured in small sizes only. (See Fig. 407 on p. 500 which shows the rotor of a 40,000-hp. turbine.) *Caption to Fig. 402*

4. A 10-lb. box of sand *A*, moving toward the left on a horizontal plane with a velocity of 50 ft. per sec., is struck by a 0.2-lb. bullet *B* which is moving toward the right with a velocity of 2000 ft. per sec. The bullet imbeds itself in the sand. What is the velocity of the sand box immediately after the impact?

Ans. — 9.8 ft. per sec. toward the left.

HAVE YOU LEARNED ALL?

Further study

No! Not even if you thoroughly understand every application in this treatment of mechanics. There is much more left to say than is said here. Good books on the subject of analytic mechanics generally contain some 300 to 400 pages and more, yet even these good books do not cover the subject in all of its aspects. If you wish to become an engineer, further study of this subject will yield most useful dividends for you. As a result of the knowledge gained here, you should not find further progress too difficult. At least you have made a good start.

Thermodynamics

THE CONTROL OF HEAT, COLD, AND POWER

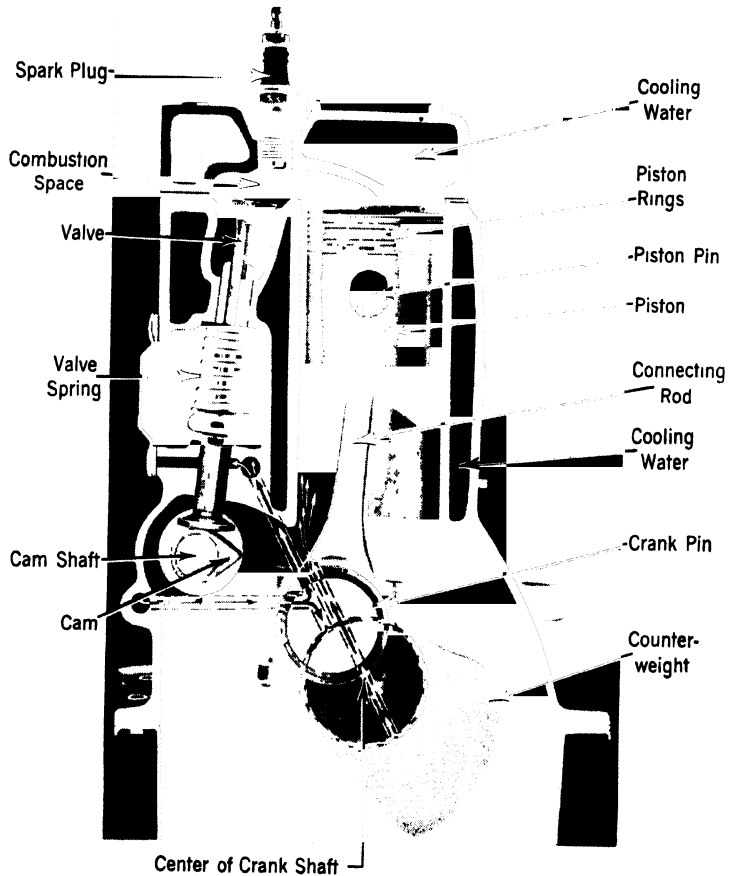
This discussion will deal with a phase of science which goes by the impressive name of *thermodynamics*. The word thermodynamics is derived from two Greek words, *therme* meaning heat, and *dynamis* meaning power. Thus, although the name of this science has a most forbidding appearance, it is simply a study of heat and power. One of the engineer's principal interests in heat is how to get work and power out of it. Our sources of heat are the sun and such fuels as coal, wood, and gasoline. At this writing, the heat from the sun is not used directly for producing work and power in commercial quantities, but it is used indirectly, as for example, in the hydraulic power plant. Without the action of the sun in evaporating water into the atmosphere, from which it is precipitated as rain, the water reservoirs above our power plants would never be filled. *Sources of heat*

HUMAN BODY AS AN ENGINE

The most important engine, to you and me, at least, is the human body, which is quite similar to but much more complicated than mechanical machines for transforming fuel energy into work and power. The food eaten is the fuel for the human body. Instead of burning (oxidizing, see pp. 65 and 80) the fuel at a high temperature, our bodies are luckily designed to "digest" (oxidize) the fuel at 98.6°F., diverting its energy to the body, whence by muscular movements we complete the conversion of heat in the fuel into work. If we just sit still, we must still supply some fuel to keep the internal bodily processes going. This fuel supply corresponds to that which must be supplied to the automotive engine while it is idling. A difference is that we can stop the automotive engine, and save the gasoline fuel, with the expectation of starting it again as desired. This procedure does not yield satisfactory results when applied to the human engine. *Fuel for body*

*Need for
oxygen*

To make the heat in the fuel available, oxygen is necessary. The automotive engine has a suction stroke during which air is drawn into the engine. The oxygen in the air mixes with the gasoline, the fuel is consumed, and some of its heat is converted into work. The burned gases, containing carbon dioxide (see page 84), then pass through the exhaust. Similarly, the human engine sucks the air into the lungs, where some of the oxygen is used immediately,



Courtesy Studebaker Corporation

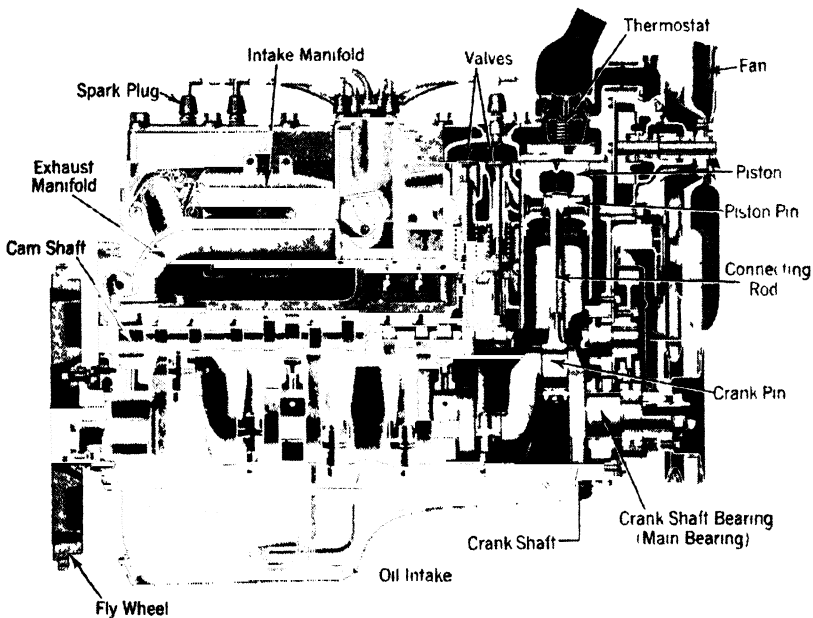
FIG. 403. TRANSVERSE SECTION OF AN AUTOMOTIVE ENGINE

This illustration shows a typical automotive engine. The combustion space is shaped to minimize detonation. The second valve is directly behind the one shown, as may be seen in Fig. 404. As the cam shaft rotates, the various cams on it open each valve at the right moment, and the valve springs close the valves as the cams permit. The arrow chains that are seen in this picture indicate the flow of oil. Study this illustration until you are thoroughly familiar with the names of the parts.

and the “burned” gases, containing carbon dioxide, are expelled (exhausted) from the lungs. This makes the analogy clear although it is an oversimplified explanation of the body processes.

THE GASOLINE ENGINE

Since a gasoline engine, one of a class of engines which are called *Automotive engines*, is a complete power plant in itself, let us look into the details of its operations. (See Fig. 403 and Fig. 404.) A young man named Otto (1832–1891) first suggested this plan of using fuel for producing mechanical work.



Courtesy Chrysler Corporation

FIG. 404. LONGITUDINAL SECTION OF AN AUTOMOTIVE ENGINE

This view gives a better idea than Fig. 403 of the relation of certain parts. Observe the two valves, side by side, in the No. 2 cylinder. The construction of the cam shaft and of the crankshaft is clearer than in Fig. 403. The intake manifold and the exhaust manifold are seen here. The intake manifold is a series of passages that lead the fuel-and-air mixture into the various cylinders. Similarly, the exhaust manifold leads the burned gases from the cylinder into the muffler (not shown), whence the gases are discharged into the atmosphere through the exhaust pipe. The thermostat causes a variation in the size of the opening to the water jackets about the cylinders and about the combustion spaces, so that more or less water will flow into these spaces depending upon whether the temperature is rising too high or is too low.

*Getting fuel
into engine*

In Fig. 405 *a*, the piston is at the top of the stroke, the intake valve is open. As the engine turns over, the piston moves down and draws in air and fuel through the intake valve. The intake valve is connected by a passage called the *intake manifold* to the carburetor where the gasoline and air are mixed. In Fig. 405 *b*,

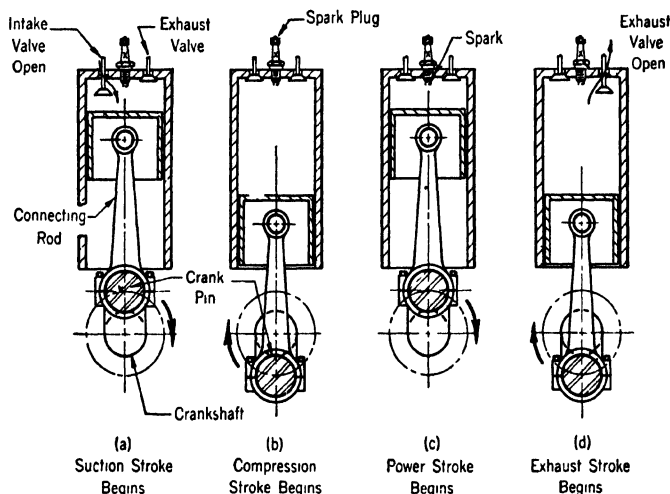


FIG. 405. OPERATION OF AN AUTOMOTIVE TYPE OF ENGINE

the suction stroke is complete, the cylinder is full of fuel and air, and the piston is about to start the compression stroke. Note that *both valves are now closed*.

*Consuming the
fuel*

As the piston moves up in Fig. 405 *b*, the fuel and air are compressed to a pressure of about 140 lb. per sq. in. When the piston has reached the top position, as in Fig. 405 *c*, a spark is caused to jump the gap of the spark plug. This spark ignites the fuel which burns very rapidly. The burned gases become quite hot and therefore tend to expand. However, if little or no expansion is permitted, the pressure rises rapidly instead. Thus, with a large pressure acting on the top of the piston in Fig. 405 *c*, the piston is forced down on the power stroke. When it reaches the bottom of the stroke, Fig. 405 *d*, the exhaust valve opens. As the piston moves up in Fig. 405 *d*, the burned gases are pushed out through the exhaust valve. Finally, when the piston reaches the top position, the exhaust valve closes, the intake valve opens, and the operation starts over from the condition pictured in Fig. 405 *a*. Present-day airplane engines operate on this same cycle of events.

Valve action

WORK AND POWER OF THE GASOLINE ENGINE

In order to get work out of a gasoline engine, fuel must be burned. Generalizing, we may say that *heat must be supplied* in order to complete a thermodynamic cycle which is designed to produce work. As an example, the events which compose the gasoline-engine cycle described above constitute a thermodynamic cycle. The work output of a cycle is often expressed as a percentage number called the *thermal efficiency* of the cycle or engine. Thus, if the efficiency of a cycle or engine is 20 per cent, the work done is 0.20 times the heat supplied. Let Q_s = the heat supplied, e = the thermal efficiency, and W = the work; then

$$\text{Work} = \text{efficiency} \times \text{heat supplied,}$$

or, we may write

$$(1) \quad W = e(Q_s).$$

For a gasoline engine, the *heat supplied*, Q_s , is the heating value of the fuel. The heating value of a fuel is approximately the heat that we can get out of it when it is burned.

Now you must know that the unit which we use for measuring heat is a British thermal unit (abbreviated B.t.u. and commonly called a *B t u*). *A British thermal unit is the amount of heat necessary to raise the temperature of 1 lb. of water through 1° temperature on the Fahrenheit scale* (see page 66 and compare in value with the calorie). The Fahrenheit scale is the one which the weatherman uses to tell us how cold or hot it is.

Heat is a form of energy and a B.t.u. is a small unit of energy. However, the B.t.u. is not so small a unit of energy as the foot-pound with which you became acquainted in the study of forces (page 481). It takes 778 ft.-lb. to make a *British thermal unit*; that is,

$$(2) \quad \text{No. of B.t.u.} \times 778 = \text{no. of ft.-lb.}; \text{ or } \frac{\text{no. of ft.-lb.}}{778} = \text{no. of B.t.u.}$$

Let us get back to heating value. In the case of liquid and solid fuels, heating values are generally expressed as so many B.t.u. per pound. For gaseous fuels, they are usually expressed in B.t.u. per cubic foot. For example, ordinary gasoline will have a heating value of about 20,000 B.t.u. per lb. The heating value of natural gas is about 1000 B.t.u. per cu. ft.

If, therefore, your automobile engine has a thermal efficiency of 20 per cent (not a bad value for this type of engine in ordinary

operation) and if you feed it 1 lb. of gasoline, the work, as obtained from equation 1, is

$$W = e(Q_s) = 0.20 \times 20,000 = 4000 \text{ B.t.u. per lb.}$$

*Computing
work*

Since gasoline weighs about 6 lb. per gallon, the work obtained from a gallon is $6 \times 4000 = 24,000$ B.t.u., approximately the work required to move your car 15 miles when it is driven at the proper speed to average 15 miles per gallon. (How many ft.-lb. is 24,000 B.t.u.? *Ans.* 18,672,000.) Let us now determine the average tractive force necessary to move the car 15 miles on level road for the conditions just stated. This tractive force is that average force which acts on the car through 15 miles. From the section on work (page 481), we recall that

$$\text{Work} = \text{force} \times \text{distance.}$$

Since the work done in moving the car is 24,000 B.t.u. = 18,672,000 ft.-lb., we may write

$$18,672,000 = \text{force} \times 15 \times 5280,$$

where the constant 5280 ft. per mile is inserted to convert 15 miles to feet so that the units on each side of the equation are the same. Solving for the force, we find

$$\text{Force} = \frac{18,672,000}{15 \times 5280} = 236 \text{ lb.}$$

Hence, a steady pull of 236 lb. would be sufficient to keep the car moving at a uniform speed under the assumed conditions.

What is power

Many people are confused by the difference between work and power; nevertheless, the real distinction is very simple. *Power is a rate of doing work.* Thus, if that pound of gasoline is used to do 4000 B.t.u. of work in 1 *min.*, the power is different from the power corresponding to 4000 B.t.u. of work done in 1 *hour*. Put it another way: suppose you drive up a mile-long hill in 1 *min.*; the power required to accomplish this is very much greater than if you take 10 *min.* to drive this mile, even neglecting the effect of the greater air resistance at the higher speed.

Horsepower

A common unit of power is a horsepower. It is defined as work at the rate of 33,000 ft.-lb. in 1 *min.*, or work at the rate of 2545 B.t.u. in 1 *hour*. This knowledge puts us in a position to calculate the horsepower of a car. Suppose you drive so that you use 3 gal. of gasoline an hour and get 24,000 B.t.u. of work from each gallon; then

Work per hr. = $3 \times 24,000 = 72,000$ B.t.u. per hr.

$$\text{Horsepower} = \frac{72,000}{2545} = 28.3,$$

which is the horsepower (hp.) being developed and delivered to the rear wheels of the automobile when it is driven at the speed which consumes 3 gal. of gasoline per hour. For an average of 15 mi. per gal., the consumption of 3 gal. of fuel per hour means $15 \frac{\text{mi.}}{\text{gal.}} \times 3 \frac{\text{gal.}}{\text{hr.}} = 45 \frac{\text{mi.}}{\text{hr.}}$ or 45 m.p.h. While the engine in your car may be *capable* of developing 100 hp., it is actually developing 28.3 hp. for the given conditions of operation.

1. An engine with a thermal efficiency of 15% is supplied with 40 B.t.u. How much work is obtained? *Problems*

Ans. 6 B.t.u.

2. An engine with a thermal efficiency of 25% is supplied with 100,000 ft.-lb. of heat. How much work is obtained?

Ans. 32.2 B.t.u.

3. An engine uses 100 lb. per hr. of a fuel which has a heating value of 2000 B.t.u. per lb. If its thermal efficiency is 18%, what horsepower is being developed?

Ans. 14.15 hp.

4. An engine developing 100 hp. is using 1 lb. per min. of fuel which has a heating value of 19,000 B.t.u. per lb. What is its thermal efficiency?

Ans. 22.35%.

5. The fuel used by a natural gas (not gasoline) engine has a heating value of 1000 B.t.u. per cu. ft. If it develops 500 hp. with a thermal efficiency of 25.45%, how many cubic feet of gas are needed per hour?

Ans. 5000 cu. ft. per hr.

6. Suppose you drive a car so that the engine is delivering 80 hp. with a thermal efficiency of 20%. The gasoline has a heating value of 20,000 B.t.u. per lb. How many gallons of gasoline are used per hour? Gasoline weighs about 6 lb. per gal. (*Note: This would be roughly equivalent to driving a 3000-lb. car uphill at 60 m.p.h. with a wide-open throttle.*)

Ans. 8.48 gal. per hr.

7. A bomber whose engines develop 8000 hp. with an efficiency of 25% makes a 6-hour trip. The gasoline has a heating value of 20,200 B.t.u. per lb. and weighs about 6.1 lb. per gal. How many gallons are needed for the trip?

Ans. 3970 gal.

CAN EFFICIENCY OF ENGINES BE INCREASED?

Yes, but not to the extent that some people think. The efficiency of an *ideal* internal combustion engine, that is, an engine with no losses in friction or heat, is expressed by the equation

$$(3) \quad e = 1 - \frac{1}{r^{k-1}},$$

where r stands for the compression ratio and k is about equal to 1.3. The *compression ratio* is equal to the volume of gases in the cylinder

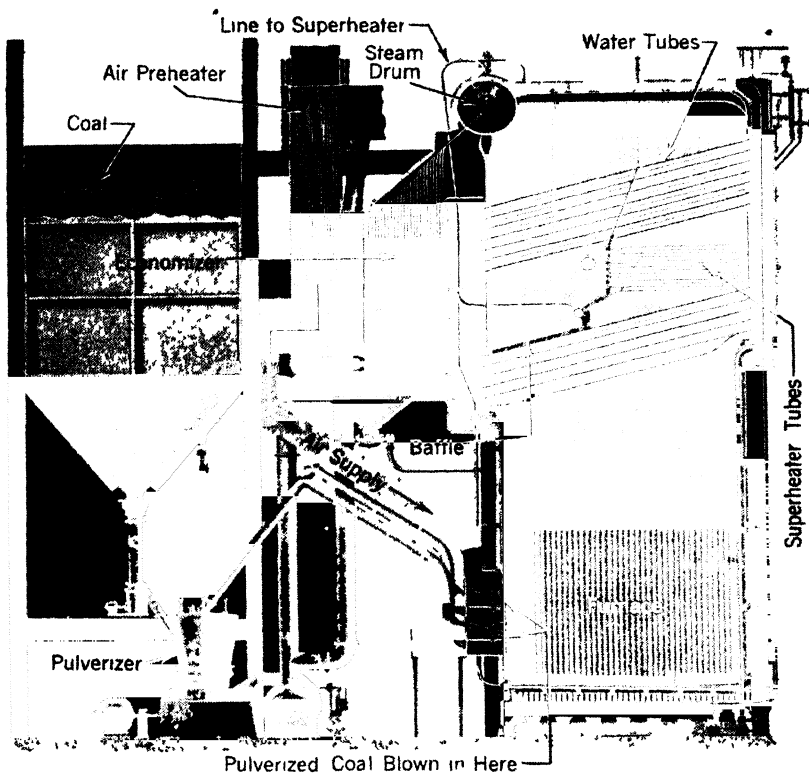
when the piston is at the bottom of the stroke (position, Fig. 405 *b*) divided by the volume of gases in the cylinder when the piston is at the top of the stroke (position, Fig. 405 *a*). If we substitute different values of r into the equation and compute the efficiency, we find that the efficiency of this ideal engine increases as the compression ratio r increases. Since experience shows that changes which increase an ideal efficiency also increase the actual efficiency of an engine, we correctly expect the efficiency of an actual engine to increase when the compression ratio is increased.

Engine knock

"Then," the reader may well say, "why don't you go ahead and run the compression ratio up so that I can get at least twice as many miles to the gallon of gasoline, or maybe three times as many?" The reason we cannot do this is because gasoline "knocks" (detonates) when the compression ratio is too high for it. This knock occurs because the fuel burns up too quickly. Since some fuels ignite easier than others and burn quicker in an engine cylinder, some fuels knock more than others. The knocking properties of a gasoline have been expressed by a number called an *octane number*. (See page 115.) For automotive engines in current use (1944), the compression ratio averages about 6.5. For this compression ratio, gasoline with an octane number of about 70 to 75 is ordinarily adequate to eliminate objectionable knocking. Gasoline of higher octane than necessary may be used in any particular engine, but there will be little if any gain in the efficiency of the engine, *unless its compression ratio is also raised*. If the gasoline has too low an octane number for an engine, the engine will knock excessively when it is hot and when the throttle is opened too wide too quickly. Excessive knocking will accelerate wear and it often results in broken pistons or other engine parts.

100 Octane

When the octane numbers were established, the highest number attainable in the laboratory was 100; but fuel with an octane number this high was so expensive that none was commercially available. During World War II, when the side with the most efficient engines was able to build the most effective aircraft, the search for high-octane fuels for use in airplane engines was so accelerated that fuels with octane numbers greater than 100 may soon become a commonplace. If this happens, our automotive engines will be designed with a much higher compression ratio and we can expect as a practical result a considerable increase in miles per gallon, that is, in the efficiency of the engine.



Courtesy Babcock and Wilcox Company

FIG. 406. STEAM-GENERATING UNIT

This illustration shows a boiler that is complete with accessories. When there are accessories, such as economizer, air preheater, and pulverizer, the device is usually called a steam-generating unit, rather than a boiler. After a brief survey of this illustration, finish the study of steam power plants and then come back to this illustration and study it in detail.

A *superheater* (see picture) heats the steam to temperatures above its boiling point, the steam coming to the superheater from the steam drum. The *economizer* heats the water entering the boiler with heat from burned gases going to the smoke-stack. This is a heat-saving process because otherwise the heat would be wasted out of the stack. The *air preheater* is also a heat-saving device, as it uses some of the heat still in the burned gases, after they have left the economizer, for the purpose of heating the air which supports combustion in the furnace. Since this air is necessarily heated in the furnace during the combustion process, all heating done outside of the furnace with heat that would otherwise be lost represents a saving in fuel costs.

Not all steam-generating units consume pulverized coal. That is a feature of this particular unit. Since pulverized coal burns much more rapidly than lump coal, the capacity of a given size of furnace for generating steam is greater with pulverized coal.

*More miles
per gallon*

For the sake of your pocketbook, observe that the use of high-octane gasoline is of no advantage unless the engine is designed with a compression ratio to fit the octane number. If your engine does not knock on gasoline of a certain octane number, do not buy a higher octane, and you might even try a lower octane gasoline.

HOW TO GET WORK OUT OF STEAM

To get work out of steam, the first problem is to produce some steam. Steam is obtained by evaporating water in a *boiler* or *steam generator* (Fig. 406). Then we need a *steam turbine* (Fig. 407 and

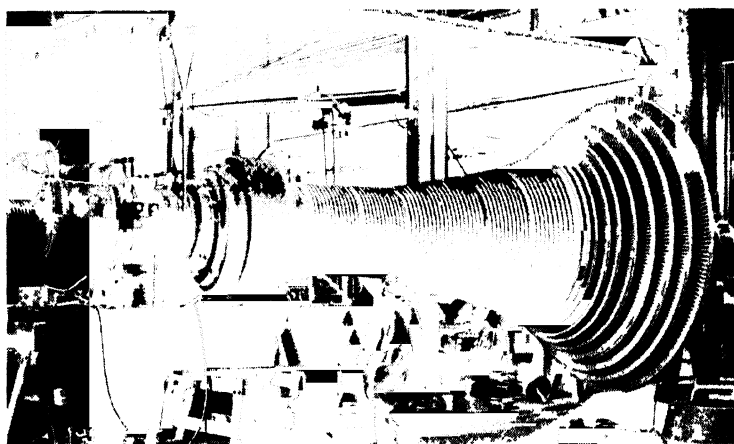


FIG. 407. ROTOR OF A TURBINE

Courtesy, Acheson-Chalmers Mfg. Company

The rotor of a turbine is the rotating element. The concentric rings which you see are rows of turbine blades. The steam is directed into these blades and therein it undergoes a change of momentum (see p. 489). Recalling that a change of momentum per unit of time represents force, we see that a force on the turbine blades is produced, thereby turning the rotor and doing work. This rotor is for a 40,000-hp. unit.

(Fig. 408) or a *steam engine* (Fig. 410). These machines are designed to take high-pressure, high-temperature steam, convert some of the energy of the steam into work, and discharge low-pressure, low-temperature steam.

*Condensing
exhaust steam*

The low-pressure steam from the turbine or engine may be discharged into the atmosphere, but in the best regulated power plants, it is discharged into a *condenser* (Fig. 409). A condenser is a device that will condense the steam by using cold water, say, from some convenient lake or river. After the steam is condensed into water, it is returned to the boiler through *pumps*, and the cycle is repeated.

Combining all of these devices we get a steam power plant as represented diagrammatically in Fig. 411. In this layout, we are assuming that all steam and water lines as well as the turbine itself are perfectly insulated so that there is no loss of heat. Observe that the steam is expanded to a pressure which is below atmospheric pressure (under vacuum) when a condenser is used; hence, the water (condensate) leaving the condenser is first pumped, by a pump called the *condensate pump* (Fig. 411), to atmospheric pressure and into a reservoir. Another pump, called the *feedwater pump*, delivers the water into the boiler. (See Fig. 412 for an idea of the physical arrangement of the elements of a power plant.)

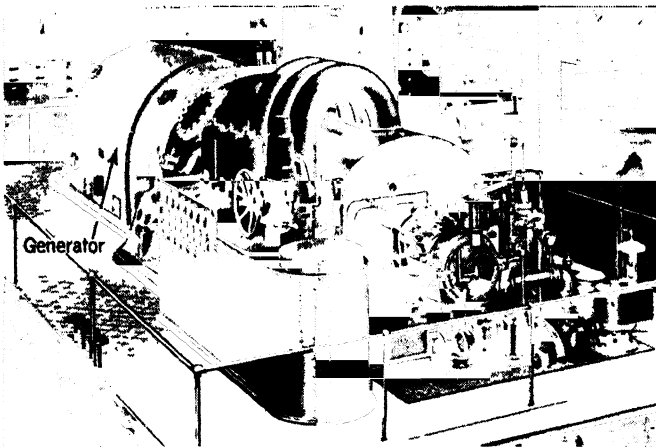


FIG. 408. MEDIUM-SIZE TURBINE

Courtesy Allis-Chalmers Mfg. Company

This is a view of an installed turbine of 10,000-hp. capacity. The linkage system at this end of the turbine is part of the governor mechanism which maintains the speed of the turbine constant at 3600 r.p.m. The curved pipes entering the top of the front part are steam lines. The condenser is below this floor level and is not in sight.

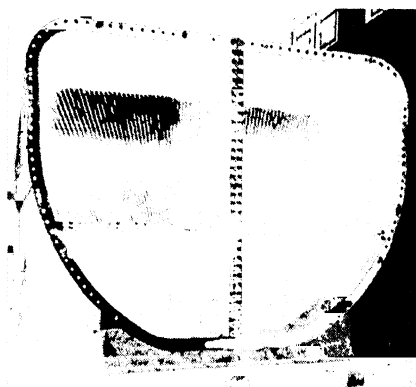
Remember the Law of Conservation of Energy (pages 55 and 483)? Applied to heat engines and heat cycles, this law goes by the formidable name of the *First Law of Thermodynamics*. Let us review some statements of this law; thus

1. Energy is indestructible.
2. Energy may be converted from one form (such as heat) to another form (such as work).
3. The energy entering a device or substance is equal to the energy leaving it (if none is stored).

Actually, the second and third statements are conclusions that follow from the first statement, but they help to simplify our mental processes.

*Applying
First Law*

In applying the *First Law* to a cycle, the only energy quantities with which we are concerned are heat and work. Then if all energy is to be accounted for, if none is stored or taken from storage, the



Courtesy Foster-Wheeler Corporation

FIG. 409. SURFACE CONDENSER

This view shows the end of a condenser with the cover removed. Just inside of the end is a sheet of metal, called the *tube sheet*, to which water tubes are attached. The lines seen on the tube sheet sloping slightly toward the center position are made by the tubes, which do not show up distinctly on this small reproduction. Water flowing through these tubes picks up heat from the exhaust steam surrounding the tubes on the other side of the tube sheet, with the result that the steam condenses into water. Observe in Figs. 411 and 412 how the units described in Figs. 406, 407, 408, 409, and 410 are combined to make up a power-plant cycle.

energy entering the steam and water is equal to the energy leaving the steam and water. Figure 411 depicts these energy quantities by heavy arrows. The heat supplied Q_s is the energy added to the water and steam from the furnace fires in the boiler.

*Output and
input*

The *output* work W_{out} is the work delivered by the turbine; it is energy leaving the steam. The heat Q_r is heat rejected from the steam to the circulating cooling (condenser) water, and it is carried away by this water. The *input* work quantities W_{in1} and W_{in2} represent the energy required to drive the pumps which force the water back into the boiler. Using the First Law of Thermodynamics (Law of the Conservation of Energy), we have

Pump work

$$\begin{aligned} \text{Energy in} &= \text{Energy out,} \\ (4) \quad Q_s + W_{in} &= Q_r + W_{out} \end{aligned}$$

Transposing, we find

$$(5) \quad W_{\text{out}} - W_{\text{in}} = W_{\text{net}} = Q_s - Q_r, \quad \text{Net work}$$

where $W_{\text{in}} = W_{\text{in1}} + W_{\text{in2}} =$ total pump work. In many practical applications, the pump work, W_{in} , is so small as to be negligible; so do not be alarmed if we fail to include it in our computations. (Be sure to study the caption to Fig. 411.)

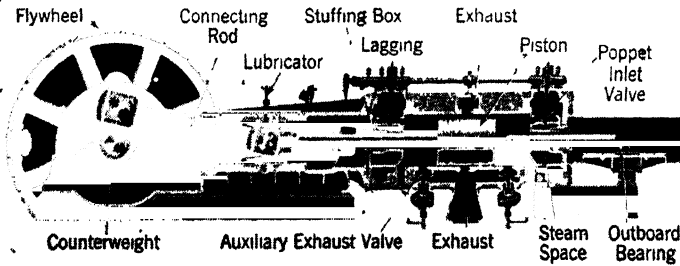


FIG. 410. RECIPROCATING STEAM ENGINE

Courtesy Skinner Engine Company

The reciprocating steam engine is often economical in small power plants. The type of engine shown, the most modern, is built in sizes up to about 1100 hp. For large amounts of power (see Fig. 407 and Fig. 408), the only competitor for the steam turbine is the water turbine. In this engine, called a *uniflow engine*, the steam enters through the inlet valve (one at each end of the cylinder) and passes out of the cylinder through exhaust ports cut around the periphery of the cylinder at its central section. The movement of the cylinder uncovers (opens) and then covers (closes) the exhaust ports. High-pressure steam acts first on one side of the piston and then the other.

Equation 5, which says that the *net work* done by a cycle is equal to the heat supplied minus the heat rejected, is a very useful form of the Law of the Conservation of Energy. It applies to *any* kind of a thermodynamic cycle. For example, we may first apply it to the gasoline engine already discussed. We have estimated the work as $W = 4000$ B.t.u. per lb. of gasoline. The heat supplied is $Q_s = 20,000$ B.t.u. per lb. Therefore, for each pound of gasoline burned, the amount of rejected heat is

$$Q_r = Q_s - W = 20,000 - 4000 = 16,000 \text{ B.t.u. per lb.}$$

Some of this heat passes through the exhaust system and some of it goes into the cooling water, but *all* of it is eventually discharged into the atmosphere. Remember that equations 1 and 5 are just as useful for a steam power plant as they are for the gasoline engine.

If 1100 B.t.u. per lb. of steam are supplied in a boiler and if 800 B.t.u. per lb. are rejected, determine the work and the thermal efficiency. Assume that 100,000 lb. of steam circulate each hour

Rejected heat

Example with steam

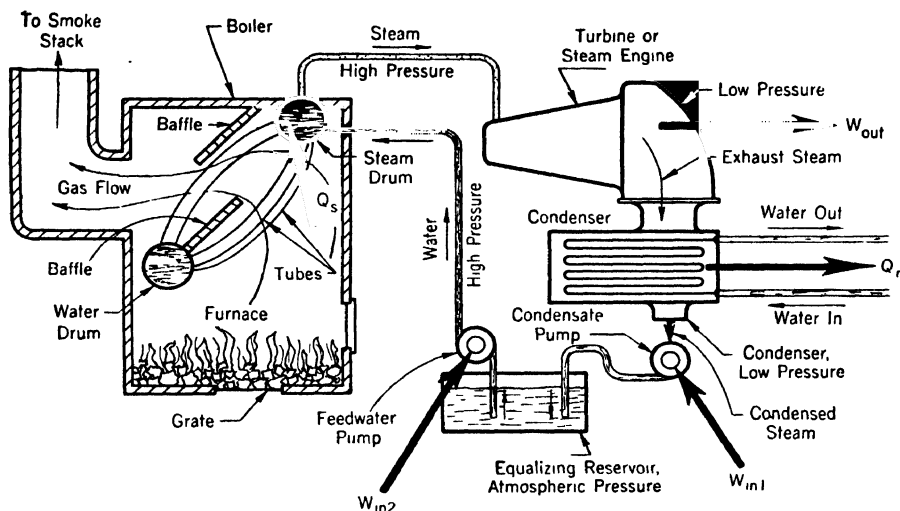


FIG. 411. DIAGRAMMATIC LAYOUT OF A STEAM POWER PLANT

The heavy arrows designate the flow of energy. From the fire in the furnace, heat Q_s flows into the water (in the water drum and tubes) causing the water to boil. Steam rises and collects at the top of the steam drum. This steam, at high pressure and temperature, flows to a turbine or reciprocating steam engine. In the turbine (Fig. 408), the steam expands to a low pressure, and at the same time, some of the energy of the steam is transformed into work output, W_{out} . Usually this work is delivered by the turbine shaft to an electric generator, where it is transformed into electricity. As you know, electricity has many uses, one of which is to do work driving other machines, for example, the pumps in this illustration. After the steam has expanded to a low pressure, it passes into a condenser (Fig. 409), through which cold "condenser" water is circulated. The condenser water, obtained from any natural source, carries away the heat Q_r taken from the steam as it is condensed. The colder the condenser water, the lower the temperature, and therefore the pressure, of the condensed steam. For this reason, a modern steam power plant is more efficient in the winter than in the summer. The condensed steam (water) from the condenser flows to a pump, called the condensate pump, and is pumped into a reservoir at atmospheric pressure. The reservoir acts as an equalizing chamber which is needed because it is a practical impossibility to keep the rate of flow of water from the condenser exactly equal to the rate of flow into the boiler.

Another accessory, the feedwater pump, forces the water back into the boiler, after which the cycle is repeated. It takes work to run these pumps, this work being an energy *input*. The pumps may be driven by electric motors which utilize a part of the electrical *output* of the generator connected to the turbine. Thus, the net work W_{net} which may be used for duties outside of the power plant is the total output, W_{out} , minus the total input, W_{in} . The pump work $W_{in1} + W_{in2}$ is small as compared to the output of the turbine and is therefore often neglected. In the furnace, the flow of the hot gases of combustion is directed by *baffles* so as to get the maximum amount of heat transferred to the water and steam. A baffle is simply a partition. Boilers are built in a large variety of forms and arrangements of drums and tubes (see Fig. 406 again).

and compute the horsepower output of the turbine. Neglect the pump work. We find immediately that

$$W = Q_s - Q_r = 1100 - 800 = 300 \text{ B.t.u. per lb.}$$

$$e = \frac{W}{Q_s} = \frac{300}{1100} = 27.3\%$$

The amount of work per hour is evidently the work per pound of steam times the pounds of steam per hour, or

$$W = 300 \times 100,000 = 30,000,000 \text{ B.t.u. per hr}$$

Then, since a horsepower is equivalent to 2545 B.t.u. per hr., the horsepower of the turbine is

*Computing
horsepower*

$$\frac{30,000,000}{2545} = 11,780 \text{ hp.}$$

This turbine would be classified as a medium-size turbine.

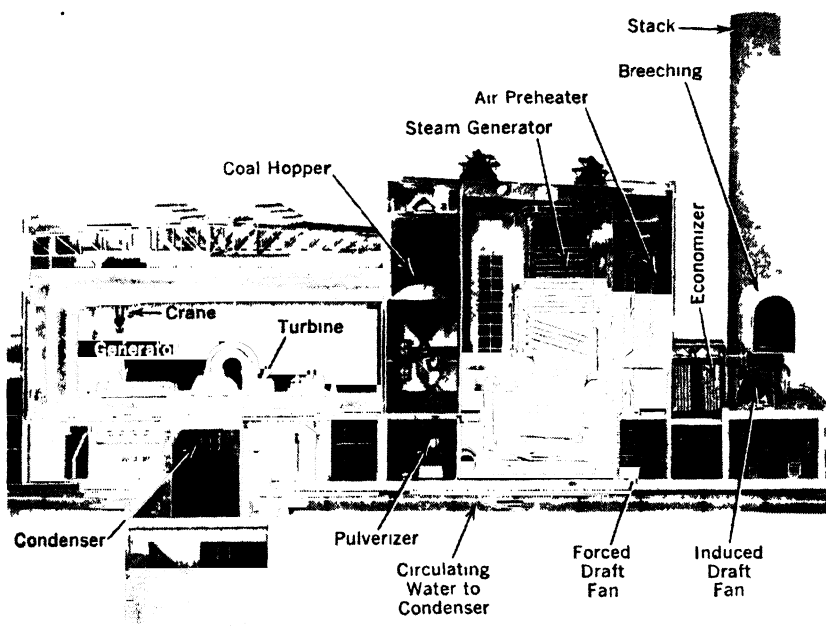


FIG. 412. A POWER PLANT

Courtesy Apex Photo Company

This illustration, which is a photograph of a model of a large steam power plant, gives you an idea of the actual arrangement of the various units. Study it carefully with frequent comparison with Fig. 411. The *induced draft fan* (right) acts to *pull* the air through the furnace, air preheater, and economizer; and the *forced draft fan* acts to push the air through these units. The use of one or more fans for the air supply makes it possible to increase the amount of air entering the furnace, and therefore to burn more fuel. The more fuel burned, the more heat released, and if the design is correct, the more steam generated.

You may have observed that the thermal efficiency in this example was not computed by using the heating value of the fuel, as it was for the internal combustion engine. Although the method given above is commonly used for computing the efficiency of a steam power cycle, the so-called over-all thermal efficiency of a power plant would be based on the heating value of the fuel consumed in the furnace.

*Over-all
efficiency*

Thus, suppose a power plant uses 2 lb. of coal, whose heating value is 13,000 B.t.u. per lb., for 3000 B.t.u. of work. The over-all thermal efficiency is then

$$e = \frac{3000}{2 \times 13,000} = 11.52\%.$$

This efficiency represents a typical value for a good medium-size plant. (If you are interested in a little more detail concerning a modern steam power plant, study Fig. 412 carefully.)

Problems

1. An inventor claims that he will be able to get 7,780,000 ft.-lb. of work from his engine in a specified time. An engineer investigating the claim finds that, during the same time, the fuel to be supplied to the engine releases 8900 B.t.u. of heat. Do you agree with the engineer's report that this is a crackpot invention? What law is violated?

2. For 1 lb. of steam, the heat supplied in a steam power engine is 1450 B.t.u. and the heat rejected is 900 B.t.u. If 1,000,000 lb. of steam are used per hour, determine the thermal efficiency and the horsepower developed.

Ans. 37.9%, 216,000 hp.

3. In a steam cycle, 1,000,000 B.t.u. per min. are supplied when the thermal efficiency is 20%. Determine the heat rejected and the horsepower developed.

Ans. 800,000 B.t.u. per min., 4720 hp.

4. The best performance obtained so far in steam power plants is about 1 hp-hr. (= 2545 B.t.u.) of work for 1 lb. of coal burned. If the coal has a heating value of 14,000 B.t.u. per lb., what is the over-all thermal efficiency?

Ans. 18.2%.

ANOTHER LAW

Waste heat

Perhaps you have speculated on the heat being rejected in the foregoing examples and wondered why something is not done to get more than, say, 20 to 25 per cent of the heat supplied converted into useful work. Some people have been heard to remark to this effect, "Of 20,000 B.t.u. in 1 lb. of gasoline, 16,000 B.t.u. are wasted. When the engineers learn how to make an engine 100 per cent efficient, we will get 75 miles to a gallon of gasoline instead of 15." However, it is not so simple as that. We may some

day drive cars 75 miles on a gallon of gasoline (motorized bicycles go that far on a gallon now), but if we do, it will not be because the motor is 100 per cent efficient. Why? Because with our present knowledge, it can be demonstrated that no conceivable heat engine can have a 100 per cent thermal efficiency — even on paper.

A young Frenchman named Carnot (1796–1832) has shown beyond doubt that the maximum conceivable thermal efficiency is *Carnot efficiency*

$$(6) \quad e_{\max} = \frac{t_1 - t_2}{t_1 + 460},$$

where t_1 is the highest temperature in the cycle and t_2 is the lowest naturally available temperature. *By naturally available temperature*, we mean a temperature provided by nature, such as the temperature of the air, or of a near-by river or lake. The law which we are leading up to is called *the Second Law of Thermodynamics* and is an outcome of Carnot's study. This law says in effect that no engine can convert all the heat supplied to it into work, that some of the heat must be rejected. As seen from equation 6, t_2 must equal -460°F. if $e = 100$ per cent; but -460°F. is absolute zero temperature, far below any naturally available temperature. To illustrate with numbers, assume that the highest temperature of the steam in a steam turbine is 1000°F. This is about the highest that we can use at the present time because we have no suitable material which can withstand a higher temperature and retain its strength, although some are on the way. Assume also that the naturally available temperature at the time is 80°F. The corresponding maximum possible efficiency is

$$e_{\max} = \frac{1000 - 80}{1000 + 460} = 63\%.$$

To be sure, this efficiency is much higher than any actually obtained, which suggests that there is room for improvement in the processes of obtaining power. There is room, and improvements are being made. However, there is not so much opportunity for improvement as the high ideal efficiency implies, because there are some inevitable losses due to friction and loss of heat, and because of the practical impossibility of adjusting the actual processes of a cycle into an ideal combination. It is probable that the best *actual thermal efficiency* will not soon exceed about one-half of this *theoretical maximum efficiency* in turbine-equipped power plants. *Room for improvement*

Problems

1. An inventor claims that his engine will produce 1000 B.t.u. of work from a certain amount of fuel. Upon investigation an engineer finds that 1500 B.t.u. are supplied to the engine and that the highest temperature in the cycle is 600°F . while the lowest natural temperature is 70°F . Could the inventor be justified in his claim? Explain.

2. The inventor mentioned in problem 1 later changes his claim to 750 B.t.u. of work from 1500 B.t.u. supplied. Is this possible? What would you guess as the work that might be obtained from this engine, assuming that it is really a good one?

Ans. Actual work, say, 375 B.t.u.

THE ELECTRIC REFRIGERATOR

Reversed cycle

For the electric refrigerator in your kitchen, you should be thankful to the man who first conceived of the thermodynamic cycle for refrigeration. This cycle is, in a sense, the reverse of cycles devised for the production of power. Whereas, in the power cycle, heat is supplied from which work is obtained, in the refrigerating cycle, work is supplied in order to obtain refrigeration. In other words, the *output* of a power cycle is work and the *input* is heat; the *output* of a refrigerating cycle is refrigeration (a removal of heat) and the *input* is work.

It is well at this stage to point out that a particular substance, water, for example, will boil at a temperature which has a definite relation to the pressure on the substance. (See page 71.) At atmospheric pressure of 14.7 lb. per sq. in., water boils at 212°F . At any other pressure, the boiling temperature of water is not 212°F .

Boiling temperatures

At a pressure of 100 lb. per sq. in. absolute,* water boils at 327.8°F . As an illustration of another substance, sulfur dioxide at 100 lb. per sq. in. boils at 110°F . Any substance that will exist in a liquid phase, say, iron or ammonia, has its own characteristic boiling temperature which is related to the pressure on the liquid.

Refrigerating with steam

Steam can be and is used for refrigeration. Steam could be, but is not, used to freeze water. When the pressure on water is reduced to 0.0885 lb. per sq. in. absolute, it will boil at 32°F . If the pressure is reduced below 0.0885 lb. per sq. in., steam will exist at a temperature below 32°F . However, except under certain favorable conditions, steam (and water) does not make a satisfactory

* A pressure gage (an instrument for reading pressures) measures the difference between the *absolute pressure* and the *atmospheric pressure*. At sea level, the atmosphere exerts a pressure of 14.7 lb. per sq. in. Thus, if a gage reads 85.3 lb. per sq. in., the absolute pressure (or the pressure measured from a state of no pressure) is $14.7 + 85.3 = 100$ lb. per sq. in.

refrigerant (a substance used in a refrigerating cycle), because, for one reason, these very low pressures (vacuums) are difficult to maintain.

A common refrigerant for household refrigerators is sulfur dioxide. (See page 100.) At 15.21 lb. per sq. in. absolute, sulfur dioxide boils at 15° F., a temperature sufficiently low for freezing water, and, as mentioned before, it boils at 110° F. when the absolute pressure is 100 lb. per sq. in. Note these pressures and temperatures carefully, as we shall use them in the following discussion.

How does a refrigerating cycle operate? Let us start at A, Fig. 413, where vaporous sulfur dioxide (SO₂), say, is entering the *compressor* at a low pressure of 15.21 lb. per sq. in. and a temperature of 15° F. The compressor pumps the SO₂ to a high pressure, for example, 100 lb. per sq. in. which raises its temperature to 110° F. Now that the refrigerant is hotter than the surrounding air (or other naturally available temperature), it passes into a *condenser*, where the refrigerant is condensed by cooling because heat flows from the hot SO₂ at 110° F. to the cooler outside air at 80° F. In larger installations, the condenser uses water, instead of air, for carrying away heat.

As the SO₂ condenses into a liquid at about 90° F., it flows into the *receiver*, which is simply a convenient equalizing reservoir. (See Fig. 411 again.) The liquid outlet of the receiver is a tube which extends nearly to the bottom of the receiver, to be sure that nothing but liquid enters it. The liquid at high pressure and 90° temperature passes through an *expansion valve*, which is a device that introduces a restriction to flow in the line — much as the ordinary water faucet is a restriction to the flow of water when it is not wide open. The expansion valve separates the high-pressure and low-pressure regions, as observed in Fig. 413, just as there is high-pressure water on the pipe side of the water faucet and low pressure on the outlet side. With a drop in pressure, there is a drop in temperature.

The liquid refrigerant then enters an *expansion coil*, in this case, the space around the freezing compartment (Fig. 413). Notice that the SO₂ is now in the low-pressure region where it will boil at a temperature, in our illustration, of 15° F. But to make it boil, heat must be supplied to it. Since the home refrigerator has an inside temperature of some 40° to 50° F., the heat flows from the contents of the box into the SO₂ in the coil, and the SO₂ therefore

Sulfur dioxide

*Compressing
the refrigerant*

*Expansion
valve*

*Boiling the
refrigerant*

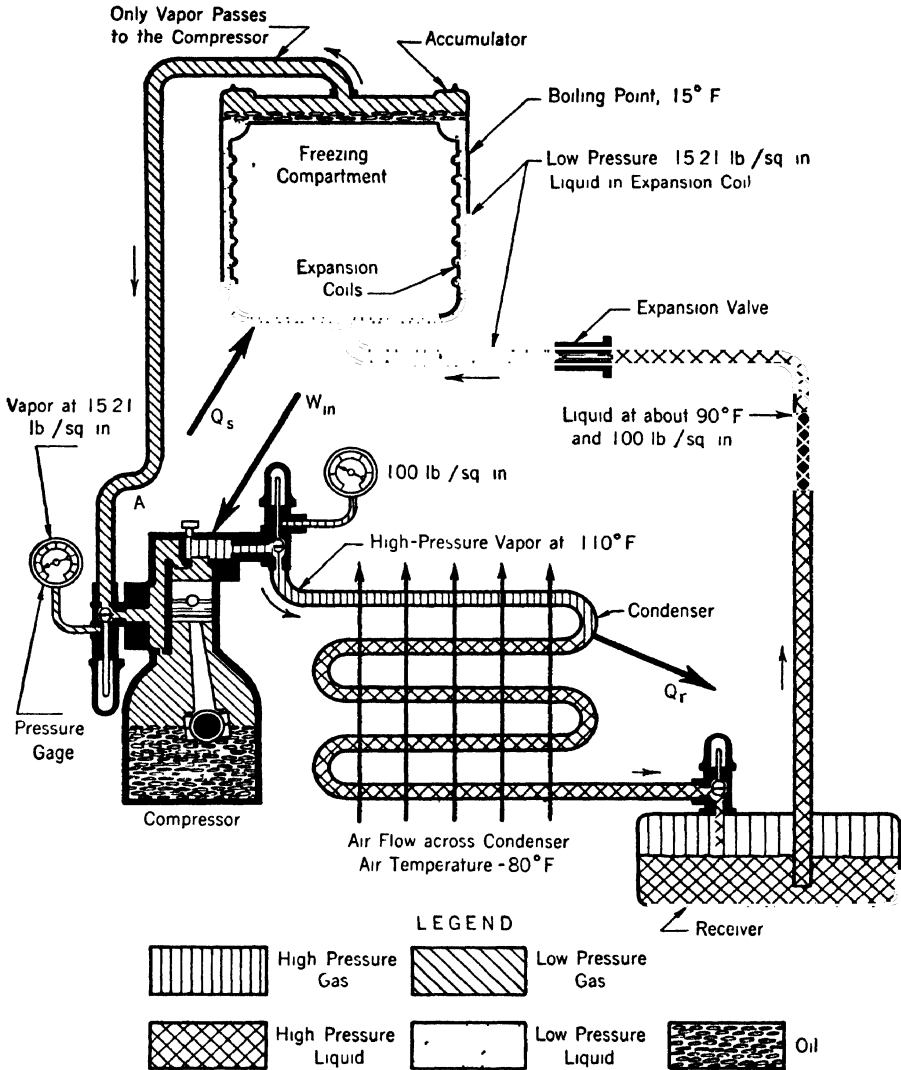


FIG. 413. DIAGRAMMATIC LAYOUT OF A REFRIGERATING UNIT

With certain refinements and controls omitted, this illustration shows how the refrigerating cycle operates. (See text.) At the condenser, heat flows from the 110° F. vapor to the 80° F. air. The condenser on a home refrigerator usually is located on the back of the box. Inside the box, heat flows from the contents of the box into the 15° F. liquid, causing it to boil (evaporate). Low-temperature vapor passes to the compressor where it is compressed to a high pressure and temperature, whence it restarts its cycle. The heavy arrows indicate the flow of energy. From the law of the conservation of energy, we see that $Q_s + W_{in} = Q_r$, or heat to be removed = heat absorbed from box plus work of compressor.

boils (evaporates). The evaporated SO_2 then flows on to the compressor, entering at A , and the cycle starts over.

You may wonder why the box does not cool on down, nearly to a temperature of 15°F . The principal reason is that automatic controls stop the operation of the compressor when the temperature reaches the point for which the controls are set. Then the pressure, and the corresponding boiling temperature, in the expansion coil is allowed to rise until the temperature in the box reaches its higher limit, at which point the compressor starts again and pumps SO_2 out of the expansion coil until the lower limit of temperature is reached again. The operation is so governed that the refrigerating cycle pumps out of the box all the heat that comes into the box from the outside and the heat which must be discharged to cool the contents. *Controlling temperature*

Observe that heat flows into the SO_2 while it is in the box (or other space to be cooled) and that the compressor serves to pump the SO_2 to a higher temperature than the naturally available temperature outside, so that the heat picked up in the box is discharged outside. The refrigerating cycle is therefore a *heat pump*, that is, a cycle which pumps heat from a low-temperature region to a high-temperature region. A moment's reflection reminds us that *heat naturally flows from the hotter body to the colder body*. You know by experience that you cannot raise the temperature of a hot stove with your finger. Of course, there are other means of obtaining a refrigerating cycle than the one described (which is the most common), but in all cases we must go to considerable trouble and expense to force the heat from a cold body to a warmer body. The arrangement of an actual domestic refrigerating unit is shown in Fig. 414. *Pumping heat*

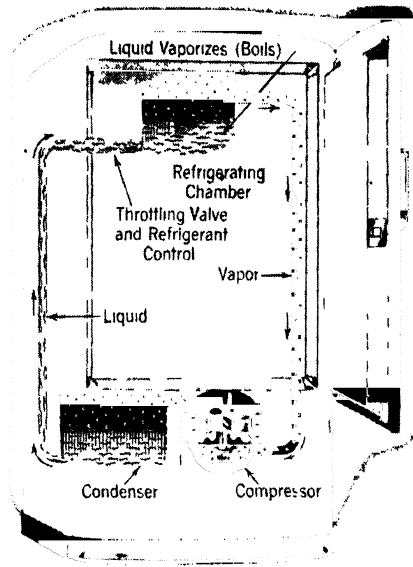
There are several substances, other than sulfur dioxide, which are used in refrigerating cycles. Ammonia (page 107) is used for large commercial ice plants since it has good thermodynamic properties for refrigerating purposes and is relatively cheap. Dichlorodifluoromethane or *freon* has found many applications, especially in air-conditioning systems, because it is not poisonous, it is not inflammable, and it too has good thermodynamic properties for the purpose. *Other refrigerants*

The reason for the existence of a power cycle is to produce work which can be devoted to many useful purposes. The modern power plant generates electricity (one form of energy) which is readily

converted into work. Some of this electricity runs the electric motor of Fig. 414 which supplies the work to keep the refrigerating cycle operating.

*Coefficient
of performance*

One nice thing about the refrigerating cycle is that by efficient design and construction we can get more B.t.u. of refrigeration



Courtesy, Frigidaire Division, General Motors

FIG. 414. DOMESTIC REFRIGERATING UNIT

*Freon
refrigerant*

This unit operates with freon as the refrigerant. After the freon evaporates in the refrigerating chamber (which is equivalent to the cooling coils in commercial machines), it enters the compressor as a vapor. This compressor, a rotary type, pumps the vapor to a high pressure and delivers it to the condenser. The natural flow of air across the condenser carries away sufficient heat to cause the freon to condense into a liquid. From the condenser, the liquid passes through the control device and is throttled to the lower pressure which exists in the refrigerating chamber. At this lower pressure, it boils, thus picking up heat from the inside of the cabinet. The vapor then starts the cycle anew, going to the compressor.

than the B.t.u. of work required. Thus, if the efficiency is thought of as being *output over input*, the output is the refrigeration and the input is the work. Since this ratio is likely to be greater than unity above 100 per cent, we do not call it efficiency. It goes by the name of the *coefficient of performance* and is abbreviated *C.O.P.* Thus, the coefficient of performance is

$$(7) \quad C.O.P. = \frac{\text{refrigeration.}}{\text{work}}$$

Suppose the refrigeration is 200 B.t.u. per min. and the work supplied is 80 B.t.u. per min. Then

$$C.O.P. = \frac{200}{80} = 2.5,$$

which means that for every B.t.u. of work done by the motor, we shall get 2.5 B.t.u. of refrigeration, that is, 2.5 B.t.u. are removed from the space being refrigerated. This is not a case of getting something for nothing since the refrigerator does not produce heat energy — it merely transports it.

AIR CONDITIONING

How would you like to have your home air conditioned? It probably is conditioned some of the time, because anything that we do to atmospheric air, such as heating it in the wintertime, is a process of conditioning air. However, no doubt when you think of air conditioning you think of cooling it for greater comfort in the summertime. As the engineer thinks of it, air conditioning means much more than either heating or cooling. A complete job of air conditioning involves the following processes:

What is air conditioning

1. Cleaning the air of dust, smoke, pollen, and odors.
2. Heating the air in the winter, or cooling it in the summer.
3. Regulating the humidity of the air — removing moisture in summer, adding moisture in winter.
4. Circulating or delivering air to the proper spaces in the proper quantities.

Let us first consider what is meant by *humidity*. All atmospheric air contains steam. Even if you are reading these pages when the outside temperature is -20°F . (20° below zero), that outside air contains steam. Of course, the pressure of the steam must be quite low, below 0.0885 lb. per sq. in. (page 508). While atmospheric pressure is about 14.7 lb. per sq. in., this *total pressure* of 14.7 lb. per sq. in. is exerted by several separate and distinct substances. Air is mostly oxygen and nitrogen, but it also contains argon, steam, and traces of several other gases. Each one of these gases exerts some part of the pressure which we call *atmospheric pressure*, so that the total pressure of the atmosphere is the sum of the pressures, called *partial pressures*, exerted by the individual gases. Hence, the steam or water vapor may and does exist at extremely low pressures under certain conditions, and it would rarely, if ever, have a partial pressure as high as 1 lb. per sq. in. in atmospheric air.

Humidity

*Adding steam
to air*

Let us consider the air in a closed room. It has some steam in it, but not so much as it might have. If we wished to add more steam, we could put a pan of water in the room, and water would evaporate just as puddles of water evaporate from the road after a rain. If we put several pans of water in the room, some of the water will evaporate, but after a certain point is reached, evaporation will be balanced by an equal condensation. When this limit is reached, the air is said to be *saturated*, meaning that it contains all the steam or moisture that it can "hold." The amount of steam that a particular amount of air can hold depends upon the temperature of the air — the higher the temperature, the greater the amount of steam possible. However, it is important to note that for each air temperature, there is a fixed maximum quantity of steam that may be retained in the air.

The importance of this limitation upon evaporation is demonstrated by an amusing experience. Someone once designed a drying room in a gymnasium for drying the wet clothes of the athletes. A number of steam pipes were installed, so that the room was extremely hot, but it was unventilated. Consequently, the clothes never got dry, because the air became saturated and then refused to take more moisture from the clothes. When the room was ventilated so that unsaturated air entered, it made an excellent drying room.

Dew point

Starting over again with our roomful of air, we shall not put any pans of water in it, but instead we shall cool the air. Since the capacity of the air for holding steam decreases as the temperature goes down, a point is eventually reached where the moisture originally in the air is enough to saturate the air. This temperature at which the cooled air becomes saturated is called the *dew point*. *If air is cooled below the dew point, some of the steam will condense out as water.* Surely you have noticed moisture on cold windowpanes in the winter or on some other cold surface in a warm room. This water is deposited on the cold surface because the air adjacent to the surface has been cooled below its dew point. If one has too much trouble with such condensation, the solution of the problem is to decrease the humidity of the air in the room or to increase the temperature of the cold surface.

*Relative
humidity*

The amount of steam that air contains is usually expressed by its *relative humidity*, which is the ratio of the weight of steam in a certain amount of air divided by the weight of steam that the

same air can hold at that temperature when it is saturated. Suppose that the relative humidity is 70 per cent ($= 0.70$). This means that the air contains 70 per cent as much steam as it can hold. If the relative humidity is 100 per cent, the air is saturated air; at 0 per cent humidity, the air contains no steam and is said to be dry air.

Although the actual amount of steam in atmospheric air is small, its presence plays an important role in our feeling of comfort. No doubt you have heard the expression, "It isn't the heat; it's the humidity." Humidity can certainly affect your comfort or discomfort; here's how it works.

Your body is perspiring even when it feels dry. If you are not exercising strenuously and if the relative humidity is not high, the perspiration evaporates as fast as it forms. This evaporation goes far toward making you feel comfortable on a warm day, because as the perspiration evaporates there is a cooling effect on the body. An alcohol rub on a feverish body lowers the temperature of the body for the same reason — because the evaporation of the alcohol, which takes place at a more rapid rate than the evaporation of water, carries away heat. Thus, on hot days of high humidity, since the air is nearly saturated, the *rate of evaporation* of perspiration is very low, moisture stays on the skin, and in the absence of the cooling effect of evaporation you feel uncomfortably warm.

The amount of moisture that the body normally gives up through perspiration is surprisingly high. An average person sitting at rest for 24 hours loses about half a gallon of water through perspiration. A man sawing wood perspires at the rate of about 4 gallons in 24 hours. Thus, one of the principal functions of air conditioning in the summer is to *remove enough moisture from the air* so that it is "dry" enough to readily evaporate the perspiration from the skin.

In the wintertime, the opposite condition often exists in heated spaces. Suppose that cold atmospheric air at 20°F. , which may have an average relative humidity of 60 per cent, is heated to a room temperature of 70°F. without the addition of moisture. Since its capacity for holding moisture increases with the temperature and since the original amount of moisture remains unchanged, the air at 70°F. will have a much lower relative humidity, in fact about 9 per cent. Such air is quite dry and is not considered to be healthful. Moreover, this dry air results in a high rate of evaporation of moisture from the skin, so that if you are not warmly clad, you will feel uncomfortable. If moisture is added

Evaporation of perspiration

Amount of perspiration

Humidity of heated air

to this 70° F. air until the relative humidity is brought back to 60 per cent, the rate of moisture evaporation from the skin decreases, less heat is carried away from the body, and comfort is obtained with much lighter clothing. In a typical five-room house, for twenty air changes daily, this increase of humidity would require the evaporation of about 8 gallons of water each day. However, it is not at all necessary for health reasons to bring the humidity up to 60 per cent. Moisture from cooking and perspiration ordinarily keeps the humidity in our homes at a reasonable level.

*Winter
conditioning*

With these phenomena in mind we see that for best results in the winter, the air needs to be not only heated but also humidified. Also, if your city is smoky and dusty, you will find further comfort and convenience when the air introduced into your living spaces is cleaned.

Clean air

Air is cleaned in various ways: (1) by passing it through filters, which operate on the same principle as the oil filter of your automobile; (2) by passing it through a water spray where the minute (invisible) particles of dirt are picked up by a fine spray of water (the water spray may also be used to add moisture for winter air conditioning); (3) by an *electric precipitator*, which shows great promise but has not yet been widely used.

PROBLEMS OF SUMMER CONDITIONING

*Cooling air by
evaporation*

First, consider the situation in dry climates where the relative humidity is low. In this case, the cooling of air is often fairly simple. It takes heat to evaporate water. If air is blown through a spray of water (even at the same temperature as the air) or through a mesh of some sort which is saturated with water, some of the water will evaporate into the air. Part of the heat necessary to vaporize it comes from the air, so that the air will be cooled by this abstraction of heat. For saturated air, none of the water will be evaporated and no cooling will occur (unless the water is colder than the air). If the air is nearly saturated, some cooling may occur, but the resulting air will have such a high relative humidity that it may not be comfortable.

*Excessive
humidity*

Possibly you have entered small theaters advertised as air conditioned, observed the cooler air all right, but when you got up to leave, found that the back and seat of the chair were covered with moisture. If so, no doubt the water was there because the air was too nearly saturated to absorb your perspiration as fast as it

occurred. However, if the air has a *low humidity*, as stated at first, some moisture may be added to it without the loss of comfort and without ending up with an excessive humidity. Yet a decided cooling of the air may be obtained at the same time by evaporating the water.

Most people in this country live in areas where the foregoing *Saturated air* system of cooling will not operate satisfactorily, because the atmospheric air already contains too much moisture. Remember, when air with a fixed amount of moisture is cooled, its *relative* humidity goes up. Suppose the outside air is at 95° F. with a relative humidity of 80 per cent, and suppose it is cooled by refrigeration to a temperature of 75° F. At this second condition, the air will be saturated. As a matter of fact, it becomes saturated at about 88° F. and stays saturated through any further cooling. The 88° F. temperature is the dew point, so that further cooling results in some of the steam condensing out. However, what we started to point out was that even though a 75° F. temperature sounded cool and comfortable (compared to 95° F. outside), saturated air at 75° F. would not be at all comfortable because *the perspiration would not leave the skin*.

Our problem then is to remove moisture from the cooled air, *Dehumidification* and it is for this reason that summer air conditioning is relatively expensive. The most common method of removing the unwanted moisture is by refrigerating the air, as described in the discussion of refrigeration, carrying the temperature down to values below the desired room temperature. Then, *sometimes, the cooled air must be artificially heated again* (yes, in the summertime) in order to obtain comfortable room conditions. We see now that the processes for complete summer conditioning would be: (1) cleaning, (2) refrigerating and drying, (3) reheating, (4) distributing, or getting the conditioned air into the right spaces. Study Fig. 415 and its caption carefully.

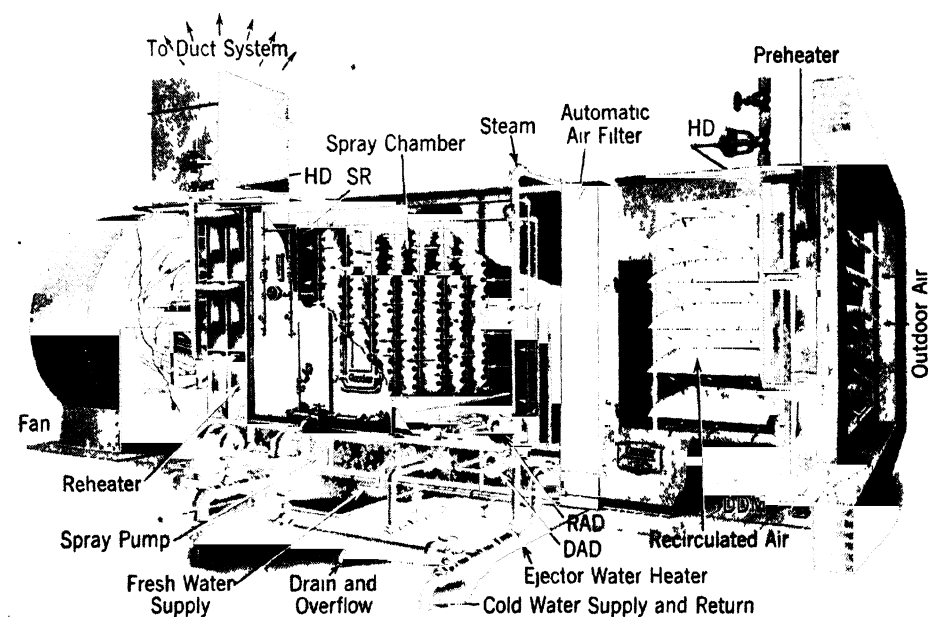
QUANTITY OF MOISTURE IN AIR

In the accompanying table, page 519, we have given the weight of steam or moisture in atmospheric air in pounds of steam per pound of dry air. Now what is *dry air*? Well, dry air is air that has no moisture in it. Since atmospheric air always contains moisture, we would have to go to some trouble to produce really dry air. Nevertheless, in engineering, it is customary to use a pound of dry air as the basis of our calculations. *Dry air*

*Weight of
moisture in air*

From the table; we find beside the 95° temperature the number 0.0365 lb. Therefore, if we add moisture to a pound of *dry air* at 95° F. until it becomes saturated, that is, until it has picked up as much moisture as it can hold, we will have added 0.0365 lb. If, however, we had added moisture only until the relative humidity reached 80 per cent, the air would have received only 80 per cent of the amount of moisture that it could hold. Hence, for each pound of dry air,

Moisture in air at 95° F. and 80% humidity = $0.80 \times 0.0365 = 0.0292$ lb.



Courtesy Carrier Engineering Corporation

FIG. 415. AIR-CONDITIONING UNIT

This picture shows a large central station unit. The air conditioning is completed here and the conditioned air is carried to various rooms through a duct system (upper left). Ordinarily, the circulated air is a mixture of fresh outside air and recirculated air (at right). This mixture passes through a filter, which removes some of the dust particles, thence across some steam coils which are used in winter conditioning as a means of regulating the humidity of the air. In summer conditioning, the spray chamber is provided with water which has been cooled by a refrigerating process. In passing through the spray chamber, the air is cooled below its dew point, so that the moisture content of the air is reduced to a predetermined level. That is, the air loses moisture while passing through the water spray in summer conditioning. (In winter conditioning, it would gain moisture.) The water spray also further cleans the air. If the air leaving the spray chamber is too cold to be introduced into living spaces, another heater is provided (reheater, on the left). The fan furnishes the motive power for distributing the conditioned air.

Suppose you wish to compute the weight of moisture in a particular room. First, compute the weight of dry air in the room. This weight may be found approximately by using the following formula. Weight of dry air is

$$(8) \quad w = 0.073 \times V,$$

where V is the volume of the room in cubic feet. Suppose the room is $12 \times 12 \times 10$ ft. Its volume is 1440 cu. ft., and the weight of dry air is nearly

$$w = 0.073 \times 1440 = 105 \text{ lb}$$

WEIGHT OF STEAM IN SATURATED AIR

| Temperature ° F. | Weight of Steam per Pound of Dry Air | Temperature ° F. | Weight of Steam per Pound of Dry Air |
|---------------------|---|---------------------|---|
| 20 | 0.00214 | 60 | 0.0110 |
| 25 | 0.00272 | 65 | 0.0132 |
| 30 | 0.00344 | 70 | 0.0158 |
| 35 | 0.00427 | 75 | 0.0187 |
| 40 | 0.00519 | 80 | 0.0223 |
| 45 | 0.00631 | 85 | 0.0263 |
| 50 | 0.00722 | 90 | 0.0311 |
| 55 | 0.00921 | 95 | 0.0365 |

Now look up in the table the temperature of the room and find the corresponding weight of moisture in saturated air. Let the temperature of the room be 75° F., and find (in table) the factor 0.0187 lb. of moisture per pound of dry air. But since we have 105 lb. of dry air, the total amount of moisture that the air can hold is

$$\text{Moisture in saturated air} = 105 \times 0.0187 = 1.96 \text{ lb.}$$

Ordinarily, however, the air is not saturated. Suppose the relative humidity is 60 per cent. (The humidity can be determined by any one of several instruments.) Then the weight of moisture in the room is

$$60\% \times 1.96 = 0.60 \times 1.96 = 1.176 \text{ lb.}$$

Let us try another example. If the outside air is at 95° F. and 80 per cent relative humidity, what would be the relative humidity if this air were cooled to 55° F. and then reheated to 80° F. (as may happen in summer air conditioning)?

Moisture in a room

Summer conditioning

The weight of moisture that enters the system with 1 lb. of outside air is

Relative humidity \times value from table at 95° F. = $0.80 \times 0.0365 = 0.0292$ lb.

*Humidity
change*

If this air is cooled to 55° F., it is then *below its dew point*, so some moisture condenses out and the air is just saturated. We see that this is true because the moisture in saturated air at 55° F. (0.00921 lb. from the table) is less than the value just computed (0.0292 lb.). Now when the saturated air at 55° F. is reheated to 80° F., the amount of moisture remains constant, inasmuch as the condensed moisture has been removed. So at 80° F. the amount of moisture is 0.00921 lb., but the amount that the air could hold is 0.0223 lb., as found from the table corresponding to 80° F. Thus, the relative humidity is

$$\frac{\text{Amount of moisture in the air (80° F.)}}{\text{Amount of moisture that the air could hold}} = \frac{0.00921}{0.0223} = 41.3\%.$$

It is only fair to warn the reader that the simple method used above is not exactly the correct way to make this computation (and others involving relative humidity), but it gives fair results if the temperature range is not very large.

ELEMENTS OF COMFORT

By way of summing up the foregoing discussion of air conditioning, let us consider the factors which enter into our feeling of comfort.

*Temperature of
air*

The lower the air temperature (below 98.6° F., the temperature of the human body), the greater the difference in temperature between your body and the air, and the faster heat flows from your body to the air — unless, of course, you insulate your body with clothing.

Moisture in air

If the temperature of the air is above 98.6° F., your body will not lose heat to the air by radiation and conduction. It will lose heat by evaporation of moisture from the skin. At normal atmospheric temperatures, relatively dry air feels cooler than more humid air, particularly if there is any movement of the air, because the drier air absorbs moisture faster. In this respect, dry air is advantageous in the summer, but disadvantageous in the winter. Because of its relative slowness in absorbing moisture, humid air has an advantage in the winter in that you feel more comfortable at a lower temperature in humid air than in dry air. This lower

temperature for comfort results in a lower heat loss from the building and therefore a saving of fuel.

When we use fans in the summer, our purpose is to circulate air around the body faster than it would circulate of its own accord. The more rapid movement of air causes a greater evaporation from the skin in a given time and thereby makes us feel cooler.

Movement of air

The temperature of the surrounding surfaces is an important factor which we have not mentioned and which is frequently overlooked as an element in our comfort. We have said that "heat naturally flows from the hotter body to the colder body." However, this statement should not be interpreted to mean that *no* heat flows from the colder body to the hotter body. What it actually means is that *more* heat flows from the hotter body to the colder, than from the colder to the hotter body.

To understand this point, imagine yourself in a room. If the walls are hotter than body temperature, your body *receives* more heat from the hot walls than it *sends* to the walls, so that the walls tend to warm you. If the walls are below body temperature, your body *sends* more heat to the cold walls than it *receives* from them, and therefore the walls tend to cool you. Note that this action is entirely separate from the effect of the temperature of the air. Thus, in cold weather, it is desirable to have warm walls for comfort.

Wall temperature

Cold walls are frequently the cause of discomfort in winter, even when the temperature of the air is so high that you think you ought to be comfortable. On the other hand, in hot summer weather, it is desirable to have cool walls for comfort. The heat stored in hot walls and furniture, including your bed mattress, is a common cause of discomfort, and this is true even at night when cool outside air is coming into the room. Wall and ceiling insulation improves both summer and winter comfort by keeping the walls and ceiling warmer in winter and cooler in summer, and for this reason, insulation adds to year-round comfort.

Insulation

1. What is the approximate weight of moisture in 1 lb. of dry air which is at 60° F. and 90% relative humidity? *Problems*

Ans. 0.0099 lb.

2. A room has the dimensions 14 × 20 × 9 ft. and the temperature in it is 90° F., while the relative humidity is 85%. What is the weight of moisture in the air of this room?

Ans. 4.86 lb.

3. The air in the room of the previous problem (90° F., 85% relative humidity) is cooled to 60° F. What weight of moisture remains in the air?

Hint. It falls below the dew point, does it not? How much moisture condenses out in cooling to 60° F.?

Ans. 2.02 lb., 2.84 lb.

4. Saturated air at 30° F. is heated to 75° F. What is the approximate relative humidity after the heating? *Ans.* 18.4%.

5. If the 75° F. air of the previous problem has moisture added to it until the relative humidity is 70%, how much moisture must be added per pound of dry air? *Ans.* 0.00965 lb.

6. A house with cubic contents of 10,000 cu. ft. is supplied with one change of air per hour. The air undergoes the process described in the previous problem. How many pounds of water must be evaporated to change the humidity as described during a period of ten hours? If the water weighs 8.33 lb. per gallon, how many gallons are necessary? *Ans.* 70.4 lb., 8.46 gal.

WHAT TO STUDY NEXT ABOUT HEAT

It is too bad that the subject of heat and power cannot be taught to everyone in six easy lessons, because it is a fascinating study. Unfortunately, however, there are a host of other ideas, not hinted at here, which added together define the science of thermodynamics. If you care to pursue further knowledge in this field, you should first acquire a good book on physics and prepare for yourself a foundation in fundamental science. You will then be prepared for a more comprehensive study of applied thermodynamics, which tells in greater detail of engineering applications of this science of heat, cold, and power. Good Luck!

Logarithmic and Trigonometric Tables

INSTRUCTIONS FOR USE OF THE MACMILLAN TABLES FOR ENGINEERING PREVIEW

| | |
|---|--|
| TABLE I <i>Pages 525-543</i> | COMMON LOGARITHMS OF NUMBERS The characteristic is given only on the first page of these tables. The mantissa is given to five places. Locate the first three digits of the number in column N and the next digit in row N. The mantissa desired is found at the intersection of the proper row and column. The Proportional Parts table is for interpolation between adjacent entries in any row. It provides the correction which is from 1/10 to 9/10 of the difference between entries. The first two digits of the mantissa, when omitted, are given above, or when starred are found below. |
| TABLE Ia <i>Page 544</i> | CONDENSED LOGARITHMS TO FIFTEEN DECIMAL PLACES To illustrate the use of this table we shall compute the logarithm of g (32.16, the average acceleration of gravity) to five places. First divide 32.16 by 3, the first digit, obtaining 10.72. Then divide 10.72 by 10 obtaining 1.072. Then divide 1.072 by 1.07 obtaining 1.0019. Accordingly, $g = 32.16 = 3(10)(1.07)(1.0019)$ The logarithms of each factor can be obtained separately and added to give $\log g = 0.47712 + 1.00000 + 0.02938 + 0.00082 = 1.50732$ This value may be checked in Table I. By following this process, and by using Table Ia, logarithms can be obtained to fifteen decimal places. |
| TABLE Ib <i>Page 545</i> | IMPORTANT CONSTANTS The values given here are needed to more than slide rule accuracy in certain calculations. The number of significant figures given varies with the probable need. |
| <i>Page 546</i> | CHART OF TRIGONOMETRIC FUNCTIONS These curves illustrate the variations of the trigonometric functions and show their positive and negative ranges for angles from $-\pi/2$ to π . |
| TABLE II <i>Pages 547-569</i> | TRIGONOMETRIC FUNCTIONS Angles below 45° are given at the top of the table. Select column headed by correct trigonometric function and move downward along left-hand minute column to locate row. Missing digits of function are found immediately above. For angles above 45° use function headings at bottom of table and minute column at right which reads upward. |
| TABLE III <i>Page 570</i> | DEGREES, MINUTES, AND SECONDS TO RADIANS An angle in degrees, minutes, and seconds can be changed into radians by the addition of equivalents. From the first three double columns, change degrees to radians. Use the fourth double column to change minutes to radians and the last double column for seconds. Then add these three values to obtain the whole angle in radians. |
| TABLE IV <i>Pages 571-572</i> | RADIAN MEASURE TRIGONOMETRIC FUNCTIONS Functions of angles in hundredths of a radian up to 1.60 radians are given in the table. The right-hand column is the equivalent angle in degrees and minutes to tenths of a minute. |
| TABLE V <i>Page 573</i> | COMPOUND INTEREST One dollar invested at the interest rate chosen from row n after a period of years chosen from column n will increase, when compounded annually, to the number of dollars found at the intersection of the proper row and column. |
| TABLE VI <i>Page 574</i> | AMOUNT OF AN ANNUITY One dollar per year invested at the interest rate chosen from row n accumulating for the number of years chosen from column n will amount to the number of dollars found at the intersection of the proper row and column. The interest is compounded annually. |

LOGARITHMIC AND TRIGONOMETRIC TABLES

TABLE I
COMMON LOGARITHMS OF NUMBERS
FROM
1 TO 10 000
TO
FIVE DECIMAL PLACES

1 — 100

| N | Log | N | Log | N | Log | N | Log | N | Log |
|-----------|-----------------|-----------|-----------------|-----------|-----------------|-----------|-----------------|-----------|-----------------|
| 0 | ———— | 20 | 1.30 103 | 40 | 1.60 206 | 60 | 1.77 815 | 80 | 1.90 309 |
| 1 | 0.00 000 | 21 | 1.32 222 | 41 | 1.61 278 | 61 | 1.78 533 | 81 | 1.90 849 |
| 2 | 0.30 103 | 22 | 1.34 242 | 42 | 1.62 325 | 62 | 1.79 239 | 82 | 1.91 381 |
| 3 | 0.47 712 | 23 | 1.36 173 | 43 | 1.63 347 | 63 | 1.79 934 | 83 | 1.91 908 |
| 4 | 0.60 206 | 24 | 1.38 021 | 44 | 1.64 345 | 64 | 1.80 618 | 84 | 1.92 428 |
| 5 | 0.69 897 | 25 | 1.39 794 | 45 | 1.65 321 | 65 | 1.81 291 | 85 | 1.92 942 |
| 6 | 0.77 815 | 26 | 1.41 497 | 46 | 1.66 276 | 66 | 1.81 954 | 86 | 1.93 450 |
| 7 | 0.84 510 | 27 | 1.43 136 | 47 | 1.67 210 | 67 | 1.82 607 | 87 | 1.93 952 |
| 8 | 0.90 309 | 28 | 1.44 716 | 48 | 1.68 124 | 68 | 1.83 251 | 88 | 1.94 448 |
| 9 | 0.95 424 | 29 | 1.46 240 | 49 | 1.69 020 | 69 | 1.83 885 | 89 | 1.94 939 |
| 10 | 1.00 000 | 30 | 1.47 712 | 50 | 1.69 897 | 70 | 1.84 510 | 90 | 1.95 424 |
| 11 | 1.04 139 | 31 | 1.49 136 | 51 | 1.70 757 | 71 | 1.85 126 | 91 | 1.95 904 |
| 12 | 1.07 918 | 32 | 1.50 515 | 52 | 1.71 600 | 72 | 1.85 733 | 92 | 1.96 379 |
| 13 | 1.11 394 | 33 | 1.51 851 | 53 | 1.72 428 | 73 | 1.86 332 | 93 | 1.96 848 |
| 14 | 1.14 613 | 34 | 1.53 148 | 54 | 1.73 239 | 74 | 1.86 923 | 94 | 1.97 313 |
| 15 | 1.17 609 | 35 | 1.54 407 | 55 | 1.74 036 | 75 | 1.87 506 | 95 | 1.97 772 |
| 16 | 1.20 412 | 36 | 1.55 630 | 56 | 1.74 819 | 76 | 1.88 081 | 96 | 1.98 227 |
| 17 | 1.23 045 | 37 | 1.56 820 | 57 | 1.75 587 | 77 | 1.88 649 | 97 | 1.98 677 |
| 18 | 1.25 527 | 38 | 1.57 978 | 58 | 1.76 343 | 78 | 1.89 209 | 98 | 1.99 123 |
| 19 | 1.27 875 | 39 | 1.59 106 | 59 | 1.77 085 | 79 | 1.89 763 | 99 | 1.99 564 |
| N | Log | N | Log | N | Log | N | Log | N | Log |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. | | | |
|-----|--------|------|------|------|------|------|------|------|------|------|------------|------|------|------|
| 100 | 00 000 | 043 | 087 | 130 | 173 | 217 | 260 | 303 | 346 | 389 | | | | |
| 101 | 432 | 475 | 518 | 561 | 604 | 647 | 689 | 732 | 775 | 817 | | | | |
| 102 | 860 | 903 | 945 | 988 | *030 | *072 | *115 | *157 | *199 | *242 | 1 | 4.4 | 4.3 | 4.2 |
| 103 | 01 284 | 326 | 368 | 410 | 452 | 494 | 536 | 578 | 620 | 662 | 2 | 8.8 | 8.6 | 8.4 |
| 104 | 703 | 745 | 787 | 828 | 870 | 912 | 953 | 995 | *036 | *078 | 3 | 13.2 | 12.9 | 12.6 |
| 105 | 02 119 | 160 | 202 | 243 | 284 | 325 | 366 | 407 | 449 | 490 | 4 | 17.6 | 17.2 | 16.8 |
| 106 | 531 | 572 | 612 | 653 | 694 | 735 | 776 | 816 | 857 | 898 | 5 | 22.0 | 21.5 | 21.0 |
| 107 | 938 | 979 | *019 | *060 | *100 | *141 | *181 | *222 | *262 | *302 | 6 | 26.4 | 25.8 | 25.2 |
| 108 | 03 342 | 383 | 423 | 463 | 503 | 543 | 583 | 623 | 663 | 703 | 7 | 30.8 | 30.1 | 29.4 |
| 109 | 743 | 782 | 822 | 862 | 902 | 941 | 981 | *021 | *060 | *100 | 8 | 35.2 | 34.4 | 33.6 |
| 110 | 04 139 | 179 | 218 | 258 | 297 | 336 | 376 | 415 | 454 | 493 | 9 | 39.6 | 38.7 | 37.8 |
| 111 | 532 | 571 | 610 | 650 | 689 | 727 | 766 | 805 | 844 | 883 | | | | |
| 112 | 922 | 961 | 999 | *038 | *077 | *115 | *154 | *192 | *231 | *269 | 1 | 4.1 | 4.0 | 3.9 |
| 113 | 05 308 | 346 | 385 | 423 | 461 | 500 | 538 | 576 | 614 | 652 | 2 | 8.2 | 8.0 | 7.8 |
| 114 | 690 | 729 | 767 | 805 | 843 | 881 | 918 | 956 | 994 | *032 | 3 | 12.3 | 12.0 | 11.7 |
| 115 | 06 070 | 108 | 145 | 183 | 221 | 258 | 296 | 333 | 371 | 408 | 4 | 16.4 | 16.0 | 15.6 |
| 116 | 446 | 483 | 521 | 558 | 595 | 633 | 670 | 707 | 744 | 781 | 5 | 20.5 | 20.0 | 19.5 |
| 117 | 819 | 856 | 893 | 930 | 967 | *004 | *041 | *078 | *115 | *151 | 6 | 24.6 | 24.0 | 23.4 |
| 118 | 07 188 | 225 | 262 | 298 | 335 | 372 | 408 | 445 | 482 | 518 | 7 | 28.7 | 28.0 | 27.3 |
| 119 | 555 | 591 | 628 | 664 | 700 | 737 | 773 | 809 | 846 | 882 | 8 | 32.8 | 32.0 | 31.2 |
| 120 | 918 | 954 | 990 | *027 | *063 | *099 | *135 | *171 | *207 | *243 | 9 | 36.9 | 36.0 | 35.1 |
| 121 | 08 279 | 314 | 350 | 386 | 422 | 458 | 493 | 529 | 565 | 600 | | | | |
| 122 | 636 | 672 | 707 | 743 | 778 | 814 | 849 | 884 | 920 | 955 | 1 | 3.8 | 3.7 | 3.6 |
| 123 | 991 | *026 | *061 | *096 | *132 | *167 | *202 | *237 | *272 | *307 | 2 | 7.6 | 7.4 | 7.2 |
| 124 | 09 342 | 377 | 412 | 447 | 482 | 517 | 552 | 587 | 621 | 656 | 3 | 11.4 | 11.1 | 10.8 |
| 125 | 691 | 726 | 760 | 795 | 830 | 864 | 899 | 934 | 968 | *003 | 4 | 15.2 | 14.8 | 14.4 |
| 126 | 10 037 | 072 | 106 | 140 | 175 | 209 | 243 | 278 | 312 | 346 | 5 | 19.0 | 18.5 | 18.0 |
| 127 | 380 | 415 | 449 | 483 | 517 | 551 | 585 | 619 | 653 | 687 | 6 | 22.8 | 22.2 | 21.6 |
| 128 | 721 | 755 | 789 | 823 | 857 | 890 | 924 | 958 | 992 | *025 | 7 | 26.6 | 25.9 | 25.2 |
| 129 | 11 059 | 093 | 126 | 160 | 193 | 227 | 261 | 294 | 327 | 361 | 8 | 30.4 | 29.6 | 28.8 |
| 130 | 394 | 428 | 461 | 494 | 528 | 561 | 594 | 628 | 661 | 694 | 9 | 34.2 | 33.3 | 32.4 |
| 131 | 727 | 760 | 793 | 826 | 860 | 893 | 926 | 959 | 992 | *024 | | | | |
| 132 | 12 057 | 090 | 123 | 156 | 189 | 222 | 254 | 287 | 320 | 352 | 1 | 3.5 | 3.4 | 3.3 |
| 133 | 385 | 418 | 450 | 483 | 516 | 548 | 581 | 613 | 646 | 678 | 2 | 7.0 | 6.8 | 6.6 |
| 134 | 710 | 743 | 775 | 808 | 840 | 872 | 905 | 937 | 969 | *001 | 3 | 10.5 | 10.2 | 9.9 |
| 135 | 13 033 | 066 | 098 | 130 | 162 | 194 | 226 | 258 | 290 | 322 | 4 | 14.0 | 13.6 | 13.2 |
| 136 | 354 | 386 | 418 | 450 | 481 | 513 | 545 | 577 | 609 | 640 | 5 | 17.5 | 17.0 | 16.5 |
| 137 | 672 | 704 | 735 | 767 | 799 | 830 | 862 | 893 | 925 | 956 | 6 | 21.0 | 20.4 | 19.8 |
| 138 | 988 | *019 | *051 | *082 | *114 | *145 | *176 | *208 | *239 | *270 | 7 | 24.5 | 23.8 | 23.1 |
| 139 | 14 301 | 333 | 364 | 395 | 426 | 457 | 489 | 520 | 551 | 582 | 8 | 28.0 | 27.2 | 26.4 |
| 140 | 613 | 644 | 675 | 706 | 737 | 768 | 799 | 829 | 860 | 891 | 9 | 31.5 | 30.6 | 29.7 |
| 141 | 922 | 953 | 983 | *014 | *045 | *076 | *106 | *137 | *168 | *198 | | | | |
| 142 | 15 229 | 259 | 290 | 320 | 351 | 381 | 412 | 442 | 473 | 503 | 1 | 3.2 | 3.1 | 3.0 |
| 143 | 534 | 564 | 594 | 625 | 655 | 685 | 715 | 746 | 776 | 806 | 2 | 6.4 | 6.2 | 6.0 |
| 144 | 836 | 866 | 897 | 927 | 957 | 987 | *017 | *047 | *077 | *107 | 3 | 9.6 | 9.3 | 9.0 |
| 145 | 16 137 | 167 | 197 | 227 | 256 | 286 | 316 | 346 | 376 | 406 | 4 | 12.8 | 12.4 | 12.0 |
| 146 | 435 | 465 | 495 | 524 | 554 | 584 | 613 | 643 | 673 | 702 | 5 | 16.0 | 15.5 | 15.0 |
| 147 | 732 | 761 | 791 | 820 | 850 | 879 | 909 | 938 | 967 | 997 | 6 | 19.2 | 18.6 | 18.0 |
| 148 | 17 026 | 056 | 085 | 114 | 143 | 173 | 202 | 231 | 260 | 289 | 7 | 22.4 | 21.7 | 21.0 |
| 149 | 319 | 348 | 377 | 406 | 435 | 464 | 493 | 522 | 551 | 580 | 8 | 25.6 | 24.8 | 24.0 |
| 150 | 609 | 638 | 667 | 696 | 725 | 754 | 782 | 811 | 840 | 869 | 9 | 28.8 | 27.9 | 27.0 |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. | | | |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|------------|--------|-----|------|------|------|------|------|------|------|------|---------------------|
| 150 | 17 609 | 638 | 667 | 696 | 725 | 754 | 782 | 811 | 840 | 869 | |
| 151 | 898 | 926 | 955 | 984 | *013 | *011 | *070 | *090 | *127 | *156 | 29 28 |
| 152 | 18 184 | 213 | 241 | 270 | 298 | 327 | 355 | 384 | 412 | 441 | 1 2 9 2 8 |
| 153 | 469 | 498 | 526 | 554 | 583 | 611 | 639 | 667 | 696 | 724 | 2 5 8 5 6 |
| 154 | 752 | 780 | 808 | 837 | 865 | 893 | 921 | 949 | 977 | *005 | 3 8 7 8 4 |
| 155 | 19 033 | 061 | 089 | 117 | 145 | 173 | 201 | 229 | 257 | 285 | 4 11 6 11 2 |
| 156 | 312 | 340 | 368 | 396 | 424 | 451 | 479 | 507 | 535 | 562 | 5 11 5 14 0 |
| 157 | 590 | 618 | 645 | 673 | 700 | 728 | 756 | 783 | 811 | 838 | 6 17 1 16 8 |
| 158 | 866 | 893 | 921 | 948 | 976 | *003 | *030 | *058 | *085 | *112 | 7 20 3 19 6 |
| 159 | 20 140 | 167 | 194 | 222 | 249 | 276 | 303 | 330 | 358 | 385 | 8 23 2 22 4 |
| 160 | 412 | 439 | 466 | 493 | 520 | 548 | 575 | 602 | 629 | 656 | 9 26 1 25 2 |
| 161 | 683 | 710 | 737 | 763 | 790 | 817 | 844 | 871 | 898 | 925 | 27 26 |
| 162 | 952 | 978 | *005 | *032 | *059 | *085 | *112 | *139 | *165 | *192 | 1 2 7 2 6 |
| 163 | 21 219 | 245 | 272 | 299 | 325 | 352 | 378 | 405 | 431 | 458 | 2 5 4 5 2 |
| 164 | 484 | 511 | 537 | 564 | 590 | 617 | 643 | 669 | 696 | 722 | 3 8 1 7 8 |
| 165 | 748 | 775 | 801 | 827 | 854 | 880 | 906 | 932 | 958 | 984 | 4 10 8 10 4 |
| 166 | 22 011 | 037 | 063 | 089 | 115 | 141 | 167 | 194 | 220 | 246 | 5 13 5 13 0 |
| 167 | 272 | 298 | 324 | 350 | 376 | 401 | 427 | 453 | 479 | 505 | 6 16 2 15 6 |
| 168 | 531 | 557 | 583 | 608 | 634 | 660 | 686 | 712 | 737 | 763 | 7 18 9 15 2 |
| 169 | 789 | 814 | 840 | 866 | 891 | 917 | 943 | 968 | 994 | *019 | 8 21 6 20 8 |
| 170 | 23 015 | 070 | 096 | 121 | 147 | 172 | 198 | 223 | 249 | 274 | 9 24 3 23 1 |
| 171 | 300 | 325 | 350 | 376 | 401 | 426 | 452 | 477 | 502 | 528 | 25 24 |
| 172 | 553 | 578 | 603 | 629 | 654 | 679 | 704 | 729 | 754 | 779 | 1 2 5 2 4 |
| 173 | 805 | 830 | 855 | 880 | 905 | 930 | 955 | 980 | *005 | *030 | 2 5 0 4 8 |
| 174 | 24 055 | 080 | 105 | 130 | 155 | 180 | 204 | 229 | 254 | 279 | 3 7 5 7 2 |
| 175 | 304 | 329 | 353 | 378 | 403 | 428 | 452 | 477 | 502 | 527 | 4 10 0 9 6 |
| 176 | 551 | 576 | 601 | 625 | 650 | 674 | 699 | 724 | 748 | 773 | 5 12 5 12 0 |
| 177 | 797 | 822 | 846 | 871 | 895 | 920 | 944 | 969 | 993 | *018 | 6 15 0 11 4 |
| 178 | 25 042 | 066 | 091 | 115 | 139 | 163 | 188 | 212 | 237 | 261 | 7 17 5 16 8 |
| 179 | 285 | 310 | 334 | 358 | 382 | 406 | 431 | 455 | 479 | 503 | 8 20 0 19 2 |
| 180 | 527 | 551 | 575 | 600 | 624 | 648 | 672 | 696 | 720 | 744 | 9 22 5 21 6 |
| 181 | 765 | 792 | 816 | 840 | 864 | 888 | 912 | 935 | 959 | 983 | 23 22 |
| 182 | 26 007 | 031 | 055 | 079 | 102 | 126 | 150 | 174 | 198 | 221 | 1 2 3 2 2 |
| 183 | 245 | 269 | 293 | 316 | 340 | 364 | 387 | 411 | 435 | 458 | 2 4 6 4 1 |
| 184 | 482 | 505 | 529 | 553 | 576 | 600 | 623 | 647 | 670 | 694 | 3 6 0 6 6 |
| 185 | 717 | 741 | 764 | 788 | 811 | 834 | 858 | 881 | 905 | 928 | 4 9 2 8 8 |
| 186 | 951 | 975 | 998 | *021 | *045 | *068 | *091 | *114 | *138 | *161 | 5 11 5 11 0 |
| 187 | 27 184 | 207 | 231 | 254 | 277 | 300 | 323 | 346 | 370 | 393 | 6 13 8 13 2 |
| 188 | 416 | 439 | 462 | 485 | 508 | 531 | 554 | 577 | 600 | 623 | 7 16 1 15 4 |
| 189 | 646 | 669 | 692 | 715 | 738 | 761 | 784 | 807 | 830 | 852 | 8 18 1 17 6 |
| 190 | 875 | 898 | 921 | 944 | 967 | 989 | *012 | *035 | *058 | *081 | 9 20 7 19 8 |
| 191 | 28 103 | 126 | 149 | 171 | 194 | 217 | 240 | 262 | 285 | 307 | 21 |
| 192 | 330 | 353 | 375 | 398 | 421 | 443 | 466 | 488 | 511 | 533 | 1 2 1 2 1 |
| 193 | 556 | 578 | 601 | 623 | 646 | 668 | 691 | 713 | 735 | 758 | 2 4 2 4 2 |
| 194 | 780 | 803 | 825 | 847 | 870 | 892 | 914 | 937 | 959 | 981 | 3 6 3 6 3 |
| 195 | 29 003 | 026 | 048 | 070 | 092 | 115 | 137 | 159 | 181 | 203 | 4 8 4 8 4 |
| 196 | 226 | 248 | 270 | 292 | 314 | 336 | 358 | 380 | 403 | 425 | 5 10 5 10 5 |
| 197 | 447 | 469 | 491 | 513 | 535 | 557 | 579 | 601 | 623 | 645 | 6 12 6 12 6 |
| 198 | 667 | 688 | 710 | 732 | 754 | 776 | 798 | 820 | 842 | 863 | 7 11 7 11 7 |
| 199 | 885 | 907 | 929 | 951 | 973 | 994 | *016 | *038 | *060 | *081 | 8 16 8 16 8 |
| 200 | 30 103 | 125 | 146 | 168 | 190 | 211 | 233 | 255 | 276 | 298 | 9 18 9 18 9 |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|------|------|------|------|------|------|------|------|------|------------------------|
| 200 | 30 103 | 125 | 146 | 168 | 190 | 211 | 233 | 255 | 276 | 298 | |
| 201 | 320 | 341 | 363 | 384 | 406 | 428 | 449 | 471 | 492 | 514 | |
| 202 | 535 | 557 | 578 | 600 | 621 | 643 | 664 | 685 | 707 | 728 | |
| 203 | 750 | 771 | 792 | 814 | 835 | 856 | 878 | 899 | 920 | 942 | log 2 =.30102 99957 |
| 204 | 963 | 984 | *006 | *027 | *048 | *069 | *091 | *112 | *133 | *154 | |
| 205 | 31 175 | 197 | 218 | 239 | 260 | 281 | 302 | 323 | 345 | 366 | |
| 206 | 387 | 408 | 429 | 450 | 471 | 492 | 513 | 534 | 555 | 576 | |
| 207 | 597 | 618 | 639 | 660 | 681 | 702 | 723 | 744 | 765 | 785 | |
| 208 | 806 | 827 | 848 | 869 | 890 | 911 | 931 | 952 | 973 | 994 | |
| 209 | 32 015 | 035 | 056 | 077 | 098 | 118 | 139 | 160 | 181 | 201 | |
| 210 | 222 | 243 | 263 | 284 | 305 | 325 | 346 | 366 | 387 | 408 | |
| 211 | 428 | 449 | 469 | 490 | 510 | 531 | 552 | 572 | 593 | 613 | |
| 212 | 634 | 654 | 675 | 695 | 715 | 736 | 756 | 777 | 797 | 818 | |
| 213 | 838 | 858 | 879 | 899 | 919 | 940 | 960 | 980 | *001 | *021 | |
| 214 | 33 041 | 062 | 082 | 102 | 122 | 143 | 163 | 183 | 203 | 224 | |
| 215 | 244 | 264 | 284 | 304 | 325 | 345 | 365 | 385 | 405 | 425 | |
| 216 | 445 | 465 | 486 | 506 | 526 | 546 | 566 | 586 | 606 | 626 | |
| 217 | 646 | 666 | 686 | 706 | 726 | 746 | 766 | 786 | 806 | 826 | |
| 218 | 846 | 866 | 885 | 905 | 925 | 945 | 965 | 985 | *005 | *025 | |
| 219 | 34 044 | 064 | 084 | 104 | 124 | 143 | 163 | 183 | 203 | 223 | |
| 220 | 242 | 262 | 282 | 301 | 321 | 341 | 361 | 380 | 400 | 420 | |
| 221 | 439 | 459 | 479 | 498 | 518 | 537 | 557 | 577 | 596 | 616 | |
| 222 | 635 | 655 | 674 | 694 | 713 | 733 | 753 | 772 | 792 | 811 | |
| 223 | 830 | 850 | 869 | 889 | 908 | 928 | 947 | 967 | 986 | *005 | |
| 224 | 35 025 | 044 | 064 | 083 | 102 | 122 | 141 | 160 | 180 | 199 | |
| 225 | 218 | 238 | 257 | 276 | 295 | 315 | 334 | 353 | 372 | 392 | |
| 226 | 411 | 430 | 449 | 468 | 488 | 507 | 526 | 545 | 564 | 583 | |
| 227 | 603 | 622 | 641 | 660 | 679 | 698 | 717 | 736 | 755 | 774 | |
| 228 | 793 | 813 | 832 | 851 | 870 | 889 | 908 | 927 | 946 | 965 | |
| 229 | 984 | *003 | *021 | *040 | *059 | *078 | *097 | *116 | *135 | *154 | |
| 230 | 36 173 | 192 | 211 | 229 | 248 | 267 | 286 | 305 | 324 | 342 | |
| 231 | 361 | 380 | 399 | 418 | 436 | 455 | 474 | 493 | 511 | 530 | |
| 232 | 549 | 568 | 586 | 605 | 624 | 642 | 661 | 680 | 698 | 717 | |
| 233 | 736 | 754 | 773 | 791 | 810 | 829 | 847 | 866 | 884 | 903 | |
| 234 | 922 | 940 | 959 | 977 | 996 | *014 | *033 | *051 | *070 | *088 | |
| 235 | 37 107 | 125 | 144 | 162 | 181 | 199 | 218 | 236 | 254 | 273 | |
| 236 | 291 | 310 | 328 | 346 | 365 | 383 | 401 | 420 | 438 | 457 | |
| 237 | 475 | 493 | 511 | 530 | 548 | 566 | 585 | 603 | 621 | 639 | |
| 238 | 658 | 676 | 694 | 712 | 731 | 749 | 767 | 785 | 803 | 822 | |
| 239 | 840 | 858 | 876 | 894 | 912 | 931 | 949 | 967 | 985 | *003 | |
| 240 | 38 021 | 039 | 057 | 075 | 093 | 112 | 130 | 148 | 166 | 184 | |
| 241 | 202 | 220 | 238 | 256 | 274 | 292 | 310 | 328 | 346 | 364 | |
| 242 | 382 | 399 | 417 | 435 | 453 | 471 | 489 | 507 | 525 | 543 | |
| 243 | 561 | 578 | 596 | 614 | 632 | 650 | 668 | 686 | 703 | 721 | |
| 244 | 739 | 757 | 775 | 792 | 810 | 828 | 846 | 863 | 881 | 899 | |
| 245 | 917 | 934 | 952 | 970 | 987 | *005 | *023 | *041 | *058 | *076 | |
| 246 | 39 094 | 111 | 129 | 146 | 164 | 182 | 199 | 217 | 235 | 252 | |
| 247 | 270 | 287 | 305 | 322 | 340 | 358 | 375 | 393 | 410 | 428 | |
| 248 | 445 | 463 | 480 | 498 | 515 | 533 | 550 | 568 | 585 | 602 | |
| 249 | 620 | 637 | 655 | 672 | 690 | 707 | 724 | 742 | 759 | 777 | |
| 250 | 794 | 811 | 829 | 846 | 863 | 881 | 898 | 915 | 933 | 950 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|------|------|------|------|------|------|------|------|------|------------|
| 250 | 39 794 | 811 | 829 | 846 | 863 | 881 | 898 | 915 | 933 | 950 | |
| 251 | 967 | 985 | *002 | *019 | *037 | *054 | *071 | *088 | *106 | *123 | |
| 252 | 40 140 | 157 | 175 | 192 | 209 | 226 | 243 | 261 | 278 | 295 | |
| 253 | 312 | 329 | 346 | 364 | 381 | 398 | 415 | 432 | 449 | 466 | |
| 254 | 483 | 500 | 518 | 535 | 552 | 569 | 586 | 603 | 620 | 637 | |
| 255 | 654 | 671 | 688 | 705 | 722 | 739 | 756 | 773 | 790 | 807 | |
| 256 | 824 | 841 | 858 | 875 | 892 | 909 | 926 | 943 | 960 | 976 | |
| 257 | 993 | *010 | *027 | *044 | *061 | *078 | *095 | *111 | *128 | *145 | |
| 258 | 41 162 | 179 | 196 | 212 | 229 | 246 | 263 | 280 | 296 | 313 | |
| 259 | 330 | 347 | 363 | 380 | 397 | 414 | 430 | 447 | 464 | 481 | |
| 260 | 497 | 514 | 531 | 547 | 564 | 581 | 597 | 614 | 631 | 647 | |
| 261 | 664 | 681 | 697 | 714 | 731 | 747 | 764 | 780 | 797 | 814 | |
| 262 | 830 | 847 | 863 | 880 | 896 | 913 | 929 | 946 | 963 | 979 | |
| 263 | 996 | *012 | *029 | *045 | *062 | *078 | *095 | *111 | *127 | *144 | |
| 264 | 42 160 | 177 | 193 | 210 | 226 | 243 | 259 | 275 | 292 | 308 | |
| 265 | 325 | 341 | 357 | 371 | 390 | 406 | 423 | 439 | 455 | 472 | |
| 266 | 488 | 504 | 521 | 537 | 553 | 570 | 586 | 602 | 619 | 635 | |
| 267 | 651 | 667 | 684 | 700 | 716 | 732 | 749 | 765 | 781 | 797 | |
| 268 | 813 | 830 | 846 | 862 | 878 | 894 | 911 | 927 | 943 | 959 | |
| 269 | 975 | 991 | *008 | *024 | *040 | *056 | *072 | *088 | *104 | *120 | |
| 270 | 43 136 | 152 | 169 | 185 | 201 | 217 | 233 | 249 | 265 | 281 | |
| 271 | 297 | 313 | 329 | 345 | 361 | 377 | 393 | 409 | 425 | 441 | |
| 272 | 457 | 473 | 489 | 505 | 521 | 537 | 553 | 569 | 584 | 600 | |
| 273 | 616 | 632 | 648 | 664 | 680 | 696 | 712 | 727 | 743 | 759 | |
| 274 | 775 | 791 | 807 | 823 | 838 | 854 | 870 | 886 | 902 | 917 | |
| 275 | 933 | 949 | 965 | 981 | 996 | *012 | *028 | *044 | *059 | *075 | |
| 276 | 44 091 | 107 | 122 | 138 | 154 | 170 | 185 | 201 | 217 | 232 | |
| 277 | 248 | 264 | 279 | 295 | 311 | 326 | 342 | 358 | 373 | 389 | |
| 278 | 404 | 420 | 436 | 451 | 467 | 483 | 498 | 514 | 529 | 545 | |
| 279 | 560 | 576 | 592 | 607 | 623 | 638 | 654 | 669 | 685 | 700 | |
| 280 | 716 | 731 | 747 | 762 | 778 | 793 | 809 | 824 | 840 | 855 | |
| 281 | 871 | 886 | 902 | 917 | 932 | 948 | 963 | 979 | 994 | *010 | |
| 282 | 45 025 | 040 | 056 | 071 | 086 | 102 | 117 | 133 | 148 | 163 | |
| 283 | 179 | 194 | 209 | 225 | 240 | 255 | 271 | 286 | 301 | 317 | |
| 284 | 332 | 347 | 362 | 378 | 393 | 408 | 423 | 439 | 454 | 469 | |
| 285 | 484 | 500 | 515 | 530 | 545 | 561 | 576 | 591 | 606 | 621 | |
| 286 | 637 | 652 | 667 | 682 | 697 | 712 | 728 | 743 | 758 | 773 | |
| 287 | 788 | 803 | 818 | 834 | 849 | 864 | 879 | 894 | 909 | 924 | |
| 288 | 939 | 954 | 969 | 984 | *000 | *015 | *030 | *045 | *060 | *075 | |
| 289 | 46 090 | 105 | 120 | 135 | 150 | 165 | 180 | 195 | 210 | 225 | |
| 290 | 240 | 255 | 270 | 285 | 300 | 315 | 330 | 345 | 359 | 374 | |
| 291 | 389 | 404 | 419 | 434 | 449 | 464 | 479 | 494 | 509 | 523 | |
| 292 | 538 | 553 | 568 | 583 | 598 | 613 | 627 | 642 | 657 | 672 | |
| 293 | 687 | 702 | 716 | 731 | 746 | 761 | 776 | 790 | 805 | 820 | |
| 294 | 835 | 850 | 864 | 879 | 894 | 909 | 923 | 938 | 953 | 967 | |
| 295 | 982 | 997 | *012 | *026 | *041 | *056 | *070 | *085 | *100 | *114 | |
| 296 | 47 129 | 144 | 159 | 173 | 188 | 202 | 217 | 232 | 246 | 261 | |
| 297 | 276 | 290 | 305 | 319 | 334 | 349 | 363 | 378 | 392 | 407 | |
| 298 | 422 | 436 | 451 | 465 | 480 | 494 | 509 | 524 | 538 | 553 | |
| 299 | 567 | 582 | 596 | 611 | 625 | 640 | 654 | 669 | 683 | 698 | |
| 300 | 712 | 727 | 741 | 756 | 770 | 784 | 799 | 813 | 828 | 842 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| | | |
|---|------|------|
| | 18 | 17 |
| 1 | 1.8 | 1.7 |
| 2 | 3.6 | 3.4 |
| 3 | 5.4 | 5.1 |
| 4 | 7.2 | 6.8 |
| 5 | 9.0 | 8.5 |
| 6 | 10.8 | 10.2 |
| 7 | 12.6 | 11.9 |
| 8 | 14.4 | 13.6 |
| 9 | 16.2 | 15.3 |

M
 $=\log_{10} e$
 $=\log_{10} 2.718\ldots$
 $=.43429\ 44819$

| | | |
|---|------|------|
| | 16 | 15 |
| 1 | 1.6 | 1.5 |
| 2 | 3.2 | 3.0 |
| 3 | 4.8 | 4.5 |
| 4 | 6.4 | 6.0 |
| 5 | 8.0 | 7.5 |
| 6 | 9.6 | 9.0 |
| 7 | 11.2 | 10.5 |
| 8 | 12.8 | 12.0 |
| 9 | 14.4 | 13.5 |

| | |
|---|------|
| | 14 |
| 1 | 1.4 |
| 2 | 2.8 |
| 3 | 4.2 |
| 4 | 5.6 |
| 5 | 7.0 |
| 6 | 8.4 |
| 7 | 9.8 |
| 8 | 11.2 |
| 9 | 12.6 |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|------|------|------|------|------|------|------|------|------|------------|
| 300 | 47 712 | 727 | 741 | 756 | 770 | 784 | 799 | 813 | 828 | 842 | |
| 301 | 857 | 871 | 885 | 900 | 914 | 929 | 943 | 958 | 972 | 986 | |
| 302 | 48 001 | 015 | 029 | 044 | 058 | 073 | 087 | 101 | 116 | 130 | |
| 303 | 144 | 159 | 173 | 187 | 202 | 216 | 230 | 244 | 259 | 273 | |
| 304 | 287 | 302 | 316 | 330 | 344 | 359 | 373 | 387 | 401 | 416 | |
| 305 | 430 | 444 | 458 | 473 | 487 | 501 | 515 | 530 | 544 | 558 | |
| 306 | 572 | 586 | 601 | 615 | 629 | 643 | 657 | 671 | 686 | 700 | |
| 307 | 714 | 728 | 742 | 756 | 770 | 785 | 799 | 813 | 827 | 841 | |
| 308 | 855 | 869 | 883 | 897 | 911 | 926 | 940 | 954 | 968 | 982 | |
| 309 | 996 | *010 | *024 | *038 | *052 | *066 | *080 | *094 | *108 | *122 | |
| 310 | 49 136 | 150 | 164 | 178 | 192 | 206 | 220 | 234 | 248 | 262 | |
| 311 | 276 | 290 | 304 | 318 | 332 | 346 | 360 | 374 | 388 | 402 | |
| 312 | 415 | 429 | 443 | 457 | 471 | 485 | 499 | 513 | 527 | 541 | |
| 313 | 554 | 568 | 582 | 596 | 610 | 624 | 638 | 651 | 665 | 679 | |
| 314 | 693 | 707 | 721 | 734 | 748 | 762 | 776 | 790 | 803 | 817 | |
| 315 | 831 | 845 | 859 | 872 | 886 | 900 | 914 | 927 | 941 | 955 | |
| 316 | 969 | 982 | 996 | *010 | *024 | *037 | *051 | *065 | *079 | *092 | |
| 317 | 50 106 | 120 | 133 | 147 | 161 | 174 | 188 | 202 | 215 | 229 | |
| 318 | 243 | 256 | 270 | 284 | 297 | 311 | 325 | 338 | 352 | 365 | |
| 319 | 379 | 393 | 406 | 420 | 433 | 447 | 461 | 474 | 488 | 501 | |
| 320 | 515 | 529 | 542 | 556 | 569 | 583 | 596 | 610 | 623 | 637 | |
| 321 | 651 | 664 | 678 | 691 | 705 | 718 | 732 | 745 | 759 | 772 | |
| 322 | 786 | 799 | 813 | 826 | 840 | 853 | 866 | 880 | 893 | 907 | |
| 323 | 920 | 934 | 947 | 961 | 974 | 987 | *001 | *014 | *028 | *041 | |
| 324 | 51 055 | 068 | 081 | 095 | 108 | 121 | 135 | 148 | 162 | 175 | |
| 325 | 188 | 202 | 215 | 228 | 242 | 255 | 268 | 282 | 295 | 308 | |
| 326 | 322 | 335 | 348 | 362 | 375 | 388 | 402 | 415 | 428 | 441 | |
| 327 | 455 | 468 | 481 | 495 | 508 | 521 | 534 | 548 | 561 | 574 | |
| 328 | 587 | 601 | 614 | 627 | 640 | 654 | 667 | 680 | 693 | 706 | |
| 329 | 720 | 733 | 746 | 759 | 772 | 786 | 799 | 812 | 825 | 838 | |
| 330 | 851 | 865 | 878 | 891 | 904 | 917 | 930 | 943 | 957 | 970 | |
| 331 | 983 | 996 | *009 | *022 | *035 | *048 | *061 | *075 | *088 | *101 | |
| 332 | 52 114 | 127 | 140 | 153 | 166 | 179 | 192 | 205 | 218 | 231 | |
| 333 | 244 | 257 | 270 | 284 | 297 | 310 | 323 | 336 | 349 | 362 | |
| 334 | 375 | 388 | 401 | 414 | 427 | 440 | 453 | 466 | 479 | 492 | |
| 335 | 504 | 517 | 530 | 543 | 556 | 569 | 582 | 595 | 608 | 621 | |
| 336 | 631 | 647 | 660 | 673 | 686 | 699 | 711 | 724 | 737 | 750 | |
| 337 | 763 | 776 | 789 | 802 | 815 | 827 | 840 | 853 | 866 | 879 | |
| 338 | 892 | 905 | 917 | 930 | 943 | 956 | 969 | 982 | 994 | *007 | |
| 339 | 53 020 | 033 | 046 | 058 | 071 | 084 | 097 | 110 | 122 | 135 | |
| 340 | 148 | 161 | 173 | 186 | 199 | 212 | 224 | 237 | 250 | 263 | |
| 341 | 275 | 288 | 301 | 314 | 326 | 339 | 352 | 364 | 377 | 390 | |
| 342 | 403 | 415 | 428 | 441 | 453 | 466 | 479 | 491 | 504 | 517 | |
| 343 | 529 | 542 | 555 | 567 | 580 | 593 | 605 | 618 | 631 | 643 | |
| 344 | 656 | 668 | 681 | 694 | 706 | 719 | 732 | 744 | 757 | 769 | |
| 345 | 782 | 794 | 807 | 820 | 832 | 845 | 857 | 870 | 882 | 895 | |
| 346 | 908 | 920 | 933 | 945 | 958 | 970 | 983 | 995 | *008 | *020 | |
| 347 | 54 033 | 045 | 058 | 070 | 083 | 095 | 108 | 120 | 133 | 145 | |
| 348 | 158 | 170 | 183 | 195 | 208 | 220 | 233 | 245 | 258 | 270 | |
| 349 | 283 | 295 | 307 | 320 | 332 | 345 | 357 | 370 | 382 | 394 | |
| 350 | 407 | 419 | 432 | 444 | 456 | 469 | 481 | 494 | 506 | 518 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

log 3
= .47712 12547
log π
= .49714 98727

| | 15 | 14 |
|---|------|------|
| 1 | 1.5 | 1.4 |
| 2 | 3.0 | 2.8 |
| 3 | 4.5 | 4.2 |
| 4 | 6.0 | 5.6 |
| 5 | 7.5 | 7.0 |
| 6 | 9.0 | 8.4 |
| 7 | 10.5 | 9.8 |
| 8 | 12.0 | 11.2 |
| 9 | 13.5 | 12.6 |

| | 13 | 12 |
|---|------|------|
| 1 | 1.3 | 1.2 |
| 2 | 2.6 | 2.4 |
| 3 | 3.9 | 3.6 |
| 4 | 5.2 | 4.8 |
| 5 | 6.5 | 6.0 |
| 6 | 7.8 | 7.2 |
| 7 | 9.1 | 8.4 |
| 8 | 10.4 | 9.6 |
| 9 | 11.7 | 10.8 |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|------------|--------|------|------|------|------|------|------|------|------|------|------------|
| 350 | 54 407 | 419 | 432 | 444 | 456 | 469 | 481 | 494 | 506 | 518 | |
| 351 | 531 | 543 | 555 | 568 | 580 | 593 | 605 | 617 | 630 | 642 | |
| 352 | 654 | 667 | 679 | 691 | 704 | 716 | 728 | 741 | 753 | 765 | |
| 353 | 777 | 790 | 802 | 814 | 827 | 839 | 851 | 864 | 876 | 888 | |
| 354 | 900 | 913 | 925 | 937 | 949 | 962 | 974 | 986 | 998 | *011 | |
| 355 | 55 023 | 035 | 047 | 060 | 072 | 084 | 096 | 108 | 121 | 133 | |
| 356 | 145 | 157 | 169 | 182 | 194 | 206 | 218 | 230 | 242 | 255 | |
| 357 | 267 | 279 | 291 | 303 | 315 | 328 | 340 | 352 | 364 | 376 | |
| 358 | 388 | 400 | 413 | 425 | 437 | 449 | 461 | 473 | 485 | 497 | |
| 359 | 509 | 522 | 534 | 546 | 558 | 570 | 582 | 594 | 606 | 618 | |
| 360 | 630 | 642 | 654 | 666 | 678 | 691 | 703 | 715 | 727 | 739 | |
| 361 | 751 | 763 | 775 | 787 | 799 | 811 | 823 | 835 | 847 | 859 | |
| 362 | 871 | 883 | 895 | 907 | 919 | 931 | 943 | 955 | 967 | 979 | |
| 363 | 991 | *003 | *015 | *027 | *038 | *050 | *062 | *074 | *086 | *098 | |
| 364 | 56 110 | 122 | 134 | 146 | 158 | 170 | 182 | 194 | 205 | 217 | |
| 365 | 229 | 241 | 253 | 265 | 277 | 289 | 301 | 312 | 324 | 336 | |
| 366 | 348 | 360 | 372 | 384 | 396 | 407 | 419 | 431 | 443 | 455 | |
| 367 | 467 | 478 | 490 | 502 | 514 | 526 | 538 | 549 | 561 | 573 | |
| 368 | 585 | 597 | 608 | 620 | 632 | 644 | 656 | 667 | 679 | 691 | |
| 369 | 703 | 714 | 726 | 738 | 750 | 761 | 773 | 785 | 797 | 808 | |
| 370 | 820 | 832 | 844 | 855 | 867 | 879 | 891 | 902 | 914 | 926 | |
| 371 | 937 | 949 | 961 | 972 | 981 | 996 | *008 | *019 | *031 | *043 | |
| 372 | 57 054 | 066 | 078 | 089 | 101 | 113 | 124 | 136 | 148 | 159 | |
| 373 | 171 | 183 | 194 | 206 | 217 | 229 | 241 | 252 | 264 | 276 | |
| 374 | 287 | 299 | 310 | 322 | 334 | 345 | 357 | 368 | 380 | 392 | |
| 375 | 403 | 415 | 426 | 438 | 449 | 461 | 473 | 484 | 496 | 507 | |
| 376 | 519 | 530 | 542 | 553 | 565 | 576 | 588 | 600 | 611 | 623 | |
| 377 | 634 | 646 | 657 | 669 | 680 | 692 | 703 | 715 | 726 | 738 | |
| 378 | 749 | 761 | 772 | 784 | 795 | 807 | 818 | 830 | 841 | 852 | |
| 379 | 864 | 875 | 887 | 898 | 910 | 921 | 933 | 944 | 955 | 967 | |
| 380 | 978 | 990 | *001 | *013 | *024 | *035 | *047 | *058 | *070 | *081 | |
| 381 | 58 092 | 104 | 115 | 127 | 138 | 149 | 161 | 172 | 184 | 195 | |
| 382 | 206 | 218 | 229 | 240 | 252 | 263 | 274 | 286 | 297 | 309 | |
| 383 | 320 | 331 | 343 | 354 | 365 | 377 | 388 | 399 | 410 | 422 | |
| 384 | 433 | 444 | 456 | 467 | 478 | 490 | 501 | 512 | 524 | 535 | |
| 385 | 546 | 557 | 569 | 580 | 591 | 602 | 614 | 625 | 636 | 647 | |
| 386 | 659 | 670 | 681 | 692 | 704 | 715 | 726 | 737 | 749 | 760 | |
| 387 | 771 | 782 | 794 | 805 | 816 | 827 | 838 | 850 | 861 | 872 | |
| 388 | 883 | 894 | 906 | 917 | 928 | 939 | 950 | 961 | 973 | 984 | |
| 389 | 995 | *006 | *017 | *028 | *040 | *051 | *062 | *073 | *084 | *095 | |
| 390 | 59 106 | 118 | 129 | 140 | 151 | 162 | 173 | 184 | 195 | 207 | |
| 391 | 218 | 229 | 240 | 251 | 262 | 273 | 284 | 295 | 306 | 318 | |
| 392 | 329 | 340 | 351 | 362 | 373 | 384 | 395 | 406 | 417 | 428 | |
| 393 | 439 | 450 | 461 | 472 | 483 | 494 | 506 | 517 | 528 | 539 | |
| 394 | 550 | 561 | 572 | 583 | 594 | 605 | 616 | 627 | 638 | 649 | |
| 395 | 660 | 671 | 682 | 693 | 704 | 715 | 726 | 737 | 748 | 759 | |
| 396 | 770 | 780 | 791 | 802 | 813 | 824 | 835 | 846 | 857 | 868 | |
| 397 | 879 | 890 | 901 | 912 | 923 | 934 | 945 | 956 | 966 | 977 | |
| 398 | 988 | 999 | *010 | *021 | *032 | *043 | *054 | *065 | *076 | *086 | |
| 399 | 60 097 | 108 | 119 | 130 | 141 | 152 | 163 | 173 | 184 | 195 | |
| 400 | 206 | 217 | 228 | 239 | 249 | 260 | 271 | 282 | 293 | 304 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| | 13 | 12 |
|---|------|------|
| 1 | 1.3 | 1.2 |
| 2 | 2.6 | 2.4 |
| 3 | 3.9 | 3.6 |
| 4 | 5.2 | 4.8 |
| 5 | 6.5 | 6.0 |
| 6 | 7.8 | 7.2 |
| 7 | 9.1 | 8.4 |
| 8 | 10.4 | 9.6 |
| 9 | 11.7 | 10.8 |

| | 11 | 10 |
|---|-----|-----|
| 1 | 1.1 | 1.0 |
| 2 | 2.2 | 2.0 |
| 3 | 3.3 | 3.0 |
| 4 | 4.4 | 4.0 |
| 5 | 5.5 | 5.0 |
| 6 | 6.6 | 6.0 |
| 7 | 7.7 | 7.0 |
| 8 | 8.8 | 8.0 |
| 9 | 9.9 | 9.0 |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|-----|-----|-----|------|------|------|------|------|------|------------|
| 400 | 60 206 | 217 | 228 | 239 | 249 | 260 | 271 | 282 | 293 | 304 | |
| 401 | 314 | 325 | 336 | 347 | 358 | 369 | 379 | 390 | 401 | 412 | |
| 402 | 423 | 433 | 444 | 455 | 466 | 477 | 487 | 498 | 509 | 520 | |
| 403 | 531 | 541 | 552 | 563 | 574 | 584 | 595 | 606 | 617 | 627 | |
| 404 | 638 | 649 | 660 | 670 | 681 | 692 | 703 | 713 | 724 | 735 | |
| 405 | 746 | 756 | 767 | 778 | 788 | 799 | 810 | 821 | 831 | 842 | |
| 406 | 853 | 863 | 874 | 885 | 895 | 906 | 917 | 927 | 938 | 949 | |
| 407 | 959 | 970 | 981 | 991 | *002 | *013 | *023 | *034 | *045 | *055 | |
| 408 | 61 066 | 077 | 087 | 098 | 109 | 119 | 130 | 140 | 151 | 162 | |
| 409 | 172 | 183 | 194 | 204 | 215 | 225 | 236 | 247 | 257 | 268 | |
| 410 | 278 | 289 | 300 | 310 | 321 | 331 | 342 | 352 | 363 | 374 | |
| 411 | 384 | 395 | 405 | 416 | 426 | 437 | 448 | 458 | 469 | 479 | 11 1.1 1.0 |
| 412 | 490 | 500 | 511 | 521 | 532 | 542 | 553 | 563 | 574 | 584 | 2 2.2 2.0 |
| 413 | 595 | 606 | 616 | 627 | 637 | 648 | 658 | 669 | 679 | 690 | 3 3.3 3.0 |
| 414 | 700 | 711 | 721 | 731 | 742 | 752 | 763 | 773 | 784 | 794 | 4 4.4 4.0 |
| 415 | 805 | 815 | 826 | 836 | 847 | 857 | 868 | 878 | 888 | 899 | 5 5.5 5.0 |
| 416 | 909 | 920 | 930 | 941 | 951 | 962 | 972 | 982 | 993 | *003 | 6 6.6 6.0 |
| 417 | 62 014 | 024 | 034 | 045 | 055 | 066 | 076 | 086 | 097 | 107 | 7 7.7 7.0 |
| 418 | 118 | 128 | 138 | 149 | 159 | 170 | 180 | 190 | 201 | 211 | 8 8.8 8.0 |
| 419 | 221 | 232 | 242 | 252 | 263 | 273 | 284 | 294 | 304 | 315 | 9 9.9 9.0 |
| 420 | 325 | 335 | 346 | 356 | 366 | 377 | 387 | 397 | 408 | 418 | |
| 421 | 428 | 439 | 449 | 459 | 469 | 480 | 490 | 500 | 511 | 521 | |
| 422 | 531 | 542 | 552 | 562 | 572 | 583 | 593 | 603 | 613 | 624 | |
| 423 | 634 | 644 | 655 | 665 | 675 | 685 | 696 | 706 | 716 | 726 | |
| 424 | 737 | 747 | 757 | 767 | 778 | 788 | 798 | 808 | 818 | 829 | |
| 425 | 839 | 849 | 859 | 870 | 880 | 890 | 900 | 910 | 921 | 931 | |
| 426 | 941 | 951 | 961 | 972 | 982 | 992 | *002 | *012 | *022 | *033 | |
| 427 | 63 043 | 053 | 063 | 073 | 083 | 094 | 104 | 114 | 124 | 134 | |
| 428 | 144 | 155 | 165 | 175 | 185 | 195 | 205 | 215 | 225 | 236 | |
| 429 | 246 | 256 | 266 | 276 | 286 | 296 | 306 | 317 | 327 | 337 | |
| 430 | 347 | 357 | 367 | 377 | 387 | 397 | 407 | 417 | 428 | 438 | |
| 431 | 448 | 458 | 468 | 478 | 488 | 498 | 508 | 518 | 528 | 538 | |
| 432 | 548 | 558 | 568 | 579 | 589 | 599 | 609 | 619 | 629 | 639 | |
| 433 | 649 | 659 | 669 | 679 | 689 | 699 | 709 | 719 | 729 | 739 | |
| 434 | 749 | 759 | 769 | 779 | 789 | 799 | 809 | 819 | 829 | 839 | |
| 435 | 849 | 859 | 869 | 879 | 889 | 899 | 909 | 919 | 929 | 939 | |
| 436 | 949 | 959 | 969 | 979 | 988 | 998 | *008 | *018 | *028 | *038 | |
| 437 | 64 048 | 058 | 068 | 078 | 088 | 098 | 108 | 118 | 128 | 137 | |
| 438 | 147 | 157 | 167 | 177 | 187 | 197 | 207 | 217 | 227 | 237 | |
| 439 | 216 | 226 | 236 | 246 | 256 | 266 | 276 | 286 | 296 | 306 | |
| 440 | 345 | 355 | 365 | 375 | 385 | 395 | 405 | 414 | 424 | 434 | |
| 441 | 444 | 454 | 464 | 473 | 483 | 493 | 503 | 513 | 523 | 532 | |
| 442 | 542 | 552 | 562 | 572 | 582 | 591 | 601 | 611 | 621 | 631 | |
| 443 | 640 | 650 | 660 | 670 | 680 | 689 | 699 | 709 | 719 | 729 | |
| 444 | 738 | 748 | 758 | 768 | 777 | 787 | 797 | 807 | 816 | 826 | |
| 445 | 836 | 846 | 856 | 865 | 875 | 885 | 895 | 904 | 914 | 924 | |
| 446 | 933 | 943 | 953 | 963 | 972 | 982 | 992 | *002 | *011 | *021 | |
| 447 | 65 031 | 040 | 050 | 060 | 070 | 079 | 089 | 099 | 108 | 118 | |
| 448 | 128 | 137 | 147 | 157 | 167 | 176 | 186 | 196 | 205 | 215 | |
| 449 | 225 | 234 | 244 | 254 | 263 | 273 | 283 | 292 | 302 | 312 | |
| 450 | 321 | 331 | 341 | 350 | 360 | 369 | 379 | 389 | 398 | 408 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

$\log M$
 $= \log [\log e]$
 $= 9.63778431$
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9
 1 0.9
 2 1.8
 3 2.7
 4 3.6
 5 4.5
 6 5.4
 7 6.3
 8 7.2
 9 8.1

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|------|------|------|------|------|------|------|------|------|------------|
| 450 | 65 321 | 331 | 341 | 350 | 360 | 369 | 379 | 389 | 398 | 408 | |
| 451 | 418 | 427 | 437 | 447 | 456 | 466 | 475 | 485 | 495 | 504 | |
| 452 | 514 | 523 | 533 | 543 | 552 | 562 | 571 | 581 | 591 | 600 | |
| 453 | 610 | 619 | 629 | 639 | 648 | 658 | 667 | 677 | 686 | 696 | |
| 454 | 706 | 715 | 725 | 734 | 744 | 753 | 763 | 772 | 782 | 792 | |
| 455 | 801 | 811 | 820 | 830 | 839 | 849 | 858 | 868 | 877 | 887 | |
| 456 | 896 | 906 | 916 | 925 | 935 | 944 | 954 | 963 | 973 | 982 | |
| 457 | 992 | *001 | *011 | *020 | *030 | *039 | *049 | *058 | *068 | *077 | |
| 458 | 66 087 | 096 | 106 | 115 | 124 | 134 | 143 | 153 | 162 | 172 | |
| 459 | 181 | 191 | 200 | 210 | 219 | 229 | 238 | 247 | 257 | 266 | |
| 460 | 276 | 285 | 295 | 304 | 314 | 324 | 332 | 342 | 351 | 361 | |
| 461 | 370 | 380 | 389 | 398 | 408 | 417 | 427 | 436 | 445 | 455 | |
| 462 | 464 | 474 | 483 | 492 | 502 | 511 | 521 | 530 | 539 | 549 | |
| 463 | 558 | 567 | 577 | 586 | 596 | 605 | 614 | 624 | 633 | 642 | |
| 464 | 652 | 661 | 671 | 680 | 689 | 699 | 708 | 717 | 727 | 736 | |
| 465 | 745 | 755 | 764 | 773 | 783 | 792 | 801 | 811 | 820 | 829 | |
| 466 | 839 | 848 | 857 | 867 | 876 | 885 | 894 | 904 | 913 | 922 | |
| 467 | 932 | 941 | 950 | 960 | 969 | 978 | 987 | 997 | *006 | *015 | |
| 468 | 67 025 | 034 | 043 | 052 | 062 | 071 | 080 | 089 | 099 | 108 | |
| 469 | 117 | 127 | 136 | 145 | 154 | 164 | 173 | 182 | 191 | 201 | |
| 470 | 210 | 219 | 228 | 237 | 247 | 256 | 265 | 274 | 284 | 293 | |
| 471 | 302 | 311 | 321 | 330 | 339 | 348 | 357 | 367 | 376 | 385 | |
| 472 | 394 | 403 | 413 | 422 | 431 | 440 | 449 | 459 | 468 | 477 | |
| 473 | 486 | 495 | 504 | 514 | 523 | 532 | 541 | 550 | 560 | 569 | |
| 474 | 578 | 587 | 596 | 605 | 614 | 624 | 633 | 642 | 651 | 660 | |
| 475 | 669 | 679 | 688 | 697 | 706 | 715 | 724 | 733 | 742 | 752 | |
| 476 | 761 | 770 | 779 | 788 | 797 | 806 | 815 | 824 | 833 | 843 | |
| 477 | 852 | 861 | 870 | 879 | 888 | 897 | 906 | 916 | 925 | 934 | |
| 478 | 945 | 954 | 963 | 972 | 981 | 990 | 997 | *006 | *015 | *024 | |
| 479 | 68 034 | 043 | 052 | 061 | 070 | 079 | 088 | 097 | 106 | 115 | |
| 480 | 124 | 133 | 142 | 151 | 160 | 169 | 178 | 187 | 196 | 205 | |
| 481 | 215 | 224 | 233 | 242 | 251 | 260 | 269 | 278 | 287 | 296 | |
| 482 | 305 | 314 | 323 | 332 | 341 | 350 | 359 | 368 | 377 | 386 | |
| 483 | 395 | 404 | 413 | 422 | 431 | 440 | 449 | 458 | 467 | 476 | |
| 484 | 485 | 494 | 502 | 511 | 520 | 529 | 538 | 547 | 556 | 565 | |
| 485 | 574 | 583 | 592 | 601 | 610 | 619 | 628 | 637 | 646 | 655 | |
| 486 | 664 | 673 | 681 | 690 | 699 | 708 | 717 | 726 | 735 | 744 | |
| 487 | 753 | 762 | 771 | 780 | 789 | 797 | 806 | 815 | 824 | 833 | |
| 488 | 842 | 851 | 860 | 869 | 878 | 886 | 895 | 904 | 913 | 922 | |
| 489 | 931 | 940 | 949 | 958 | 966 | 975 | 984 | 993 | *002 | *011 | |
| 490 | 69 020 | 028 | 037 | 046 | 055 | 064 | 073 | 082 | 090 | 099 | |
| 491 | 108 | 117 | 126 | 135 | 144 | 152 | 161 | 170 | 179 | 188 | |
| 492 | 197 | 205 | 214 | 223 | 232 | 241 | 249 | 258 | 267 | 276 | |
| 493 | 285 | 294 | 302 | 311 | 320 | 329 | 338 | 346 | 355 | 364 | |
| 494 | 373 | 381 | 390 | 399 | 408 | 417 | 425 | 434 | 443 | 452 | |
| 495 | 461 | 469 | 478 | 487 | 496 | 504 | 513 | 522 | 531 | 539 | |
| 496 | 548 | 557 | 566 | 574 | 583 | 592 | 601 | 609 | 618 | 627 | |
| 497 | 636 | 644 | 653 | 662 | 671 | 679 | 688 | 697 | 705 | 714 | |
| 498 | 723 | 732 | 740 | 749 | 758 | 767 | 775 | 784 | 793 | 801 | |
| 499 | 810 | 819 | 827 | 836 | 845 | 854 | 862 | 871 | 880 | 888 | |
| 500 | 897 | 906 | 914 | 923 | 932 | 940 | 949 | 958 | 966 | 975 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| | 10 | 9 |
|---|----|----|
| 1 | 10 | 09 |
| 2 | 20 | 18 |
| 3 | 30 | 27 |
| 4 | 40 | 36 |
| 5 | 50 | 45 |
| 6 | 60 | 54 |
| 7 | 70 | 63 |
| 8 | 80 | 72 |
| 9 | 90 | 81 |

| | 8 |
|---|----|
| 1 | 08 |
| 2 | 16 |
| 3 | 24 |
| 4 | 32 |
| 5 | 40 |
| 6 | 48 |
| 7 | 56 |
| 8 | 64 |
| 9 | 72 |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|------|------|------|------|------|------|------|------|------|-------------------------|
| 500 | 69 897 | 906 | 914 | 923 | 932 | 940 | 949 | 958 | 966 | 975 | log 5 = .69897 00043 |
| 501 | 984 | 992 | *001 | *010 | *018 | *027 | *036 | *044 | *053 | *062 | |
| 502 | 70 070 | 079 | 088 | 096 | 105 | 114 | 122 | 131 | 140 | 148 | |
| 503 | 157 | 165 | 174 | 183 | 191 | 200 | 209 | 217 | 226 | 234 | |
| 504 | 243 | 252 | 260 | 269 | 278 | 286 | 295 | 303 | 312 | 321 | |
| 505 | 329 | 338 | 346 | 355 | 364 | 372 | 381 | 389 | 398 | 406 | |
| 506 | 415 | 424 | 432 | 441 | 449 | 458 | 467 | 475 | 484 | 492 | |
| 507 | 501 | 509 | 518 | 526 | 535 | 544 | 552 | 561 | 569 | 578 | |
| 508 | 586 | 595 | 603 | 612 | 621 | 629 | 638 | 646 | 655 | 663 | |
| 509 | 672 | 680 | 689 | 697 | 706 | 714 | 723 | 731 | 740 | 749 | |
| 510 | 757 | 766 | 774 | 783 | 791 | 800 | 808 | 817 | 825 | 834 | |
| 511 | 842 | 851 | 859 | 868 | 876 | 885 | 893 | 902 | 910 | 919 | |
| 512 | 927 | 935 | 944 | 952 | 961 | 969 | 978 | 986 | 995 | *003 | |
| 513 | 71 012 | 020 | 029 | 037 | 046 | 054 | 063 | 071 | 079 | 088 | 9 |
| 514 | 096 | 105 | 113 | 122 | 130 | 139 | 147 | 155 | 164 | 172 | 1 |
| 515 | 181 | 189 | 198 | 206 | 214 | 223 | 231 | 240 | 248 | 257 | 2 |
| 516 | 265 | 273 | 282 | 290 | 299 | 307 | 315 | 324 | 332 | 341 | 3 |
| 517 | 349 | 357 | 366 | 374 | 383 | 391 | 399 | 408 | 416 | 425 | 4 |
| 518 | 433 | 441 | 450 | 458 | 466 | 475 | 483 | 492 | 500 | 508 | 5 |
| 519 | 517 | 525 | 533 | 542 | 550 | 559 | 567 | 575 | 584 | 592 | 6 |
| 520 | 600 | 609 | 617 | 625 | 634 | 642 | 650 | 659 | 667 | 675 | 7 |
| 521 | 684 | 692 | 700 | 709 | 717 | 725 | 734 | 742 | 750 | 759 | 8 |
| 522 | 767 | 775 | 784 | 792 | 800 | 809 | 817 | 825 | 834 | 842 | |
| 523 | 850 | 858 | 867 | 875 | 883 | 892 | 900 | 908 | 917 | 925 | |
| 524 | 933 | 941 | 950 | 958 | 966 | 975 | 983 | 991 | 999 | *008 | |
| 525 | 72 016 | 021 | 032 | 041 | 049 | 057 | 066 | 074 | 082 | 090 | |
| 526 | 099 | 107 | 115 | 123 | 132 | 140 | 148 | 156 | 165 | 173 | |
| 527 | 181 | 189 | 198 | 206 | 214 | 222 | 230 | 239 | 247 | 255 | |
| 528 | 263 | 272 | 280 | 288 | 296 | 304 | 313 | 321 | 329 | 337 | |
| 529 | 346 | 354 | 362 | 370 | 378 | 387 | 395 | 403 | 411 | 419 | |
| 530 | 428 | 436 | 444 | 452 | 460 | 469 | 477 | 485 | 493 | 501 | |
| 531 | 509 | 518 | 526 | 534 | 542 | 550 | 558 | 567 | 575 | 583 | 7 |
| 532 | 591 | 599 | 607 | 616 | 624 | 632 | 640 | 648 | 656 | 665 | 1 |
| 533 | 673 | 681 | 689 | 697 | 705 | 713 | 722 | 730 | 738 | 746 | 2 |
| 534 | 754 | 762 | 770 | 779 | 787 | 795 | 803 | 811 | 819 | 827 | 3 |
| 535 | 835 | 843 | 852 | 860 | 868 | 876 | 884 | 892 | 900 | 908 | 4 |
| 536 | 916 | 925 | 933 | 941 | 949 | 957 | 965 | 973 | 981 | 989 | 5 |
| 537 | 997 | *006 | *014 | *022 | *030 | *038 | *046 | *054 | *062 | *070 | 6 |
| 538 | 73 078 | 086 | 094 | 102 | 111 | 119 | 127 | 135 | 143 | 151 | 7 |
| 539 | 159 | 167 | 175 | 183 | 191 | 199 | 207 | 215 | 223 | 231 | 8 |
| 540 | 239 | 247 | 255 | 263 | 272 | 280 | 288 | 296 | 304 | 312 | 9 |
| 541 | 320 | 328 | 336 | 344 | 352 | 360 | 368 | 376 | 384 | 392 | |
| 542 | 400 | 408 | 416 | 424 | 432 | 440 | 448 | 456 | 464 | 472 | |
| 543 | 480 | 488 | 496 | 504 | 512 | 520 | 528 | 536 | 544 | 552 | |
| 544 | 560 | 568 | 576 | 584 | 592 | 600 | 608 | 616 | 624 | 632 | |
| 545 | 640 | 648 | 656 | 664 | 672 | 679 | 687 | 695 | 703 | 711 | |
| 546 | 719 | 727 | 735 | 743 | 751 | 759 | 767 | 775 | 783 | 791 | |
| 547 | 799 | 807 | 815 | 823 | 830 | 838 | 846 | 854 | 862 | 870 | |
| 548 | 878 | 886 | 894 | 902 | 910 | 918 | 926 | 933 | 941 | 949 | |
| 549 | 957 | 965 | 973 | 981 | 989 | 997 | *005 | *013 | *020 | *028 | |
| 550 | 74 036 | 044 | 052 | 060 | 068 | 076 | 084 | 092 | 099 | 107 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|-----|-----|-----|------|------|------|------|------|------|------------|
| 550 | 74 036 | 044 | 052 | 060 | 068 | 076 | 084 | 092 | 099 | 107 | |
| 551 | 115 | 123 | 131 | 139 | 147 | 155 | 162 | 170 | 178 | 186 | |
| 552 | 194 | 202 | 210 | 218 | 225 | 233 | 241 | 249 | 257 | 265 | |
| 553 | 273 | 280 | 288 | 296 | 304 | 312 | 320 | 327 | 335 | 343 | |
| 554 | 351 | 359 | 367 | 374 | 382 | 390 | 398 | 406 | 414 | 421 | |
| 555 | 429 | 437 | 445 | 453 | 461 | 468 | 476 | 484 | 492 | 500 | |
| 556 | 507 | 515 | 523 | 531 | 539 | 547 | 554 | 562 | 570 | 578 | |
| 557 | 586 | 593 | 601 | 609 | 617 | 624 | 632 | 640 | 648 | 656 | |
| 558 | 663 | 671 | 679 | 687 | 695 | 702 | 710 | 718 | 726 | 733 | |
| 559 | 741 | 749 | 757 | 764 | 772 | 780 | 788 | 796 | 803 | 811 | |
| 560 | 819 | 827 | 834 | 842 | 850 | 858 | 865 | 873 | 881 | 889 | |
| 561 | 896 | 904 | 912 | 920 | 927 | 935 | 943 | 950 | 958 | 966 | |
| 562 | 974 | 981 | 989 | 997 | *005 | *012 | *020 | *028 | *035 | *043 | |
| 563 | 75 051 | 059 | 066 | 074 | 082 | 089 | 097 | 105 | 113 | 120 | |
| 564 | 128 | 136 | 143 | 151 | 159 | 166 | 174 | 182 | 189 | 197 | |
| 565 | 205 | 213 | 220 | 228 | 236 | 243 | 251 | 259 | 266 | 274 | |
| 566 | 282 | 289 | 297 | 305 | 312 | 320 | 328 | 335 | 343 | 351 | |
| 567 | 358 | 366 | 374 | 381 | 389 | 397 | 404 | 412 | 420 | 427 | |
| 568 | 435 | 442 | 450 | 458 | 465 | 473 | 481 | 488 | 496 | 504 | |
| 569 | 511 | 519 | 526 | 534 | 542 | 549 | 557 | 565 | 572 | 580 | |
| 570 | 587 | 595 | 603 | 610 | 618 | 626 | 633 | 641 | 648 | 656 | |
| 571 | 664 | 671 | 679 | 686 | 694 | 702 | 709 | 717 | 724 | 732 | |
| 572 | 740 | 747 | 755 | 762 | 770 | 778 | 785 | 793 | 800 | 808 | |
| 573 | 815 | 823 | 831 | 838 | 846 | 853 | 861 | 868 | 876 | 884 | |
| 574 | 891 | 899 | 906 | 914 | 921 | 929 | 937 | 944 | 952 | 959 | |
| 575 | 967 | 974 | 982 | 989 | 997 | *005 | *012 | *020 | *027 | *035 | |
| 576 | 76 042 | 050 | 057 | 065 | 072 | 080 | 087 | 095 | 103 | 110 | |
| 577 | 118 | 125 | 133 | 140 | 148 | 155 | 163 | 170 | 178 | 185 | |
| 578 | 193 | 200 | 208 | 215 | 223 | 230 | 238 | 245 | 253 | 260 | |
| 579 | 268 | 275 | 283 | 290 | 298 | 305 | 313 | 320 | 328 | 335 | |
| 580 | 343 | 350 | 358 | 365 | 373 | 380 | 388 | 395 | 403 | 410 | |
| 581 | 418 | 425 | 433 | 440 | 448 | 455 | 462 | 470 | 477 | 485 | |
| 582 | 492 | 500 | 507 | 515 | 522 | 530 | 537 | 545 | 552 | 559 | |
| 583 | 567 | 574 | 582 | 589 | 597 | 604 | 612 | 619 | 626 | 634 | |
| 584 | 641 | 649 | 656 | 664 | 671 | 678 | 686 | 693 | 701 | 708 | |
| 585 | 716 | 723 | 730 | 738 | 745 | 753 | 760 | 768 | 775 | 782 | |
| 586 | 790 | 797 | 805 | 812 | 819 | 827 | 834 | 842 | 849 | 856 | |
| 587 | 864 | 871 | 879 | 886 | 893 | 901 | 908 | 916 | 923 | 930 | |
| 588 | 938 | 945 | 953 | 960 | 967 | 975 | 982 | 989 | 997 | *004 | |
| 589 | 77 012 | 019 | 026 | 034 | 041 | 048 | 056 | 063 | 070 | 078 | |
| 590 | 085 | 093 | 100 | 107 | 115 | 122 | 129 | 137 | 144 | 151 | |
| 591 | 159 | 166 | 173 | 181 | 188 | 195 | 203 | 210 | 217 | 225 | |
| 592 | 232 | 240 | 247 | 254 | 262 | 269 | 276 | 283 | 291 | 298 | |
| 593 | 305 | 313 | 320 | 327 | 335 | 342 | 349 | 357 | 364 | 371 | |
| 594 | 379 | 386 | 393 | 401 | 408 | 415 | 422 | 430 | 437 | 444 | |
| 595 | 452 | 459 | 466 | 474 | 481 | 488 | 495 | 503 | 510 | 517 | |
| 596 | 525 | 532 | 539 | 546 | 554 | 561 | 568 | 576 | 583 | 590 | |
| 597 | 597 | 605 | 612 | 619 | 627 | 634 | 641 | 648 | 656 | 663 | |
| 598 | 670 | 677 | 685 | 692 | 699 | 706 | 714 | 721 | 728 | 735 | |
| 599 | 743 | 750 | 757 | 764 | 772 | 779 | 786 | 793 | 801 | 808 | |
| 600 | 815 | 822 | 830 | 837 | 844 | 851 | 859 | 866 | 873 | 880 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| | 8 | 7 |
|---|-----|-----|
| 1 | 0.8 | 0.7 |
| 2 | 1.6 | 1.4 |
| 3 | 2.4 | 2.1 |
| 4 | 3.2 | 2.8 |
| 5 | 4.0 | 3.5 |
| 6 | 4.8 | 4.2 |
| 7 | 5.6 | 4.9 |
| 8 | 6.4 | 5.6 |
| 9 | 7.2 | 6.3 |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|-----|-----|-----|-----|-----|------|------|------|------|------------|
| 600 | 77 815 | 822 | 830 | 837 | 844 | 851 | 859 | 866 | 873 | 880 | |
| 601 | 887 | 895 | 902 | 909 | 916 | 924 | 931 | 938 | 945 | 952 | |
| 602 | 960 | 967 | 974 | 981 | 988 | 996 | *003 | *010 | *017 | *025 | |
| 603 | 78 032 | 039 | 046 | 053 | 061 | 068 | 075 | 082 | 089 | 097 | |
| 604 | 104 | 111 | 118 | 125 | 132 | 140 | 147 | 154 | 161 | 168 | |
| 605 | 176 | 183 | 190 | 197 | 204 | 211 | 219 | 226 | 233 | 240 | |
| 606 | 247 | 254 | 262 | 269 | 276 | 283 | 290 | 297 | 305 | 312 | |
| 607 | 319 | 326 | 333 | 340 | 347 | 355 | 362 | 369 | 376 | 383 | |
| 608 | 390 | 398 | 405 | 412 | 419 | 426 | 433 | 440 | 447 | 455 | |
| 609 | 462 | 469 | 476 | 483 | 490 | 497 | 504 | 512 | 519 | 526 | |
| 610 | 533 | 540 | 547 | 554 | 561 | 569 | 576 | 583 | 590 | 597 | |
| 611 | 604 | 611 | 618 | 625 | 633 | 640 | 647 | 654 | 661 | 668 | |
| 612 | 675 | 682 | 689 | 696 | 704 | 711 | 718 | 725 | 732 | 739 | |
| 613 | 746 | 753 | 760 | 767 | 774 | 781 | 789 | 796 | 803 | 810 | |
| 614 | 817 | 824 | 831 | 838 | 845 | 852 | 859 | 866 | 873 | 880 | |
| 615 | 888 | 895 | 902 | 909 | 916 | 923 | 930 | 937 | 944 | 951 | |
| 616 | 958 | 965 | 972 | 979 | 986 | 993 | *000 | *007 | *014 | *021 | |
| 617 | 79 029 | 036 | 043 | 050 | 057 | 064 | 071 | 078 | 085 | 092 | |
| 618 | 099 | 106 | 113 | 120 | 127 | 134 | 141 | 148 | 155 | 162 | |
| 619 | 169 | 176 | 183 | 190 | 197 | 204 | 211 | 218 | 225 | 232 | |
| 620 | 239 | 246 | 253 | 260 | 267 | 274 | 281 | 288 | 295 | 302 | |
| 621 | 309 | 316 | 323 | 330 | 337 | 344 | 351 | 358 | 365 | 372 | |
| 622 | 379 | 386 | 393 | 400 | 407 | 414 | 421 | 428 | 435 | 442 | |
| 623 | 449 | 456 | 463 | 470 | 477 | 484 | 491 | 498 | 505 | 511 | |
| 624 | 518 | 525 | 532 | 539 | 546 | 553 | 560 | 567 | 574 | 581 | |
| 625 | 588 | 595 | 602 | 609 | 616 | 623 | 630 | 637 | 644 | 650 | |
| 626 | 657 | 664 | 671 | 678 | 685 | 692 | 699 | 706 | 713 | 720 | |
| 627 | 727 | 734 | 741 | 748 | 754 | 761 | 768 | 775 | 782 | 789 | |
| 628 | 796 | 803 | 810 | 817 | 824 | 831 | 837 | 844 | 851 | 858 | |
| 629 | 865 | 872 | 879 | 886 | 893 | 900 | 906 | 913 | 920 | 927 | |
| 630 | 934 | 941 | 948 | 955 | 962 | 969 | 975 | 982 | 989 | 996 | |
| 631 | 80 003 | 010 | 017 | 024 | 030 | 037 | 044 | 051 | 058 | 065 | |
| 632 | 072 | 079 | 085 | 092 | 099 | 106 | 113 | 120 | 127 | 134 | |
| 633 | 140 | 147 | 154 | 161 | 168 | 175 | 182 | 188 | 195 | 202 | |
| 634 | 209 | 216 | 223 | 229 | 236 | 243 | 250 | 257 | 264 | 271 | |
| 635 | 277 | 284 | 291 | 298 | 305 | 312 | 318 | 325 | 332 | 339 | |
| 636 | 346 | 353 | 359 | 366 | 373 | 380 | 387 | 393 | 400 | 407 | |
| 637 | 414 | 421 | 428 | 434 | 441 | 448 | 455 | 462 | 468 | 475 | |
| 638 | 482 | 489 | 496 | 502 | 509 | 516 | 523 | 530 | 536 | 543 | |
| 639 | 550 | 557 | 564 | 570 | 577 | 584 | 591 | 598 | 604 | 611 | |
| 640 | 618 | 625 | 632 | 638 | 645 | 652 | 659 | 665 | 672 | 679 | |
| 641 | 686 | 693 | 699 | 706 | 713 | 720 | 726 | 733 | 740 | 747 | |
| 642 | 754 | 760 | 767 | 774 | 781 | 787 | 794 | 801 | 808 | 814 | |
| 643 | 821 | 828 | 835 | 841 | 848 | 855 | 862 | 868 | 875 | 882 | |
| 644 | 889 | 895 | 902 | 909 | 916 | 922 | 929 | 936 | 943 | 949 | |
| 645 | 956 | 963 | 969 | 976 | 983 | 990 | 996 | *003 | *010 | *017 | |
| 646 | 81 023 | 030 | 037 | 043 | 050 | 057 | 064 | 070 | 077 | 084 | |
| 647 | 090 | 097 | 104 | 111 | 117 | 124 | 131 | 137 | 144 | 151 | |
| 648 | 158 | 164 | 171 | 178 | 184 | 191 | 198 | 204 | 211 | 218 | |
| 649 | 224 | 231 | 238 | 245 | 251 | 258 | 265 | 271 | 278 | 285 | |
| 650 | 291 | 298 | 305 | 311 | 318 | 325 | 331 | 338 | 345 | 351 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| | 8 | 7 |
|---|-----|-----|
| 1 | 0.8 | 0.7 |
| 2 | 1.6 | 1.4 |
| 3 | 2.4 | 2.1 |
| 4 | 3.2 | 2.8 |
| 5 | 4.0 | 3.5 |
| 6 | 4.8 | 4.2 |
| 7 | 5.6 | 4.9 |
| 8 | 6.4 | 5.6 |
| 9 | 7.2 | 6.3 |

| | 6 |
|---|-----|
| 1 | 0.6 |
| 2 | 1.2 |
| 3 | 1.8 |
| 4 | 2.4 |
| 5 | 3.0 |
| 6 | 3.6 |
| 7 | 4.2 |
| 8 | 4.8 |
| 9 | 5.4 |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|------|------|------|------|------|------|------|------|------|------------|
| 650 | 81 291 | 298 | 305 | 311 | 318 | 325 | 331 | 338 | 345 | 351 | |
| 651 | 358 | 365 | 371 | 378 | 385 | 391 | 398 | 405 | 411 | 418 | |
| 652 | 425 | 431 | 438 | 445 | 451 | 458 | 465 | 471 | 478 | 485 | |
| 653 | 491 | 498 | 505 | 511 | 518 | 525 | 531 | 538 | 544 | 551 | |
| 654 | 558 | 564 | 571 | 578 | 584 | 591 | 598 | 604 | 611 | 617 | |
| 655 | 624 | 631 | 637 | 644 | 651 | 657 | 664 | 671 | 677 | 684 | |
| 656 | 690 | 697 | 704 | 710 | 717 | 723 | 730 | 737 | 743 | 750 | |
| 657 | 757 | 763 | 770 | 776 | 783 | 790 | 796 | 803 | 809 | 816 | |
| 658 | 823 | 829 | 836 | 842 | 849 | 856 | 862 | 869 | 875 | 882 | |
| 659 | 889 | 895 | 902 | 908 | 915 | 921 | 928 | 935 | 941 | 948 | |
| 660 | 954 | 961 | 968 | 974 | 981 | 987 | 994 | *000 | *007 | *014 | |
| 661 | 82 020 | 027 | 033 | 040 | 046 | 053 | 060 | 066 | 073 | 079 | |
| 662 | 086 | 092 | 099 | 105 | 112 | 119 | 125 | 132 | 138 | 145 | |
| 663 | 151 | 158 | 164 | 171 | 178 | 184 | 191 | 197 | 204 | 210 | |
| 664 | 217 | 223 | 230 | 236 | 243 | 249 | 256 | 263 | 269 | 276 | |
| 665 | 282 | 289 | 295 | 302 | 308 | 315 | 321 | 328 | 334 | 341 | |
| 666 | 347 | 354 | 360 | 367 | 373 | 380 | 387 | 393 | 400 | 406 | |
| 667 | 413 | 419 | 426 | 432 | 439 | 445 | 452 | 458 | 465 | 471 | |
| 668 | 478 | 484 | 491 | 497 | 504 | 510 | 517 | 523 | 530 | 536 | |
| 669 | 543 | 549 | 556 | 562 | 569 | 575 | 582 | 588 | 595 | 601 | |
| 670 | 607 | 614 | 620 | 627 | 633 | 640 | 646 | 653 | 659 | 666 | |
| 671 | 672 | 679 | 685 | 692 | 698 | 705 | 711 | 718 | 724 | 730 | |
| 672 | 737 | 743 | 750 | 756 | 763 | 769 | 776 | 782 | 789 | 795 | |
| 673 | 802 | 808 | 814 | 821 | 827 | 834 | 840 | 847 | 853 | 860 | |
| 674 | 866 | 872 | 879 | 885 | 892 | 898 | 905 | 911 | 918 | 924 | |
| 675 | 930 | 937 | 943 | 950 | 956 | 963 | 969 | 975 | 982 | 988 | |
| 676 | 995 | *001 | *008 | *014 | *020 | *027 | *033 | *040 | *046 | *052 | |
| 677 | 83 059 | 065 | 072 | 078 | 085 | 091 | 097 | 104 | 110 | 117 | |
| 678 | 123 | 129 | 136 | 142 | 149 | 155 | 161 | 168 | 174 | 181 | |
| 679 | 187 | 193 | 200 | 206 | 213 | 219 | 225 | 232 | 238 | 245 | |
| 680 | 251 | 257 | 264 | 270 | 276 | 283 | 289 | 296 | 302 | 308 | |
| 681 | 315 | 321 | 327 | 334 | 340 | 347 | 353 | 359 | 366 | 372 | |
| 682 | 378 | 385 | 391 | 398 | 404 | 410 | 417 | 423 | 429 | 436 | |
| 683 | 442 | 448 | 455 | 461 | 467 | 474 | 480 | 487 | 493 | 499 | |
| 684 | 506 | 512 | 518 | 525 | 531 | 537 | 544 | 550 | 556 | 563 | |
| 685 | 569 | 575 | 582 | 588 | 594 | 601 | 607 | 613 | 620 | 626 | |
| 686 | 632 | 639 | 645 | 651 | 658 | 664 | 670 | 677 | 683 | 689 | |
| 687 | 696 | 702 | 708 | 715 | 721 | 727 | 734 | 740 | 746 | 753 | |
| 688 | 759 | 765 | 771 | 778 | 784 | 790 | 797 | 803 | 809 | 816 | |
| 689 | 822 | 828 | 835 | 841 | 847 | 853 | 860 | 866 | 872 | 879 | |
| 690 | 885 | 891 | 897 | 904 | 910 | 916 | 923 | 929 | 935 | 942 | |
| 691 | 948 | 954 | 960 | 967 | 973 | 979 | 985 | 992 | 998 | *004 | |
| 692 | 84 011 | 017 | 023 | 029 | 036 | 042 | 048 | 055 | 061 | 067 | |
| 693 | 073 | 080 | 086 | 092 | 098 | 105 | 111 | 117 | 123 | 130 | |
| 694 | 136 | 142 | 148 | 155 | 161 | 167 | 173 | 180 | 186 | 192 | |
| 695 | 198 | 205 | 211 | 217 | 223 | 230 | 236 | 242 | 248 | 255 | |
| 696 | 261 | 267 | 273 | 280 | 286 | 292 | 298 | 305 | 311 | 317 | |
| 697 | 323 | 330 | 336 | 342 | 348 | 354 | 361 | 367 | 373 | 379 | |
| 698 | 386 | 392 | 398 | 404 | 410 | 417 | 423 | 429 | 435 | 442 | |
| 699 | 448 | 454 | 460 | 466 | 473 | 479 | 485 | 491 | 497 | 504 | |
| 700 | 510 | 516 | 522 | 528 | 535 | 541 | 547 | 553 | 559 | 566 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| | 7 | 6 |
|---|-----|-----|
| 1 | 0.7 | 0.6 |
| 2 | 1.4 | 1.2 |
| 3 | 2.1 | 1.8 |
| 4 | 2.8 | 2.4 |
| 5 | 3.5 | 3.0 |
| 6 | 4.2 | 3.6 |
| 7 | 4.9 | 4.2 |
| 8 | 5.6 | 4.8 |
| 9 | 6.3 | 5.4 |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|-----|--------|-----|-----|-----|------|------|------|------|------|------|--|--|---|---|-----|-----|-----|---|-----|-----|-----|-----|-----|---|-----|-----|-----|-----|-----|---|-----|-----|---|-----|-----|---|-----|-----|---|-----|-----|
| 700 | 84 510 | 516 | 522 | 528 | 535 | 541 | 547 | 553 | 559 | 566 | log 7 = .84509 80400 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 701 | 572 | 578 | 584 | 590 | 597 | 603 | 609 | 615 | 621 | 628 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 702 | 634 | 640 | 646 | 652 | 658 | 665 | 671 | 677 | 683 | 689 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 703 | 696 | 702 | 708 | 714 | 720 | 726 | 733 | 739 | 745 | 751 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 704 | 757 | 763 | 770 | 776 | 782 | 788 | 794 | 800 | 807 | 813 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 705 | 819 | 825 | 831 | 837 | 844 | 850 | 856 | 862 | 868 | 874 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 706 | 880 | 887 | 893 | 899 | 905 | 911 | 917 | 924 | 930 | 936 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 707 | 942 | 948 | 954 | 960 | 967 | 973 | 979 | 985 | 991 | 997 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 708 | 85 003 | 009 | 016 | 022 | 028 | 034 | 040 | 046 | 052 | 058 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 709 | 065 | 071 | 077 | 083 | 089 | 095 | 101 | 107 | 114 | 120 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 710 | 126 | 132 | 138 | 144 | 150 | 156 | 163 | 169 | 175 | 181 | <table><tr><td></td><td>7</td><td>6</td></tr><tr><td>1</td><td>0.7</td><td>0.6</td></tr><tr><td>2</td><td>1.4</td><td>1.2</td></tr><tr><td>3</td><td>2.1</td><td>1.8</td></tr><tr><td>4</td><td>2.8</td><td>2.4</td></tr><tr><td>5</td><td>3.5</td><td>3.0</td></tr><tr><td>6</td><td>4.2</td><td>3.6</td></tr><tr><td>7</td><td>4.9</td><td>4.2</td></tr><tr><td>8</td><td>5.6</td><td>4.8</td></tr><tr><td>9</td><td>6.3</td><td>5.4</td></tr></table> | | 7 | 6 | 1 | 0.7 | 0.6 | 2 | 1.4 | 1.2 | 3 | 2.1 | 1.8 | 4 | 2.8 | 2.4 | 5 | 3.5 | 3.0 | 6 | 4.2 | 3.6 | 7 | 4.9 | 4.2 | 8 | 5.6 | 4.8 | 9 | 6.3 | 5.4 |
| | 7 | 6 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 1 | 0.7 | 0.6 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 2 | 1.4 | 1.2 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 3 | 2.1 | 1.8 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 2.8 | 2.4 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 5 | 3.5 | 3.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 6 | 4.2 | 3.6 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 7 | 4.9 | 4.2 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 8 | 5.6 | 4.8 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 9 | 6.3 | 5.4 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 711 | 187 | 193 | 199 | 205 | 211 | 217 | 224 | 230 | 236 | 242 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 712 | 248 | 254 | 260 | 266 | 272 | 278 | 285 | 291 | 297 | 303 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 713 | 309 | 315 | 321 | 327 | 333 | 339 | 345 | 352 | 358 | 364 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 714 | 370 | 376 | 382 | 388 | 394 | 400 | 406 | 412 | 418 | 425 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 715 | 431 | 437 | 443 | 449 | 455 | 461 | 467 | 473 | 479 | 485 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 716 | 491 | 497 | 503 | 509 | 516 | 522 | 528 | 534 | 540 | 546 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 717 | 552 | 558 | 564 | 570 | 576 | 582 | 588 | 594 | 600 | 606 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 718 | 612 | 618 | 625 | 631 | 637 | 643 | 649 | 655 | 661 | 667 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 719 | 673 | 679 | 685 | 691 | 697 | 703 | 709 | 715 | 721 | 727 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 720 | 733 | 739 | 745 | 751 | 757 | 763 | 769 | 775 | 781 | 788 | <table><tr><td></td><td>5</td></tr><tr><td>1</td><td>0.5</td></tr><tr><td>2</td><td>1.0</td></tr><tr><td>3</td><td>1.5</td></tr><tr><td>4</td><td>2.0</td></tr><tr><td>5</td><td>2.5</td></tr><tr><td>6</td><td>3.0</td></tr><tr><td>7</td><td>3.5</td></tr><tr><td>8</td><td>4.0</td></tr><tr><td>9</td><td>4.5</td></tr></table> | | 5 | 1 | 0.5 | 2 | 1.0 | 3 | 1.5 | 4 | 2.0 | 5 | 2.5 | 6 | 3.0 | 7 | 3.5 | 8 | 4.0 | 9 | 4.5 | | | | | | | | | | |
| | 5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 1 | 0.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 2 | 1.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 3 | 1.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 2.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 5 | 2.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 6 | 3.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 7 | 3.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 8 | 4.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 9 | 4.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 721 | 794 | 800 | 806 | 812 | 818 | 824 | 830 | 836 | 842 | 848 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 722 | 854 | 860 | 866 | 872 | 878 | 884 | 890 | 896 | 902 | 908 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 723 | 914 | 920 | 926 | 932 | 938 | 944 | 950 | 956 | 962 | 968 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 724 | 974 | 980 | 986 | 992 | 998 | *004 | *010 | *016 | *022 | *028 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 725 | 86 034 | 040 | 046 | 052 | 058 | 064 | 070 | 076 | 082 | 088 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 726 | 094 | 100 | 106 | 112 | 118 | 124 | 130 | 136 | 141 | 147 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 727 | 153 | 159 | 165 | 171 | 177 | 183 | 189 | 195 | 201 | 207 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 728 | 213 | 219 | 225 | 231 | 237 | 243 | 249 | 255 | 261 | 267 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 729 | 273 | 279 | 285 | 291 | 297 | 303 | 308 | 314 | 320 | 326 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 730 | 332 | 338 | 344 | 350 | 356 | 362 | 368 | 374 | 380 | 386 | <table><tr><td></td><td>5</td></tr><tr><td>1</td><td>0.5</td></tr><tr><td>2</td><td>1.0</td></tr><tr><td>3</td><td>1.5</td></tr><tr><td>4</td><td>2.0</td></tr><tr><td>5</td><td>2.5</td></tr><tr><td>6</td><td>3.0</td></tr><tr><td>7</td><td>3.5</td></tr><tr><td>8</td><td>4.0</td></tr><tr><td>9</td><td>4.5</td></tr></table> | | 5 | 1 | 0.5 | 2 | 1.0 | 3 | 1.5 | 4 | 2.0 | 5 | 2.5 | 6 | 3.0 | 7 | 3.5 | 8 | 4.0 | 9 | 4.5 | | | | | | | | | | |
| | 5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 1 | 0.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 2 | 1.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 3 | 1.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 2.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 5 | 2.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 6 | 3.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 7 | 3.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 8 | 4.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 9 | 4.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 731 | 392 | 398 | 404 | 410 | 415 | 421 | 427 | 433 | 439 | 445 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 732 | 451 | 457 | 463 | 469 | 475 | 481 | 487 | 493 | 499 | 504 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 733 | 510 | 516 | 522 | 528 | 534 | 540 | 546 | 552 | 558 | 564 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 734 | 570 | 576 | 581 | 587 | 593 | 599 | 605 | 611 | 617 | 623 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 735 | 629 | 635 | 641 | 646 | 652 | 658 | 664 | 670 | 676 | 682 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 736 | 688 | 694 | 700 | 705 | 711 | 717 | 723 | 729 | 735 | 741 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 737 | 747 | 753 | 759 | 764 | 770 | 776 | 782 | 788 | 794 | 800 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 738 | 806 | 812 | 817 | 823 | 829 | 835 | 841 | 847 | 853 | 859 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 739 | 864 | 870 | 876 | 882 | 888 | 894 | 900 | 906 | 911 | 917 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 740 | 923 | 929 | 935 | 941 | 947 | 953 | 958 | 964 | 970 | 976 | <table><tr><td></td><td>5</td></tr><tr><td>1</td><td>0.5</td></tr><tr><td>2</td><td>1.0</td></tr><tr><td>3</td><td>1.5</td></tr><tr><td>4</td><td>2.0</td></tr><tr><td>5</td><td>2.5</td></tr><tr><td>6</td><td>3.0</td></tr><tr><td>7</td><td>3.5</td></tr><tr><td>8</td><td>4.0</td></tr><tr><td>9</td><td>4.5</td></tr></table> | | 5 | 1 | 0.5 | 2 | 1.0 | 3 | 1.5 | 4 | 2.0 | 5 | 2.5 | 6 | 3.0 | 7 | 3.5 | 8 | 4.0 | 9 | 4.5 | | | | | | | | | | |
| | 5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 1 | 0.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 2 | 1.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 3 | 1.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 4 | 2.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 5 | 2.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 6 | 3.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 7 | 3.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 8 | 4.0 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 9 | 4.5 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 741 | 982 | 988 | 994 | 999 | *005 | *011 | *017 | *023 | *029 | *035 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 742 | 87 040 | 046 | 052 | 058 | 064 | 070 | 075 | 081 | 087 | 093 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 743 | 099 | 105 | 111 | 116 | 122 | 128 | 134 | 140 | 146 | 151 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 744 | 157 | 163 | 169 | 175 | 181 | 186 | 192 | 198 | 204 | 210 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 745 | 216 | 221 | 227 | 233 | 239 | 245 | 251 | 256 | 262 | 268 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 746 | 274 | 280 | 286 | 291 | 297 | 303 | 309 | 315 | 320 | 326 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 747 | 332 | 338 | 344 | 349 | 355 | 361 | 367 | 373 | 379 | 384 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 748 | 390 | 396 | 402 | 408 | 413 | 419 | 425 | 431 | 437 | 442 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 749 | 448 | 454 | 460 | 466 | 471 | 477 | 483 | 489 | 495 | 500 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 750 | 506 | 512 | 518 | 523 | 529 | 535 | 541 | 547 | 552 | 558 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|-----|-----|------|------|------|------|------|------|------|------------|
| 750 | 87 506 | 512 | 518 | 523 | 529 | 535 | 541 | 547 | 552 | 558 | |
| 751 | 564 | 570 | 576 | 581 | 587 | 593 | 599 | 604 | 610 | 616 | |
| 752 | 622 | 628 | 633 | 639 | 645 | 651 | 656 | 662 | 668 | 674 | |
| 753 | 679 | 685 | 691 | 697 | 703 | 708 | 714 | 720 | 726 | 731 | |
| 754 | 737 | 743 | 749 | 754 | 760 | 766 | 772 | 777 | 783 | 789 | |
| 755 | 795 | 800 | 806 | 812 | 818 | 823 | 829 | 835 | 841 | 846 | |
| 756 | 852 | 858 | 864 | 869 | 875 | 881 | 887 | 892 | 898 | 904 | |
| 757 | 910 | 915 | 921 | 927 | 933 | 938 | 944 | 950 | 955 | 961 | |
| 758 | 967 | 973 | 978 | 984 | 990 | 996 | *001 | *007 | *013 | *018 | |
| 759 | 88 024 | 030 | 036 | 041 | 047 | 053 | 058 | 064 | 070 | 076 | |
| 760 | 081 | 087 | 093 | 098 | 104 | 110 | 116 | 121 | 127 | 133 | |
| 761 | 138 | 144 | 150 | 156 | 161 | 167 | 173 | 178 | 184 | 190 | |
| 762 | 195 | 201 | 207 | 213 | 218 | 224 | 230 | 235 | 241 | 247 | |
| 763 | 252 | 258 | 264 | 270 | 275 | 281 | 287 | 292 | 298 | 304 | |
| 764 | 309 | 315 | 321 | 326 | 332 | 338 | 343 | 349 | 355 | 360 | |
| 765 | 366 | 372 | 377 | 383 | 389 | 395 | 400 | 406 | 412 | 417 | |
| 766 | 423 | 429 | 434 | 440 | 446 | 451 | 457 | 463 | 468 | 474 | |
| 767 | 480 | 485 | 491 | 497 | 502 | 508 | 513 | 519 | 525 | 530 | |
| 768 | 536 | 542 | 547 | 553 | 559 | 564 | 570 | 576 | 581 | 587 | |
| 769 | 593 | 598 | 604 | 610 | 615 | 621 | 627 | 632 | 638 | 643 | |
| 770 | 649 | 655 | 660 | 666 | 672 | *677 | 683 | 689 | 694 | 700 | |
| 771 | 705 | 711 | 717 | 722 | 728 | 734 | 739 | 745 | 750 | 756 | |
| 772 | 762 | 767 | 773 | 779 | 784 | 790 | 795 | 801 | 807 | 812 | |
| 773 | 818 | 824 | 829 | 835 | 840 | 846 | 852 | 857 | 863 | 868 | |
| 774 | 874 | 880 | 885 | 891 | 897 | 902 | 908 | 913 | 919 | 925 | |
| 775 | 930 | 936 | 941 | 947 | 953 | 958 | 964 | 969 | 975 | 981 | |
| 776 | 986 | 992 | 997 | *003 | *009 | *014 | *020 | *025 | *031 | *037 | |
| 777 | 89 042 | 048 | 053 | 059 | 064 | 070 | 076 | 081 | 087 | 092 | |
| 778 | 098 | 104 | 109 | 115 | 120 | 126 | 131 | 137 | 143 | 148 | |
| 779 | 154 | 159 | 165 | 170 | 176 | 182 | 187 | 193 | 198 | 204 | |
| 780 | 209 | 215 | 221 | 226 | 232 | 237 | 243 | 248 | 254 | 260 | |
| 781 | 265 | 271 | 276 | 282 | 287 | 293 | 298 | 304 | 310 | 315 | |
| 782 | 321 | 326 | 332 | 337 | 343 | 348 | 354 | 360 | 365 | 371 | |
| 783 | 376 | 382 | 387 | 393 | 398 | 404 | 409 | 415 | 421 | 426 | |
| 784 | 432 | 437 | 443 | 448 | 454 | 459 | 465 | 470 | 476 | 481 | |
| 785 | 487 | 492 | 498 | 504 | 509 | 515 | 520 | 526 | 531 | 537 | |
| 786 | 542 | 548 | 553 | 559 | 564 | 570 | 575 | 581 | 586 | 592 | |
| 787 | 597 | 603 | 609 | 614 | 620 | 625 | 631 | 636 | 642 | 647 | |
| 788 | 653 | 658 | 664 | 669 | 675 | 680 | 686 | 691 | 697 | 702 | |
| 789 | 708 | 713 | 719 | 724 | 730 | 735 | 741 | 746 | 752 | 757 | |
| 790 | 763 | 768 | 774 | 779 | 785 | 790 | 796 | 801 | 807 | 812 | |
| 791 | 818 | 823 | 829 | 834 | 840 | 845 | 851 | 856 | 862 | 867 | |
| 792 | 873 | 878 | 883 | 889 | 894 | 900 | 905 | 911 | 916 | 922 | |
| 793 | 927 | 933 | 938 | 944 | 949 | 955 | 960 | 966 | 971 | 977 | |
| 794 | 982 | 988 | 993 | 998 | *004 | *009 | *015 | *020 | *026 | *031 | |
| 795 | 90 037 | 042 | 048 | 053 | 059 | 064 | 069 | 075 | 080 | 086 | |
| 796 | 091 | 097 | 102 | 108 | 113 | 119 | 124 | 129 | 135 | 140 | |
| 797 | 146 | 151 | 157 | 162 | 168 | 173 | 179 | 184 | 189 | 195 | |
| 798 | 200 | 206 | 211 | 217 | 222 | 227 | 233 | 238 | 244 | 249 | |
| 799 | 255 | 260 | 266 | 271 | 276 | 282 | 287 | 293 | 298 | 304 | |
| 800 | 309 | 314 | 320 | 325 | 331 | 336 | 342 | 347 | 352 | 358 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| | 6 | 5 |
|---|-----|-----|
| 1 | 0.6 | 0.5 |
| 2 | 1.2 | 1.0 |
| 3 | 1.8 | 1.5 |
| 4 | 2.4 | 2.0 |
| 5 | 3.0 | 2.5 |
| 6 | 3.6 | 3.0 |
| 7 | 4.2 | 3.5 |
| 8 | 4.8 | 4.0 |
| 9 | 5.4 | 4.5 |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|-----|-----|-----|-----|-----|-----|-----|------|------|------------|
| 800 | 90 309 | 314 | 320 | 325 | 331 | 336 | 342 | 347 | 352 | 358 | |
| 801 | 363 | 369 | 374 | 380 | 385 | 390 | 396 | 401 | 407 | 412 | |
| 802 | 417 | 423 | 428 | 434 | 439 | 445 | 450 | 455 | 461 | 466 | |
| 803 | 472 | 477 | 482 | 488 | 493 | 499 | 504 | 509 | 515 | 520 | |
| 804 | 526 | 531 | 536 | 542 | 547 | 553 | 558 | 563 | 569 | 574 | |
| 805 | 580 | 585 | 590 | 596 | 601 | 607 | 612 | 617 | 623 | 628 | |
| 806 | 634 | 639 | 644 | 650 | 655 | 660 | 666 | 671 | 677 | 682 | |
| 807 | 687 | 693 | 698 | 703 | 709 | 714 | 720 | 725 | 730 | 736 | |
| 808 | 741 | 747 | 752 | 757 | 763 | 768 | 773 | 779 | 784 | 789 | |
| 809 | 795 | 800 | 806 | 811 | 816 | 822 | 827 | 832 | 838 | 843 | |
| 810 | 849 | 854 | 859 | 865 | 870 | 875 | 881 | 886 | 891 | 897 | |
| 811 | 902 | 907 | 913 | 918 | 924 | 929 | 934 | 940 | 945 | 950 | |
| 812 | 956 | 961 | 966 | 972 | 977 | 982 | 988 | 993 | 998 | *001 | |
| 813 | 91 009 | 014 | 020 | 025 | 030 | 036 | 041 | 046 | 052 | 057 | |
| 814 | 062 | 068 | 073 | 078 | 084 | 089 | 094 | 100 | 105 | 110 | |
| 815 | 116 | 121 | 126 | 132 | 137 | 142 | 148 | 153 | 158 | 164 | |
| 816 | 169 | 174 | 180 | 185 | 190 | 196 | 201 | 206 | 212 | 217 | |
| 817 | 222 | 228 | 233 | 238 | 243 | 249 | 254 | 259 | 265 | 270 | |
| 818 | 275 | 281 | 286 | 291 | 297 | 302 | 307 | 312 | 318 | 323 | |
| 819 | 328 | 334 | 339 | 344 | 350 | 355 | 360 | 365 | 371 | 376 | |
| 820 | 381 | 387 | 392 | 397 | 403 | 408 | 413 | 418 | 424 | 429 | |
| 821 | 434 | 440 | 445 | 450 | 455 | 461 | 466 | 471 | 477 | 482 | |
| 822 | 487 | 492 | 498 | 503 | 508 | 514 | 519 | 524 | 529 | 535 | |
| 823 | 540 | 545 | 551 | 556 | 561 | 566 | 572 | 577 | 582 | 587 | |
| 824 | 593 | 598 | 603 | 609 | 614 | 619 | 624 | 630 | 635 | 640 | |
| 825 | 645 | 651 | 656 | 661 | 666 | 672 | 677 | 682 | 687 | 693 | |
| 826 | 698 | 703 | 709 | 714 | 719 | 724 | 730 | 735 | 740 | 745 | |
| 827 | 751 | 756 | 761 | 766 | 772 | 777 | 782 | 787 | 793 | 798 | |
| 828 | 803 | 808 | 814 | 819 | 824 | 829 | 834 | 840 | 845 | 850 | |
| 829 | 855 | 861 | 866 | 871 | 876 | 882 | 887 | 892 | 897 | 903 | |
| 830 | 908 | 913 | 918 | 924 | 929 | 934 | 939 | 944 | 950 | 955 | |
| 831 | 960 | 965 | 971 | 976 | 981 | 986 | 991 | 997 | *002 | *007 | |
| 832 | 92 012 | 018 | 023 | 028 | 033 | 038 | 044 | 049 | 054 | 059 | |
| 833 | 065 | 070 | 075 | 080 | 085 | 091 | 096 | 101 | 106 | 111 | |
| 834 | 117 | 122 | 127 | 132 | 137 | 143 | 148 | 153 | 158 | 163 | |
| 835 | 169 | 174 | 179 | 184 | 189 | 195 | 200 | 205 | 210 | 215 | |
| 836 | 221 | 226 | 231 | 236 | 241 | 247 | 252 | 257 | 262 | 267 | |
| 837 | 273 | 278 | 283 | 288 | 293 | 298 | 304 | 309 | 314 | 319 | |
| 838 | 324 | 330 | 335 | 340 | 345 | 350 | 355 | 361 | 366 | 371 | |
| 839 | 376 | 381 | 387 | 392 | 397 | 402 | 407 | 412 | 418 | 423 | |
| 840 | 428 | 433 | 438 | 443 | 449 | 454 | 459 | 464 | 469 | 474 | |
| 841 | 480 | 485 | 490 | 495 | 500 | 505 | 511 | 516 | 521 | 526 | |
| 842 | 531 | 536 | 542 | 547 | 552 | 557 | 562 | 567 | 572 | 578 | |
| 843 | 583 | 588 | 593 | 598 | 603 | 609 | 614 | 619 | 624 | 629 | |
| 844 | 634 | 639 | 645 | 650 | 655 | 660 | 665 | 670 | 675 | 681 | |
| 845 | 686 | 691 | 696 | 701 | 706 | 711 | 716 | 722 | 727 | 732 | |
| 846 | 737 | 742 | 747 | 752 | 758 | 763 | 768 | 773 | 778 | 783 | |
| 847 | 788 | 793 | 799 | 804 | 809 | 814 | 819 | 824 | 829 | 834 | |
| 848 | 840 | 845 | 850 | 855 | 860 | 865 | 870 | 875 | 881 | 886 | |
| 849 | 891 | 896 | 901 | 906 | 911 | 916 | 921 | 927 | 932 | 937 | |
| 850 | 942 | 947 | 952 | 957 | 962 | 967 | 973 | 978 | 983 | 988 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| | 6 | 5 |
|---|-----|-----|
| 1 | 0.6 | 0.5 |
| 2 | 1.2 | 1.0 |
| 3 | 1.8 | 1.5 |
| 4 | 2.4 | 2.0 |
| 5 | 3.0 | 2.5 |
| 6 | 3.6 | 3.0 |
| 7 | 4.2 | 3.5 |
| 8 | 4.8 | 4.0 |
| 9 | 5.4 | 4.5 |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|-----|------|------|------|------|------|------|------|------|------------|
| 850 | 92 942 | 947 | 952 | 957 | 962 | 967 | 973 | 978 | 983 | 988 | |
| 851 | 993 | 998 | *003 | *008 | *013 | *018 | *024 | *029 | *034 | *039 | |
| 852 | 93 044 | 049 | 054 | 059 | 064 | 069 | 075 | 080 | 085 | 090 | |
| 853 | 095 | 100 | 105 | 110 | 115 | 120 | 125 | 131 | 136 | 141 | |
| 854 | 146 | 151 | 156 | 161 | 166 | 171 | 176 | 181 | 186 | 192 | |
| 855 | 197 | 202 | 207 | 212 | 217 | 222 | 227 | 232 | 237 | 242 | |
| 856 | 247 | 252 | 258 | 263 | 268 | 273 | 278 | 283 | 288 | 293 | |
| 857 | 298 | 303 | 308 | 313 | 318 | 323 | 328 | 334 | 339 | 344 | |
| 858 | 349 | 354 | 359 | 364 | 369 | 374 | 379 | 384 | 389 | 394 | |
| 859 | 399 | 404 | 409 | 414 | 420 | 425 | 430 | 435 | 440 | 445 | |
| 860 | 450 | 455 | 460 | 465 | 470 | 475 | 480 | 485 | 490 | 495 | |
| 861 | 500 | 505 | 510 | 515 | 520 | 526 | 531 | 536 | 541 | 546 | |
| 862 | 551 | 556 | 561 | 566 | 571 | 576 | 581 | 586 | 591 | 596 | |
| 863 | 601 | 606 | 611 | 616 | 621 | 626 | 631 | 636 | 641 | 646 | |
| 864 | 651 | 656 | 661 | 666 | 671 | 676 | 682 | 687 | 692 | 697 | |
| 865 | 702 | 707 | 712 | 717 | 722 | 727 | 732 | 737 | 742 | 747 | |
| 866 | 752 | 757 | 762 | 767 | 772 | 777 | 782 | 787 | 792 | 797 | |
| 867 | 802 | 807 | 812 | 817 | 822 | 827 | 832 | 837 | 842 | 847 | |
| 868 | 852 | 857 | 862 | 867 | 872 | 877 | 882 | 887 | 892 | 897 | |
| 869 | 902 | 907 | 912 | 917 | 922 | 927 | 932 | 937 | 942 | 947 | |
| 870 | 952 | 957 | 962 | 967 | 972 | 977 | 982 | 987 | 992 | 997 | |
| 871 | 94 002 | 007 | 012 | 017 | 022 | 027 | 032 | 037 | 042 | 047 | |
| 872 | 052 | 057 | 062 | 067 | 072 | 077 | 082 | 086 | 091 | 096 | |
| 873 | 101 | 106 | 111 | 116 | 121 | 126 | 131 | 136 | 141 | 146 | |
| 874 | 151 | 156 | 161 | 166 | 171 | 176 | 181 | 186 | 191 | 196 | |
| 875 | 201 | 206 | 211 | 216 | 221 | 226 | 231 | 236 | 240 | 245 | |
| 876 | 250 | 255 | 260 | 265 | 270 | 275 | 280 | 285 | 290 | 295 | |
| 877 | 300 | 305 | 310 | 315 | 320 | 325 | 330 | 335 | 340 | 345 | |
| 878 | 349 | 354 | 359 | 364 | 369 | 374 | 379 | 384 | 389 | 394 | |
| 879 | 399 | 404 | 409 | 414 | 419 | 424 | 429 | 433 | 438 | 443 | |
| 880 | 448 | 453 | 458 | 463 | 468 | 473 | 478 | 483 | 488 | 493 | |
| 881 | 498 | 503 | 507 | 512 | 517 | 522 | 527 | 532 | 537 | 542 | |
| 882 | 547 | 552 | 557 | 562 | 567 | 571 | 576 | 581 | 586 | 591 | |
| 883 | 596 | 601 | 606 | 611 | 616 | 621 | 626 | 630 | 635 | 640 | |
| 884 | 645 | 650 | 655 | 660 | 665 | 670 | 675 | 680 | 685 | 689 | |
| 885 | 694 | 699 | 704 | 709 | 714 | 719 | 724 | 729 | 734 | 738 | |
| 886 | 743 | 748 | 753 | 758 | 763 | 768 | 773 | 778 | 783 | 787 | |
| 887 | 792 | 797 | 802 | 807 | 812 | 817 | 822 | 827 | 832 | 836 | |
| 888 | 841 | 846 | 851 | 856 | 861 | 866 | 871 | 876 | 880 | 885 | |
| 889 | 890 | 895 | 900 | 905 | 910 | 915 | 919 | 924 | 929 | 934 | |
| 890 | 939 | 944 | 949 | 954 | 959 | 963 | 968 | 973 | 978 | 983 | |
| 891 | 988 | 993 | 998 | *002 | *007 | *012 | *017 | *022 | *027 | *032 | |
| 892 | 95 036 | 041 | 046 | 051 | 056 | 061 | 066 | 071 | 075 | 080 | |
| 893 | 085 | 090 | 095 | 100 | 105 | 109 | 114 | 119 | 124 | 129 | |
| 894 | 134 | 139 | 143 | 148 | 153 | 158 | 163 | 168 | 173 | 177 | |
| 895 | 182 | 187 | 192 | 197 | 202 | 207 | 211 | 216 | 221 | 226 | |
| 896 | 231 | 236 | 240 | 245 | 250 | 255 | 260 | 265 | 270 | 274 | |
| 897 | 279 | 284 | 289 | 294 | 299 | 303 | 308 | 313 | 318 | 323 | |
| 898 | 328 | 332 | 337 | 342 | 347 | 352 | 357 | 361 | 366 | 371 | |
| 899 | 376 | 381 | 386 | 390 | 395 | 400 | 405 | 410 | 415 | 419 | |
| 900 | 424 | 429 | 434 | 439 | 444 | 448 | 453 | 458 | 463 | 468 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| | 6 | 5 |
|---|-----|-----|
| 1 | 0.6 | 0.5 |
| 2 | 1.2 | 1.0 |
| 3 | 1.8 | 1.5 |
| 4 | 2.4 | 2.0 |
| 5 | 3.0 | 2.5 |
| 6 | 3.6 | 3.0 |
| 7 | 4.2 | 3.5 |
| 8 | 4.8 | 4.0 |
| 9 | 5.4 | 4.5 |

| | 4 |
|---|-----|
| 1 | 0.4 |
| 2 | 0.8 |
| 3 | 1.2 |
| 4 | 1.6 |
| 5 | 2.0 |
| 6 | 2.4 |
| 7 | 2.8 |
| 8 | 3.2 |
| 9 | 3.6 |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|-----|--------|------|------|------|------|------|------|------|------|------|------------|
| 900 | 95 424 | 429 | 434 | 439 | 444 | 448 | 453 | 458 | 463 | 468 | |
| 901 | 472 | 477 | 482 | 487 | 492 | 497 | 501 | 506 | 511 | 516 | |
| 902 | 521 | 525 | 530 | 535 | 540 | 545 | 550 | 554 | 559 | 564 | |
| 903 | 569 | 574 | 578 | 583 | 588 | 593 | 598 | 602 | 607 | 612 | |
| 904 | 617 | 622 | 626 | 631 | 636 | 641 | 646 | 650 | 655 | 660 | |
| 905 | 665 | 670 | 674 | 679 | 684 | 689 | 694 | 698 | 703 | 708 | |
| 906 | 713 | 718 | 722 | 727 | 732 | 737 | 742 | 746 | 751 | 756 | |
| 907 | 761 | 766 | 770 | 775 | 780 | 785 | 789 | 794 | 799 | 804 | |
| 908 | 809 | 813 | 818 | 823 | 828 | 832 | 837 | 842 | 847 | 852 | |
| 909 | 856 | 861 | 866 | 871 | 875 | 880 | 885 | 890 | 895 | 899 | |
| 910 | 904 | 909 | 914 | 918 | 923 | 928 | 933 | 938 | 942 | 947 | |
| 911 | 952 | 957 | 961 | 966 | 971 | 976 | 980 | 985 | 990 | 995 | |
| 912 | 999 | *004 | *009 | *014 | *019 | *023 | *028 | *033 | *038 | *042 | |
| 913 | 96 047 | 052 | 057 | 061 | 066 | 071 | 076 | 080 | 085 | 090 | |
| 914 | 095 | 099 | 104 | 109 | 114 | 118 | 123 | 128 | 133 | 137 | |
| 915 | 142 | 147 | 152 | 156 | 161 | 166 | 171 | 175 | 180 | 185 | |
| 916 | 190 | 194 | 199 | 204 | 209 | 213 | 218 | 223 | 227 | 232 | |
| 917 | 237 | 242 | 246 | 251 | 256 | 261 | 265 | 270 | 275 | 280 | |
| 918 | 284 | 289 | 294 | 298 | 303 | 308 | 313 | 317 | 322 | 327 | |
| 919 | 332 | 336 | 341 | 346 | 350 | 355 | 360 | 365 | 369 | 374 | |
| 920 | 379 | 384 | 388 | 393 | 398 | 402 | 407 | 412 | 417 | 421 | |
| 921 | 426 | 431 | 435 | 440 | 445 | 450 | 454 | 459 | 464 | 468 | |
| 922 | 473 | 478 | 483 | 487 | 492 | 497 | 501 | 506 | 511 | 515 | |
| 923 | 520 | 525 | 530 | 534 | 539 | 544 | 548 | 553 | 558 | 562 | |
| 924 | 567 | 572 | 577 | 581 | 586 | 591 | 595 | 600 | 605 | 609 | |
| 925 | 614 | 619 | 624 | 628 | 633 | 638 | 642 | 647 | 652 | 656 | |
| 926 | 661 | 666 | 670 | 675 | 680 | 685 | 689 | 694 | 699 | 703 | |
| 927 | 708 | 713 | 717 | 722 | 727 | 731 | 736 | 741 | 745 | 750 | |
| 928 | 755 | 759 | 764 | 769 | 774 | 778 | 783 | 788 | 792 | 797 | |
| 929 | 802 | 806 | 811 | 816 | 820 | 825 | 830 | 834 | 839 | 844 | |
| 930 | 848 | 853 | 858 | 862 | 867 | 872 | 876 | 881 | 886 | 890 | |
| 931 | 895 | 900 | 904 | 909 | 911 | 918 | 923 | 928 | 932 | 937 | |
| 932 | 942 | 946 | 951 | 956 | 960 | 965 | 970 | 974 | 979 | 984 | |
| 933 | 988 | 993 | 997 | *002 | *007 | *011 | *016 | *021 | *025 | *030 | |
| 934 | 97 035 | 039 | 044 | 049 | 053 | 058 | 063 | 067 | 072 | 077 | |
| 935 | 081 | 086 | 090 | 095 | 100 | 104 | 109 | 114 | 118 | 123 | |
| 936 | 128 | 132 | 137 | 142 | 146 | 151 | 155 | 160 | 165 | 169 | |
| 937 | 174 | 179 | 183 | 188 | 192 | 197 | 202 | 206 | 211 | 216 | |
| 938 | 220 | 225 | 230 | 234 | 239 | 243 | 248 | 253 | 257 | 262 | |
| 939 | 267 | 271 | 276 | 280 | 285 | 290 | 294 | 299 | 304 | 308 | |
| 940 | 313 | 317 | 322 | 327 | 331 | 336 | 340 | 345 | 350 | 354 | |
| 941 | 359 | 364 | 368 | 373 | 377 | 382 | 387 | 391 | 396 | 400 | |
| 942 | 405 | 410 | 414 | 419 | 424 | 428 | 433 | 437 | 442 | 447 | |
| 943 | 451 | 456 | 460 | 465 | 470 | 474 | 479 | 483 | 488 | 493 | |
| 944 | 497 | 502 | 506 | 511 | 516 | 520 | 525 | 529 | 534 | 539 | |
| 945 | 543 | 548 | 552 | 557 | 562 | 566 | 571 | 575 | 580 | 585 | |
| 946 | 589 | 594 | 598 | 603 | 607 | 612 | 617 | 621 | 626 | 630 | |
| 947 | 635 | 640 | 644 | 649 | 653 | 658 | 663 | 667 | 672 | 676 | |
| 948 | 681 | 685 | 690 | 695 | 699 | 704 | 708 | 713 | 717 | 722 | |
| 949 | 727 | 731 | 736 | 740 | 745 | 749 | 754 | 759 | 763 | 768 | |
| 950 | 772 | 777 | 782 | 786 | 791 | 795 | 800 | 804 | 809 | 813 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

| | 5 | 4 |
|---|-----|-----|
| 1 | 0.5 | 0.4 |
| 2 | 1.0 | 0.8 |
| 3 | 1.5 | 1.2 |
| 4 | 2.0 | 1.6 |
| 5 | 2.5 | 2.0 |
| 6 | 3.0 | 2.4 |
| 7 | 3.5 | 2.8 |
| 8 | 4.0 | 3.2 |
| 9 | 4.5 | 3.6 |

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |
|------|--------|-----|-----|------|------|------|------|------|------|------|------------|
| 950 | 97 772 | 777 | 782 | 786 | 791 | 795 | 800 | 804 | 809 | 813 | |
| 951 | 818 | 823 | 827 | 832 | 836 | 841 | 845 | 850 | 855 | 859 | |
| 952 | 864 | 868 | 873 | 877 | 882 | 886 | 891 | 896 | 900 | 905 | |
| 953 | 909 | 914 | 918 | 923 | 928 | 932 | 937 | 941 | 946 | 950 | |
| 954 | 955 | 959 | 964 | 968 | 973 | 978 | 982 | 987 | 991 | 996 | |
| 955 | 98 000 | 005 | 009 | 014 | 019 | 023 | 028 | 032 | 037 | 041 | |
| 956 | 046 | 050 | 055 | 059 | 064 | 068 | 073 | 078 | 082 | 087 | |
| 957 | 091 | 096 | 100 | 105 | 109 | 114 | 118 | 123 | 127 | 132 | |
| 958 | 137 | 141 | 146 | 150 | 155 | 159 | 164 | 168 | 173 | 177 | |
| 959 | 182 | 186 | 191 | 195 | 200 | 204 | 209 | 214 | 218 | 223 | |
| 960 | 227 | 232 | 236 | 241 | 245 | 250 | 254 | 259 | 263 | 268 | |
| 961 | 272 | 277 | 281 | 286 | 290 | 295 | 299 | 304 | 308 | 313 | |
| 962 | 318 | 322 | 327 | 331 | 336 | 340 | 345 | 349 | 354 | 358 | |
| 963 | 363 | 367 | 372 | 376 | 381 | 385 | 390 | 394 | 399 | 403 | |
| 964 | 408 | 412 | 417 | 421 | 426 | 430 | 435 | 439 | 444 | 448 | |
| 965 | 453 | 457 | 462 | 466 | 471 | 475 | 480 | 484 | 489 | 493 | |
| 966 | 498 | 502 | 507 | 511 | 516 | 520 | 525 | 529 | 534 | 538 | |
| 967 | 543 | 547 | 552 | 556 | 561 | 565 | 570 | 574 | 579 | 583 | |
| 968 | 588 | 592 | 597 | 601 | 605 | 610 | 614 | 619 | 623 | 628 | |
| 969 | 632 | 637 | 641 | 646 | 650 | 655 | 659 | 664 | 668 | 673 | |
| 970 | 677 | 682 | 686 | 691 | 695 | 700 | 704 | 709 | 713 | 717 | |
| 971 | 722 | 726 | 731 | 735 | 740 | 744 | 749 | 753 | 758 | 762 | 5 4 |
| 972 | 767 | 771 | 776 | 780 | 784 | 789 | 793 | 798 | 802 | 807 | 1 0.5 0.4 |
| 973 | 811 | 816 | 820 | 825 | 829 | 834 | 838 | 843 | 847 | 851 | 2 1.0 0.8 |
| 974 | 856 | 860 | 865 | 869 | 874 | 878 | 883 | 887 | 892 | 896 | 3 1.5 1.2 |
| 975 | 900 | 905 | 909 | 914 | 918 | 923 | 927 | 932 | 936 | 941 | 4 2.0 1.6 |
| 976 | 945 | 949 | 954 | 958 | 963 | 967 | 972 | 976 | 981 | 985 | 5 2.5 2.0 |
| 977 | 989 | 994 | 998 | *003 | *007 | *012 | *016 | *021 | *025 | *029 | 6 3.0 2.4 |
| 978 | 99 034 | 038 | 043 | 047 | 052 | 056 | 061 | 065 | 069 | 074 | 7 3.5 2.8 |
| 979 | 078 | 083 | 087 | 092 | 096 | 100 | 105 | 109 | 114 | 118 | 8 4.0 3.2 |
| 980 | 123 | 127 | 131 | 136 | 140 | 145 | 149 | 154 | 158 | 162 | 9 4.5 3.6 |
| 981 | 167 | 171 | 176 | 180 | 185 | 189 | 193 | 198 | 202 | 207 | |
| 982 | 211 | 216 | 220 | 224 | 229 | 233 | 238 | 242 | 247 | 251 | |
| 983 | 255 | 260 | 264 | 269 | 273 | 277 | 282 | 286 | 291 | 295 | |
| 984 | 300 | 304 | 308 | 313 | 317 | 322 | 326 | 330 | 335 | 339 | |
| 985 | 344 | 348 | 352 | 357 | 361 | 366 | 370 | 374 | 379 | 383 | |
| 986 | 388 | 392 | 396 | 401 | 405 | 410 | 414 | 419 | 423 | 427 | |
| 987 | 432 | 436 | 441 | 445 | 449 | 454 | 458 | 463 | 467 | 471 | |
| 988 | 476 | 480 | 484 | 489 | 493 | 498 | 502 | 506 | 511 | 515 | |
| 989 | 520 | 524 | 528 | 533 | 537 | 542 | 546 | 550 | 555 | 559 | |
| 990 | 564 | 568 | 572 | 577 | 581 | 585 | 590 | 594 | 599 | 603 | |
| 991 | 607 | 612 | 616 | 621 | 625 | 629 | 634 | 638 | 642 | 647 | |
| 992 | 651 | 656 | 660 | 664 | 669 | 673 | 677 | 682 | 686 | 691 | |
| 993 | 695 | 699 | 704 | 708 | 712 | 717 | 721 | 726 | 730 | 734 | |
| 994 | 739 | 743 | 747 | 752 | 756 | 760 | 765 | 769 | 774 | 778 | |
| 995 | 782 | 787 | 791 | 795 | 800 | 804 | 808 | 813 | 817 | 822 | |
| 996 | 826 | 830 | 835 | 839 | 843 | 848 | 852 | 856 | 861 | 865 | |
| 997 | 870 | 874 | 878 | 883 | 887 | 891 | 896 | 900 | 904 | 909 | |
| 998 | 913 | 917 | 922 | 926 | 930 | 935 | 939 | 944 | 948 | 952 | |
| 999 | 957 | 961 | 965 | 970 | 974 | 978 | 983 | 987 | 991 | 996 | |
| 1000 | 00 000 | 004 | 009 | 013 | 017 | 022 | 026 | 030 | 035 | 039 | |
| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts. |

544 Table Ia — Condensed Logarithms and Antilogarithms [Ia]

CONDENSED LOGARITHMS TO FIFTEEN DECIMAL PLACES

[The first digits of n are given in the first row at the top, the last digit of n in the left-hand column. The first column of logarithms are those of 1, 2, 3, ..., 9. The remaining columns give $\log(1+x)$ where $x = (0.1)^k$ times 1, 2, ..., 9.]

| Last Digit } | First Digit of $n \rightarrow$ | 1 | 10 | 100 |
|-----------------|-----------------------------------|--|-------------------|-----------------|
| | Log n | First Digits of log $n \rightarrow$ | 0 | .00 |
| 1 | 00000 00000 00000 | 01139 268 51 5822 | 0132 137 57 82643 | 013 40774 79319 |
| 2 | 30102 999 56 6 981 | 07918 12160 47625 | 0860 01717 61918 | 086 77215 31227 |
| 3 | 47712 12 17 19662 | 11394 33 23 06857 | 1283 72217 05172 | 130 09330 20118 |
| 4 | 60205 9991 5 27962 | 14612 80356 78258 | 1703 33592 98780 | 173 37128 09001 |
| 5 | 69897 00045 36019 | 17609 12590 55681 | 2118 92990 69938 | 216 60617 56508 |
| 6 | 77815 12503 83614 | 20411 99826 53925 | 2530 58652 61770 | 259 79807 19909 |
| 7 | 84509 80400 14257 | 23041 89213 78274 | 2958 37776 85210 | 302 94705 53618 |
| 8 | 90308 99869 91944 | 25527 20051 03506 | 3342 37554 86950 | 346 05321 09506 |
| 9 | 95424 2094 39325 | 27875 36009 52829 | 3742 64979 40624 | 389 11662 36911 |

(cont next p.)

| | 1 000 | 1 0000 | 1 00000 | 1 000000 | 1 0000000 | 1 00000000 |
|---|----------------|---------------|-------------|------------|-----------|------------|
| | 000 | 0000 | 00000 | 000000 | 0000000 | 00000000 |
| 1 | 04 51272 76865 | 0 15429 23101 | 01312 91265 | 0154 29146 | 013 42945 | 01 34294 |
| 2 | 08 68502 11619 | 0 86858 02780 | 08685 88015 | 0868 58888 | 086 85890 | 08 68589 |
| 3 | 15 02688 05227 | 1 30286 39028 | 13028 81491 | 1302 88325 | 130 28834 | 13 02883 |
| 4 | 17 36830 58465 | 1 75714 31850 | 17571 74153 | 1737 17758 | 173 71779 | 17 37178 |
| 5 | 21 70929 72250 | 2 17111 81215 | 21711 66581 | 2171 47187 | 217 11724 | 21 71472 |
| 6 | 26 04985 47390 | 2 60568 87215 | 26057 50071 | 2605 76611 | 260 57668 | 26 05767 |
| 7 | 30 38997 84812 | 3 09955 49761 | 30400 5075 | 3040 06031 | 304 00613 | 30 40061 |
| 8 | 34 72066 85561 | 3 47421 68881 | 34713 41158 | 3474 35147 | 347 43557 | 34 74356 |
| 9 | 39 06512 49910 | 3 90847 44554 | 39086 52745 | 3908 64555 | 390 86502 | 39 08550 |

[For $x < 0.0000001$ $\log(1+x) = x/M$ to within 3 in the 17th place where $M = 0.43429448$. Hence the last column gives multiples of M except for the decimal place. All the columns that would follow have the same significant digits displaced each time one place.]

CONDENSED ANTILOGARITHMS TO TEN DECIMAL PLACES

[The first digits of n are given in the first row at the top, $n = (0.1)^k$, $k = 1, 2, 3, \dots, 9$ are given in the left-hand column. The first digits in 10^x are given in the second row at the top.]

| x | $n = 0.1x$ | 0 01x | 0 001x | 0 0001x | (0.1) ² x | (0.1) ³ x | (0.1) ⁴ x |
|-----|-----------------|-------------|------------|-----------|----------------------|----------------------|----------------------|
| | 10 ^x | 1 | 10 | 1 00 | 1 000 | 1 0000 | 1 00000 |
| 1 | 1 25892 54118 | 02 429 2992 | 0230 52 81 | 023 02850 | 02 30261 | 0 23026 | 02303 |
| 2 | 1 58489 3132 | 04712 85481 | 0461 57903 | 046 06231 | 04 60528 | 0 46052 | 04605 |
| 3 | 1 99526 23150 | 07151 93022 | 0693 16689 | 069 10142 | 06 90799 | 0 69078 | 06908 |
| 4 | 2 51188 64315 | 09647 81961 | 0925 28861 | 092 14583 | 09 21076 | 0 92104 | 09210 |
| 5 | 3 16227 76602 | 12201 84433 | 1157 91543 | 115 19555 | 11 51359 | 1 15130 | 11513 |
| 6 | 3 98107 17055 | 14851 36215 | 1391 13857 | 138 25058 | 13 81646 | 1 38156 | 13816 |
| 7 | 5 1187 3363 | 17189 75549 | 1624 86929 | 161 31092 | 16 11939 | 1 61182 | 16118 |
| 8 | 6 0957 34448 | 20226 44346 | 1859 13881 | 184 37627 | 18 42238 | 1 84209 | 18421 |
| 9 | 7 94328 23472 | 23026 87708 | 2093 94837 | 207 44753 | 20 72541 | 2 07235 | 20723 |

[For $n < 0.000001$ $10^{-n} = 1 + n(1/M)$ to within 3 in the 12th decimal place where $(1/M) = 2.302585$. Hence the last column gives multiples of $(1/M)$ except for the decimal place. All the columns that would follow contain the same significant digits displaced one place for each new column.]

LOGARITHMS OF IMPORTANT CONSTANTS

| n = NUMBER | VALUE OF n | $\text{LOG}_{10} n$ |
|---------------------------------------|--|---------------------|
| π | 3.14159265 | 0.49714987 |
| $1 \div \pi$ | 0.31830989 | 9.50285013 |
| π^2 | 9.86960440 | 0.99429975 |
| $\sqrt{\pi}$ | 1.77245385 | 0.24857494 |
| e = Napierian Base | 2.71828183 | 0.43429448 |
| $M = \log_{10} e$ | 0.43429448 | 9.63778431 |
| $1 \div M = \log_e 10$ | 2.30258509 | 0.36221569 |
| $180 \div \pi$ = degrees in 1 radian | 57.2957795 | 1.75812263 |
| $\pi \div 180$ = radians in 1° | 0.01745329 | 8.24187737 |
| $\pi \div 10800$ = radians in $1'$ | 0.000290882 | 6.46372612 |
| $\pi \div 648000$ = radians in $1''$ | 0.000004848136811095 | 4.68557487 |
| $\sin 1''$ | 0.000004848136811076 | 4.68557487 |
| $\tan 1''$ | 0.000004848136811133 | 4.68557487 |
| centimeters in 1 ft. | 30.480 | 1.4840158 |
| feet in 1 cm. | 0.032808 | 8.5159842 |
| inches in 1 m. | 39.37 (exact legal value) | 1.5951654 |
| pounds in 1 kg. | 2.20462 | 0.3433340 |
| kilograms in 1 lb. | 0.453593 | 9.6566660 |
| g (average value) | 32.16 ft./sec./sec. = 981 cm./sec./sec. | 1.5073 2.9916690 |
| weight of 1 cu. ft. of water | 62.425 lb. (max. density) | 1.7953586 |
| weight of 1 cu. ft. of air | 0.0807 lb. (at 32°F.) | 8.907 |
| cu. in. in 1 (U. S.) gallon | 231 (exact legal value) | 2.3636120 |
| ft. lb. per sec. in 1 H. P. | 550 (exact legal value) | 2.7403627 |
| kg. m. per sec. in 1 H. P. | 76.0404 | 1.10445 |
| watts in 1 H. P. | 745.957 | 2.8727135 |

SEVERAL NUMBERS VERY ACCURATELY

| | | | | | | |
|-----------------|-----------|-------|-------|-------|-------|-------|
| π | = 3.14159 | 26535 | 89793 | 23846 | 26433 | 83280 |
| e | = 2.71828 | 18284 | 59045 | 23536 | 02871 | 71353 |
| M | = 0.43429 | 44819 | 03251 | 82765 | 11289 | 18917 |
| $1 \div M$ | = 2.30258 | 50929 | 94045 | 68401 | 79914 | 51684 |
| $\log_{10} \pi$ | = 0.49714 | 98726 | 94133 | 85435 | 12682 | 88291 |
| $\log_{10} M$ | = 9.63778 | 43113 | 00536 | 78912 | | |

CERTAIN CONVENIENT VALUES FOR $n = 1$ TO $n = 10$

| n | $1/n$ | $\sqrt[n]{n}$ | $\sqrt[n]{1}$ | $n!$ | $1/n!$ | $\text{LOG}_{10} n$ |
|-----|----------|---------------|---------------|---------|-----------|---------------------|
| 1 | 1.000000 | 1.00000 | 1.00000 | 1 | 1.0000000 | 0.000000000 |
| 2 | 0.500000 | 1.41421 | 1.25992 | 2 | 0.5000000 | 0.301029996 |
| 3 | 0.333333 | 1.73205 | 1.44225 | 6 | 0.1666667 | 0.477121255 |
| 4 | 0.250000 | 2.00000 | 1.58740 | 24 | 0.0416667 | 0.602059991 |
| 5 | 0.200000 | 2.23607 | 1.70998 | 120 | 0.0083333 | 0.698970004 |
| 6 | 0.166667 | 2.44949 | 1.81712 | 720 | 0.0013889 | 0.778151250 |
| 7 | 0.142857 | 2.64575 | 1.91293 | 5040 | 0.0001984 | 0.845098040 |
| 8 | 0.125000 | 2.82843 | 2.00000 | 40320 | 0.0000248 | 0.903089987 |
| 9 | 0.111111 | 3.00000 | 2.08008 | 362880 | 0.0000028 | 0.954242509 |
| 10 | 0.100000 | 3.16228 | 2.15443 | 3628800 | 0.0000003 | 1.000000000 |

TABLE II

ACTUAL VALUES

OF THE

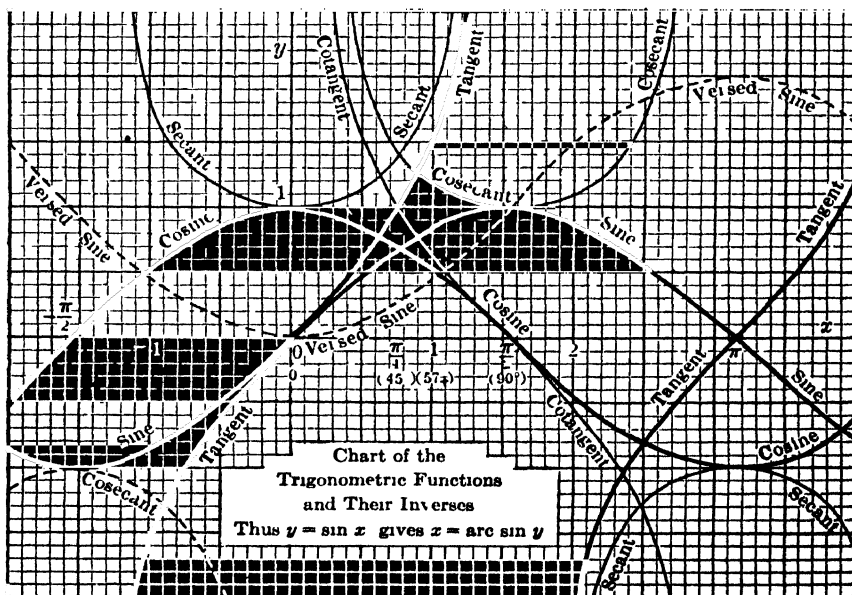
TRIGONOMETRIC FUNCTIONS

FROM

0° TO 90° AT INTERVALS OF ONE MINUTE

TO

FIVE DECIMAL PLACES



| ' | Sin | Tan | Ctn | Cos | ' | Sin | Tan | Ctn | Cos | ' |
|----|--------|--------|--------|--------|----|--------|--------|--------|--------|--------|
| 0 | .00000 | .00000 | — | 1.0000 | 60 | .01745 | .01746 | 57.290 | .99985 | 60 |
| 1 | 029 | 029 | 3437.7 | 000 59 | 1 | 774 | 775 | 56.351 | 984 59 | 1 |
| 2 | 058 | 058 | 1718.9 | 000 58 | 2 | 803 | 804 | 55.442 | 984 58 | 2 |
| 3 | 087 | 087 | 1145.9 | 000 57 | 3 | 832 | 833 | 54.561 | 983 57 | 3 |
| 4 | 116 | 116 | 859.44 | 000 56 | 4 | 862 | 862 | 53.709 | 983 56 | 4 |
| 5 | .00145 | .00145 | 687.55 | 1.0000 | 55 | .01891 | .01891 | 52.882 | .99982 | 55 |
| 6 | 175 | 175 | 572.96 | 000 54 | 6 | 920 | 920 | 52.081 | 982 54 | 6 |
| 7 | 204 | 204 | 491.11 | 000 53 | 7 | 949 | 949 | 51.303 | 981 53 | 7 |
| 8 | 233 | 233 | 429.72 | 000 52 | 8 | .01978 | .01978 | 50.549 | 980 52 | 8 |
| 9 | 262 | 262 | 381.97 | 000 51 | 9 | .02007 | .02007 | 49.816 | 980 51 | 9 |
| 10 | .00291 | .00291 | 343.77 | 1.0000 | 50 | .02036 | .02036 | 49.104 | .99979 | 50 |
| 11 | 320 | 320 | 312.52 | .99999 | 49 | 11 | 065 | 066 | 48.412 | 979 49 |
| 12 | 349 | 349 | 286.48 | 999 48 | 12 | 094 | 095 | 47.740 | 978 48 | 12 |
| 13 | 378 | 378 | 264.44 | 999 47 | 13 | 123 | 124 | 47.085 | 977 47 | 13 |
| 14 | 407 | 407 | 245.55 | 999 46 | 14 | 152 | 153 | 46.449 | 977 46 | 14 |
| 15 | .00436 | .00436 | 229.18 | .99999 | 45 | .02181 | .02182 | 45.829 | .99976 | 45 |
| 16 | 465 | 465 | 214.86 | 999 44 | 16 | 211 | 211 | 45.226 | 976 44 | 16 |
| 17 | 495 | 495 | 202.22 | 999 43 | 17 | 240 | 240 | 44.639 | 975 43 | 17 |
| 18 | 524 | 524 | 190.98 | 999 42 | 18 | 269 | 269 | 44.066 | 974 42 | 18 |
| 19 | 553 | 553 | 180.93 | 998 41 | 19 | 298 | 298 | 43.508 | 974 41 | 19 |
| 20 | .00582 | .00582 | 171.89 | .99998 | 40 | .02327 | .02328 | 42.964 | .99973 | 40 |
| 21 | 611 | 611 | 163.70 | 998 39 | 21 | 356 | 357 | 42.433 | 972 39 | 21 |
| 22 | 640 | 640 | 156.26 | 998 38 | 22 | 385 | 386 | 41.916 | 972 38 | 22 |
| 23 | 669 | 669 | 149.47 | 998 37 | 23 | 414 | 415 | 41.411 | 971 37 | 23 |
| 24 | 698 | 698 | 143.24 | 998 36 | 24 | 443 | 444 | 40.917 | 970 36 | 24 |
| 25 | .00727 | .00727 | 137.51 | .99997 | 35 | .02472 | .02473 | 40.436 | .99969 | 35 |
| 26 | 756 | 756 | 132.22 | 997 34 | 26 | 501 | 502 | 39.965 | 969 34 | 26 |
| 27 | 785 | 785 | 127.32 | 997 33 | 27 | 530 | 531 | 39.506 | 968 33 | 27 |
| 28 | 814 | 815 | 122.77 | 997 32 | 28 | 560 | 560 | 39.057 | 967 32 | 28 |
| 29 | 844 | 844 | 118.54 | 996 31 | 29 | 589 | 589 | 38.618 | 966 31 | 29 |
| 30 | .00873 | .00873 | 114.59 | .99996 | 30 | .02618 | .02619 | 38.188 | .99966 | 30 |
| 31 | 902 | 902 | 110.89 | 996 29 | 31 | 617 | 618 | 37.769 | 965 29 | 31 |
| 32 | 931 | 931 | 107.43 | 996 28 | 32 | 676 | 677 | 37.358 | 964 28 | 32 |
| 33 | 960 | 960 | 104.17 | 995 27 | 33 | 705 | 706 | 36.956 | 963 27 | 33 |
| 34 | .00989 | .00989 | 101.11 | 995 26 | 34 | 734 | 735 | 36.563 | 963 26 | 34 |
| 35 | .01018 | .01018 | 98.218 | .99995 | 25 | .02763 | .02764 | 36.178 | .99962 | 25 |
| 36 | 047 | 047 | 95.489 | 995 24 | 36 | 792 | 793 | 35.801 | 961 24 | 36 |
| 37 | 076 | 076 | 92.908 | 994 23 | 37 | 821 | 822 | 35.431 | 960 23 | 37 |
| 38 | 105 | 105 | 90.463 | 994 22 | 38 | 850 | 851 | 35.070 | 959 22 | 38 |
| 39 | 134 | 135 | 88.144 | 994 21 | 39 | 879 | 881 | 34.715 | 959 21 | 39 |
| 40 | .01164 | .01164 | 85.940 | .99993 | 20 | .02908 | .02910 | 34.368 | .99958 | 20 |
| 41 | 193 | 193 | 83.844 | 993 19 | 41 | 938 | 939 | 34.027 | 957 19 | 41 |
| 42 | 222 | 222 | 81.847 | 993 18 | 42 | 967 | 968 | 33.694 | 956 18 | 42 |
| 43 | 251 | 251 | 79.943 | 992 17 | 43 | .02996 | .02997 | 33.366 | 955 17 | 43 |
| 44 | 280 | 280 | 78.126 | 992 16 | 44 | .03025 | .03026 | 33.045 | 954 16 | 44 |
| 45 | .01309 | .01309 | 76.390 | .99991 | 15 | .03054 | .03055 | 32.730 | .99953 | 15 |
| 46 | 338 | 338 | 74.729 | 991 14 | 46 | 083 | 084 | 32.421 | 952 14 | 46 |
| 47 | 367 | 367 | 73.139 | 991 13 | 47 | 112 | 114 | 32.118 | 952 13 | 47 |
| 48 | 396 | 396 | 71.615 | 990 12 | 48 | 141 | 143 | 31.821 | 951 12 | 48 |
| 49 | 425 | 425 | 70.153 | 990 11 | 49 | 170 | 172 | 31.528 | 950 11 | 49 |
| 50 | .01454 | .01455 | 68.750 | .99989 | 10 | .03199 | .03201 | 31.242 | .99949 | 10 |
| 51 | 483 | 484 | 67.402 | 989 9 | 51 | 228 | 230 | 30.960 | 948 9 | 51 |
| 52 | 513 | 513 | 66.105 | 989 8 | 52 | 257 | 259 | 30.683 | 947 8 | 52 |
| 53 | 542 | 542 | 64.858 | 988 7 | 53 | 286 | 288 | 30.412 | 946 7 | 53 |
| 54 | 571 | 571 | 63.657 | 988 6 | 54 | 316 | 317 | 30.145 | 945 6 | 54 |
| 55 | .01600 | .01600 | 62.499 | .99987 | 5 | .03345 | .03346 | 29.882 | .99944 | 5 |
| 56 | 629 | 629 | 61.383 | 987 4 | 56 | 374 | 376 | 29.624 | 943 4 | 56 |
| 57 | 658 | 658 | 60.306 | 986 3 | 57 | 403 | 405 | 29.371 | 942 3 | 57 |
| 58 | 687 | 687 | 59.266 | 986 2 | 58 | 432 | 434 | 29.122 | 941 2 | 58 |
| 59 | 716 | 716 | 58.261 | 985 1 | 59 | 461 | 463 | 28.877 | 940 1 | 59 |
| 60 | .01745 | .01746 | 57.290 | .99985 | 0 | .03490 | .03492 | 28.636 | .99939 | 0 |
| | Cos | Ctn | Tan | Sin | | Cos | Ctn | Tan | Sin | |

89°

88°

| ' | Sin | Tan | Ctn | Cos | ' | Sin | Tan | Ctn | Cos | ' | |
|----|--------|--------|--------|--------|----|-----|--------|--------|--------|--------|----|
| 0 | .03490 | .03492 | 28.636 | .99939 | 60 | 0 | .05234 | .05241 | 19.081 | .99863 | 60 |
| 1 | 519 | 521 | .399 | 938 | 59 | 1 | 263 | 270 | 18.976 | 861 | 59 |
| 2 | 548 | 550 | 28.166 | 937 | 58 | 2 | 292 | 299 | .871 | 860 | 58 |
| 3 | 577 | 579 | 27.937 | 936 | 57 | 3 | 321 | 328 | .768 | 858 | 57 |
| 4 | 606 | 609 | .712 | 935 | 56 | 4 | 350 | 357 | .666 | 857 | 56 |
| 5 | .03635 | .03638 | 27.490 | .99934 | 55 | 5 | .05379 | .05387 | 18.564 | .99855 | 55 |
| 6 | 661 | 667 | .271 | 933 | 54 | 6 | 408 | 416 | .464 | 851 | 54 |
| 7 | 693 | 696 | 27.057 | 932 | 53 | 7 | 437 | 445 | .366 | 852 | 53 |
| 8 | 723 | 725 | 26.845 | 931 | 52 | 8 | 466 | 474 | .268 | 851 | 52 |
| 9 | 752 | 751 | .637 | 930 | 51 | 9 | 495 | 503 | .171 | 849 | 51 |
| 10 | .03781 | .03783 | 26.432 | .99929 | 50 | 10 | .05524 | .05533 | 18.075 | .99847 | 50 |
| 11 | 810 | 812 | .230 | 927 | 49 | 11 | 553 | 562 | 17.980 | 846 | 49 |
| 12 | 839 | 842 | 26.031 | 926 | 48 | 12 | 582 | 591 | .886 | 844 | 48 |
| 13 | 868 | 871 | 25.835 | 925 | 47 | 13 | 611 | 620 | .793 | 842 | 47 |
| 14 | 897 | 900 | .612 | 924 | 46 | 14 | 640 | 649 | .702 | 841 | 46 |
| 15 | .03926 | .03929 | 25.452 | .99923 | 45 | 15 | .05669 | .05678 | 17.611 | .99839 | 45 |
| 16 | 955 | 958 | .261 | 922 | 44 | 16 | 698 | 708 | .521 | 838 | 44 |
| 17 | .03984 | .03987 | 25.080 | 921 | 43 | 17 | 727 | 737 | .431 | 836 | 43 |
| 18 | .04013 | .04016 | 24.898 | 919 | 42 | 18 | 756 | 766 | .343 | 834 | 42 |
| 19 | 012 | 016 | .719 | 918 | 41 | 19 | 785 | 795 | .256 | 833 | 41 |
| 20 | .04071 | .04075 | 24.542 | .99917 | 40 | 20 | .05811 | .05824 | 17.169 | .99831 | 40 |
| 21 | 100 | 104 | .368 | 916 | 39 | 21 | 811 | 824 | 17.081 | 829 | 39 |
| 22 | 129 | 133 | .196 | 915 | 38 | 22 | 843 | 853 | 16.999 | 827 | 38 |
| 23 | 159 | 162 | 24.026 | 913 | 37 | 23 | 902 | 912 | .915 | 826 | 37 |
| 24 | 188 | 191 | 23.859 | 912 | 36 | 24 | 931 | 941 | .832 | 824 | 36 |
| 25 | .04217 | .04220 | 23.695 | .99911 | 35 | 25 | .05960 | .05970 | 16.750 | .99822 | 35 |
| 26 | 246 | 250 | .532 | 910 | 34 | 26 | .05989 | .05999 | .668 | 821 | 34 |
| 27 | 275 | 279 | .372 | 909 | 33 | 27 | .06018 | .06029 | .587 | 819 | 33 |
| 28 | 304 | 308 | .211 | 907 | 32 | 28 | 017 | 058 | .507 | 817 | 32 |
| 29 | 333 | 337 | 23.058 | 906 | 31 | 29 | 076 | 087 | .428 | 815 | 31 |
| 30 | .04362 | .04366 | 22.804 | .99905 | 30 | 30 | .06105 | .06116 | 16.350 | .99813 | 30 |
| 31 | 391 | 395 | .752 | 904 | 29 | 31 | 131 | 145 | .272 | 812 | 29 |
| 32 | 420 | 424 | .602 | 902 | 28 | 32 | 163 | 175 | .195 | 810 | 28 |
| 33 | 449 | 454 | .454 | 901 | 27 | 33 | 192 | 204 | .119 | 808 | 27 |
| 34 | 478 | 483 | .308 | 900 | 26 | 34 | 221 | 233 | 16.043 | 806 | 26 |
| 35 | .04507 | .04512 | 22.164 | .99898 | 25 | 35 | .06250 | .06262 | 15.969 | .99804 | 25 |
| 36 | 536 | 541 | 22.022 | 897 | 24 | 36 | 279 | 291 | .895 | 803 | 24 |
| 37 | 565 | 570 | 21.881 | 896 | 23 | 37 | 308 | 321 | .821 | 801 | 23 |
| 38 | 594 | 599 | .713 | 894 | 22 | 38 | 337 | 350 | .748 | 799 | 22 |
| 39 | 623 | 628 | .606 | 893 | 21 | 39 | 366 | 379 | .676 | 797 | 21 |
| 40 | .04653 | .04658 | 21.470 | .99892 | 20 | 40 | .06395 | .06408 | 15.605 | .99795 | 20 |
| 41 | 682 | 687 | .337 | 890 | 19 | 41 | 421 | 438 | .534 | 793 | 19 |
| 42 | 711 | 716 | .205 | 889 | 18 | 42 | 453 | 467 | .464 | 792 | 18 |
| 43 | 740 | 745 | 21.075 | 888 | 17 | 43 | 482 | 496 | .394 | 790 | 17 |
| 44 | 769 | 774 | 20.946 | 886 | 16 | 44 | 511 | 525 | .325 | 788 | 16 |
| 45 | .04798 | .04803 | 20.819 | .99885 | 15 | 45 | .06540 | .06551 | 15.257 | .99786 | 15 |
| 46 | 827 | 833 | .693 | 883 | 14 | 46 | 569 | 581 | .189 | 784 | 14 |
| 47 | 856 | 862 | .569 | 882 | 13 | 47 | 598 | 613 | .122 | 782 | 13 |
| 48 | 885 | 891 | .446 | 881 | 12 | 48 | 627 | 642 | 15.056 | 780 | 12 |
| 49 | 914 | 920 | .325 | 879 | 11 | 49 | 656 | 671 | 14.990 | 778 | 11 |
| 50 | .04943 | .04949 | 20.206 | .99878 | 10 | 50 | .06685 | .06700 | 14.924 | .99776 | 10 |
| 51 | .04972 | .04978 | 20.087 | 876 | 9 | 51 | 711 | 730 | .860 | 774 | 9 |
| 52 | .05001 | .05007 | 19.970 | 875 | 8 | 52 | 743 | 759 | .795 | 772 | 8 |
| 53 | 030 | 037 | .855 | 873 | 7 | 53 | 773 | 788 | .732 | 770 | 7 |
| 54 | 059 | 066 | .740 | 872 | 6 | 54 | 802 | 817 | .669 | 768 | 6 |
| 55 | .05088 | .05095 | 19.627 | .99870 | 5 | 55 | .06831 | .06847 | 14.606 | .99766 | 5 |
| 56 | 117 | 124 | .516 | 869 | 4 | 56 | 860 | 876 | .544 | 764 | 4 |
| 57 | 146 | 153 | .405 | 867 | 3 | 57 | 889 | 905 | .482 | 762 | 3 |
| 58 | 175 | 182 | .296 | 866 | 2 | 58 | 918 | 934 | .421 | 760 | 2 |
| 59 | 205 | 212 | .188 | 864 | 1 | 59 | 947 | 963 | .361 | 758 | 1 |
| 60 | .05234 | .05241 | 19.081 | .99863 | 0 | 60 | .06976 | .06993 | 14.301 | .99756 | 0 |
| | Cos | Ctn | Tan | Sin | ' | | Cos | Ctn | Tan | Sin | ' |

87°

86°

| | Sin | Tan | Ctn | Cos | | Sin | Tan | Ctn | Cos | |
|----|--------|--------|--------|--------|----|-----|--------|--------|--------|--------|
| 0 | .06976 | .06993 | 14.301 | .99756 | 60 | 0 | .08716 | .08749 | 11.430 | .99619 |
| 1 | .07005 | .07022 | .241 | 754 | 59 | 1 | 745 | .778 | .392 | 617 |
| 2 | 034 | .051 | .182 | 752 | 58 | 2 | 774 | .807 | .354 | 614 |
| 3 | 063 | .080 | .124 | 750 | 57 | 3 | 803 | .837 | .316 | 612 |
| 4 | 092 | .110 | .065 | 748 | 56 | 4 | 831 | .866 | .279 | 609 |
| 5 | .07121 | .07139 | 14.008 | .99746 | 55 | 5 | .08860 | .08895 | 11.242 | .99607 |
| 6 | 150 | .168 | 13.951 | 744 | 54 | 6 | 889 | .925 | .205 | 604 |
| 7 | 179 | .197 | .894 | 742 | 53 | 7 | 918 | .954 | .168 | 602 |
| 8 | 208 | .227 | .838 | 740 | 52 | 8 | 947 | .08983 | .132 | 599 |
| 9 | 237 | .256 | .782 | 738 | 51 | 9 | .08976 | .09013 | .095 | 596 |
| 10 | .07266 | .07285 | 13.727 | .99736 | 50 | 10 | .09005 | .09042 | 11.059 | .99594 |
| 11 | 295 | .314 | .672 | 734 | 49 | 11 | 011 | .074 | 11.024 | 591 |
| 12 | 324 | .344 | .617 | 731 | 48 | 1 | 063 | .101 | 10.988 | 588 |
| 13 | 353 | .373 | .563 | 729 | 47 | 1 | .092 | .139 | .953 | 586 |
| 14 | 382 | .402 | .510 | 727 | 46 | 11 | 121 | .139 | .918 | 583 |
| 15 | .07411 | .07431 | 13.457 | .99725 | 45 | 15 | .09150 | .09189 | 10.883 | .99580 |
| 16 | 440 | .461 | .404 | 723 | 44 | 16 | 179 | .218 | .848 | 578 |
| 17 | 469 | .490 | .352 | 721 | 43 | 17 | 208 | .247 | .814 | 575 |
| 18 | 498 | .519 | .300 | 719 | 42 | 18 | 237 | .277 | .780 | 572 |
| 19 | 527 | .548 | .248 | 716 | 41 | 19 | 266 | .306 | .746 | 570 |
| 20 | .07556 | .07578 | 13.197 | .99714 | 40 | 20 | .09295 | .09335 | 10.712 | .99567 |
| 21 | 555 | .607 | .146 | 712 | 39 | 21 | 324 | .365 | .678 | 564 |
| 22 | 614 | .636 | .096 | 710 | 38 | 22 | 353 | .394 | .645 | 562 |
| 23 | 643 | .665 | 13.016 | 708 | 37 | 23 | 382 | .423 | .612 | 559 |
| 24 | 672 | .695 | 12.996 | 705 | 36 | 24 | 411 | .453 | .579 | 556 |
| 25 | .07701 | .07724 | 12.947 | .99703 | 35 | 25 | .09440 | .09482 | 10.546 | .99553 |
| 26 | 730 | .753 | .898 | 701 | 34 | 26 | 469 | .511 | .544 | 551 |
| 27 | 759 | .782 | .850 | 699 | 33 | 27 | 498 | .541 | .511 | 548 |
| 28 | 788 | .812 | .801 | 696 | 32 | 28 | 527 | .570 | .449 | .532 |
| 29 | 817 | .841 | .754 | 694 | 31 | 29 | 556 | .600 | .417 | 542 |
| 30 | .07846 | .07870 | 12.706 | .99692 | 30 | 30 | .09583 | .09629 | 10.385 | .99540 |
| 31 | 875 | .899 | .659 | 689 | 29 | 31 | 614 | .658 | .354 | 537 |
| 32 | 904 | .929 | .612 | 687 | 28 | 32 | 642 | .688 | .322 | 534 |
| 33 | 933 | .958 | .566 | 685 | 27 | 33 | 671 | .717 | .291 | 531 |
| 34 | 962 | .07987 | .520 | 683 | 26 | 34 | 700 | .746 | .260 | 528 |
| 35 | .07991 | .08017 | 12.474 | .99680 | 25 | 35 | .09729 | .09776 | 10.229 | .99526 |
| 36 | .08020 | .046 | .429 | 678 | 24 | 36 | 758 | .805 | .199 | 523 |
| 37 | 019 | .075 | .384 | 676 | 23 | 37 | 787 | .834 | .168 | 520 |
| 38 | 078 | .104 | .349 | 673 | 22 | 38 | 816 | .864 | .138 | 517 |
| 39 | 107 | .134 | .295 | 671 | 21 | 39 | 845 | .893 | .108 | 514 |
| 40 | .08136 | .08163 | 12.251 | .99668 | 20 | 40 | .09874 | .09923 | 10.078 | .99511 |
| 41 | 165 | .192 | .207 | 666 | 19 | 41 | 903 | .952 | .048 | 508 |
| 42 | 194 | .221 | .163 | 664 | 18 | 42 | .09981 | 10.019 | .506 | 18 |
| 43 | 223 | .251 | .120 | 661 | 17 | 43 | 961 | .10011 | 9.9893 | 503 |
| 44 | 252 | .280 | .077 | 659 | 16 | 44 | .09990 | .040 | .9601 | 500 |
| 45 | .08281 | .08309 | 12.035 | .99657 | 15 | 45 | .10019 | .10069 | 9.9310 | .99497 |
| 46 | 310 | .339 | 11.992 | 654 | 14 | 46 | 018 | .099 | .9021 | 494 |
| 47 | 339 | .368 | .950 | 652 | 13 | 47 | 077 | .128 | .8731 | 491 |
| 48 | 368 | .397 | .909 | 649 | 12 | 48 | 106 | .158 | .8448 | 488 |
| 49 | 397 | .427 | .867 | 647 | 11 | 49 | 135 | .187 | .8164 | 485 |
| 50 | .08426 | .08456 | 11.826 | .99644 | 10 | 50 | .10164 | .10216 | 9.7882 | .99482 |
| 51 | 455 | .485 | .785 | 642 | 9 | 51 | 192 | .216 | .7601 | 479 |
| 52 | 484 | .514 | .745 | 639 | 8 | 52 | 221 | .275 | .7322 | 476 |
| 53 | 513 | .544 | .705 | 637 | 7 | 53 | 250 | .305 | .7044 | 473 |
| 54 | 542 | .573 | .664 | 635 | 6 | 54 | 279 | .334 | .6768 | 470 |
| 55 | .08571 | .08602 | 11.625 | .99632 | 5 | 55 | .10308 | .10363 | 9.6493 | .99467 |
| 56 | 600 | .632 | .585 | 630 | 4 | 56 | 337 | .393 | .6220 | 464 |
| 57 | 629 | .661 | .546 | 627 | 3 | 57 | 366 | .422 | .5919 | 461 |
| 58 | 658 | .690 | .507 | 625 | 2 | 58 | 395 | .452 | .5679 | 458 |
| 59 | 687 | .720 | .468 | 622 | 1 | 59 | 424 | .481 | .5411 | 455 |
| 60 | .08716 | .08749 | 11.430 | .99619 | 0 | 60 | .10453 | .10510 | 9.5144 | .99452 |
| | Cos | Ctn | Tan | Sin | | | Cos | Ctn | Tan | Sin |

85°

84°

| ' | Sin | Tan | Ctn | Cos | ' | Sin | Tan | Ctn | Cos | ' |
|----|--------|--------|--------|--------|----|--------|--------|--------|--------|----|
| 0 | .10453 | .10510 | 9.5144 | .99452 | 60 | .12187 | .12278 | 8.1443 | .99255 | 60 |
| 1 | 482 | 540 | .4878 | 449 | 59 | 216 | 308 | .1248 | 251 | 59 |
| 2 | 511 | 569 | .4614 | 446 | 58 | 245 | 338 | .1054 | 248 | 58 |
| 3 | 540 | 599 | .4352 | 443 | 57 | 274 | 367 | .0860 | 244 | 57 |
| 4 | 569 | 628 | .4090 | 440 | 56 | 302 | 397 | .0667 | 240 | 56 |
| 5 | .10597 | .10657 | 9.3831 | .99437 | 55 | .12331 | .12426 | 8.0476 | .99237 | 55 |
| 6 | 626 | 687 | .3572 | 434 | 54 | 360 | 456 | .0285 | 233 | 54 |
| 7 | 655 | 716 | .3315 | 431 | 53 | 389 | 485 | 8.0095 | 230 | 53 |
| 8 | 684 | 746 | .3060 | 428 | 52 | 418 | 515 | 7.9906 | 226 | 52 |
| 9 | 713 | 775 | .2806 | 424 | 51 | 447 | 544 | .9718 | 222 | 51 |
| 10 | .10742 | .10805 | 9.2553 | .99421 | 50 | .12476 | .12574 | 7.9530 | .99219 | 50 |
| 11 | 771 | 834 | .2302 | 418 | 49 | 504 | 603 | .9344 | 215 | 49 |
| 12 | 800 | 863 | .2052 | 415 | 48 | 533 | 633 | .9158 | 211 | 48 |
| 13 | 829 | 893 | .1803 | 412 | 47 | 562 | 662 | .8973 | 208 | 47 |
| 14 | 858 | 922 | .1555 | 409 | 46 | 591 | 692 | .8789 | 204 | 46 |
| 15 | .10887 | .10952 | 9.1309 | .99406 | 45 | .12620 | .12722 | 7.8606 | .99200 | 45 |
| 16 | 916 | .10981 | .1065 | 402 | 44 | 649 | 751 | .8424 | 197 | 44 |
| 17 | 945 | .11011 | .0821 | 399 | 43 | 678 | 781 | .8243 | 193 | 43 |
| 18 | .10973 | 040 | .0579 | 396 | 42 | 706 | 810 | .8062 | 189 | 42 |
| 19 | .11002 | 070 | .0338 | 393 | 41 | 735 | 840 | .7882 | 186 | 41 |
| 20 | .11031 | .11099 | 9.0098 | .99390 | 40 | .12764 | .12869 | 7.7704 | .99182 | 40 |
| 21 | 060 | 128 | 8.9860 | 386 | 39 | 793 | 899 | .7525 | 178 | 39 |
| 22 | 089 | 158 | .9623 | 383 | 38 | 822 | 929 | .7348 | 175 | 38 |
| 23 | 118 | 187 | .9357 | 380 | 37 | 851 | 958 | .7171 | 171 | 37 |
| 24 | 147 | 217 | .9152 | 377 | 36 | 880 | .12988 | .6996 | 167 | 36 |
| 25 | .11176 | .11246 | 8.8919 | .99374 | 35 | .12908 | .13017 | 7.6821 | .99163 | 35 |
| 26 | 205 | 276 | .8686 | 370 | 34 | 937 | 047 | .6647 | 160 | 34 |
| 27 | 234 | 305 | .8455 | 367 | 33 | 966 | 076 | .6473 | 156 | 33 |
| 28 | 263 | 335 | .8225 | 364 | 32 | .12995 | 106 | .6301 | 152 | 32 |
| 29 | 291 | 364 | .7996 | 360 | 31 | .13024 | 136 | .6129 | 148 | 31 |
| 30 | .11320 | .11394 | 8.7769 | .99357 | 30 | .13053 | .13165 | 7.5958 | .99144 | 30 |
| 31 | 349 | 423 | .7542 | 354 | 29 | 081 | 195 | .5787 | 141 | 29 |
| 32 | 378 | 452 | .7317 | 351 | 28 | 110 | 224 | .5618 | 137 | 28 |
| 33 | 407 | 482 | .7093 | 347 | 27 | 139 | 254 | .5449 | 133 | 27 |
| 34 | 436 | 511 | .6870 | 344 | 26 | 168 | 284 | .5281 | 129 | 26 |
| 35 | .11465 | .11541 | 8.6648 | .99341 | 25 | .13197 | .13313 | 7.5113 | .99125 | 25 |
| 36 | 494 | .570 | .6427 | 337 | 24 | 226 | 343 | .4947 | 122 | 24 |
| 37 | 523 | 600 | .6208 | 334 | 23 | 254 | 372 | .4781 | 118 | 23 |
| 38 | 552 | 629 | .5989 | 331 | 22 | 283 | 402 | .4615 | 114 | 22 |
| 39 | 580 | 659 | .5772 | 327 | 21 | 312 | 432 | .4451 | 110 | 21 |
| 40 | .11609 | .11688 | 8.5555 | .99324 | 20 | .13341 | .13461 | 7.4287 | .99106 | 20 |
| 41 | 638 | 718 | .5340 | 320 | 19 | 370 | 491 | .4124 | 102 | 19 |
| 42 | 667 | 747 | .5126 | 317 | 18 | 427 | 550 | .3800 | 094 | 17 |
| 43 | 696 | 777 | .4913 | 314 | 17 | 456 | 580 | .3639 | 091 | 16 |
| 44 | 725 | 806 | .4701 | 310 | 16 | .13485 | .13609 | 7.3479 | .99087 | 15 |
| 45 | .11754 | .11836 | 8.4490 | .99307 | 15 | 514 | 639 | .3319 | 083 | 14 |
| 46 | 753 | 865 | .4280 | 303 | 14 | 543 | 669 | .3160 | 079 | 13 |
| 47 | 812 | 895 | .4071 | 300 | 13 | 572 | 698 | .3002 | 075 | 12 |
| 48 | 840 | 924 | .3863 | 297 | 12 | 600 | 728 | .2844 | 071 | 11 |
| 49 | 869 | 954 | .3656 | 293 | 11 | .13629 | .13758 | 7.2687 | .99067 | 10 |
| 50 | .11898 | .11983 | 8.3450 | .99290 | 10 | 658 | 787 | .2531 | 063 | 9 |
| 51 | 927 | .12013 | .3245 | 286 | 9 | 687 | 817 | .2375 | 059 | 8 |
| 52 | 956 | 042 | .3041 | 283 | 8 | 716 | 846 | .2220 | 055 | 7 |
| 53 | .11985 | 072 | .2838 | 279 | 7 | 744 | 876 | .2066 | 051 | 6 |
| 54 | .12014 | 101 | .2636 | 276 | 6 | .13773 | .13906 | 7.1912 | .99047 | 5 |
| 55 | .12043 | .12131 | 8.2434 | .99272 | 5 | 802 | 935 | .1759 | 043 | 4 |
| 56 | 071 | 160 | .2234 | 269 | 4 | 831 | 965 | .1607 | 039 | 3 |
| 57 | 100 | 190 | .2035 | 265 | 3 | 860 | .13995 | .1455 | 035 | 2 |
| 58 | 129 | 219 | .1837 | 262 | 2 | 889 | .14024 | .1304 | 031 | 1 |
| 59 | 158 | 249 | .1640 | 258 | 1 | .13917 | .14054 | 7.1154 | .99027 | 0 |
| 60 | .12187 | .12278 | 8.1443 | .99255 | 0 | | | | | |
| | Cos | Ctn | Tan | Sin | | Cos | Ctn | Tan | Sin | |

83°

82°

| ' | Sin | Tan | Ctn | Cos | ' | Sin | Tan | Ctn | Cos | ' | |
|----|--------|--------|--------|--------|----|-----|--------|--------|--------|--------|----|
| 0 | .13917 | .14054 | 7.1154 | .99027 | 60 | 0 | .15643 | .15838 | 6.3138 | .98769 | 60 |
| 1 | .946 | .084 | .1004 | .023 | 59 | 1 | .672 | .868 | .3019 | .764 | 59 |
| 2 | .13975 | .113 | .0855 | .019 | 58 | 2 | .701 | .898 | .2901 | .760 | 58 |
| 3 | .14004 | .143 | .0706 | .015 | 57 | 3 | .730 | .928 | .2783 | .755 | 57 |
| 4 | .033 | .173 | .0558 | .011 | 56 | 4 | .758 | .958 | .2666 | .751 | 56 |
| 5 | .14061 | .14202 | 7.0410 | .99006 | 55 | 5 | .15787 | .15988 | 6.2549 | .98746 | 55 |
| 6 | .090 | .232 | .0264 | .99002 | 54 | 6 | .816 | .16017 | .2432 | .741 | 54 |
| 7 | .119 | .262 | 7.0117 | .98998 | 53 | 7 | .845 | .047 | .2316 | .737 | 53 |
| 8 | .148 | .291 | 6.9972 | .994 | 52 | 8 | .873 | .077 | .2200 | .732 | 52 |
| 9 | .177 | .321 | .9827 | .990 | 51 | 9 | .902 | .107 | .2085 | .728 | 51 |
| 10 | .14205 | .14351 | 6.9682 | .98986 | 50 | 10 | .15931 | .16137 | 6.1970 | .98723 | 50 |
| 11 | .234 | .381 | .9538 | .982 | 49 | 11 | .959 | .167 | .1856 | .718 | 49 |
| 12 | .263 | .410 | .9395 | .978 | 48 | 12 | .15988 | .196 | .1742 | .714 | 48 |
| 13 | .292 | .440 | .9252 | .973 | 47 | 13 | .16017 | .226 | .1628 | .709 | 47 |
| 14 | .320 | .470 | .9110 | .969 | 46 | 14 | .046 | .256 | .1515 | .704 | 46 |
| 15 | .14349 | .14499 | 6.8969 | .98965 | 45 | 15 | .16074 | .16286 | 6.1402 | .98700 | 45 |
| 16 | .378 | .529 | .8828 | .961 | 44 | 16 | .103 | .316 | .1290 | .695 | 44 |
| 17 | .407 | .559 | .8687 | .957 | 43 | 17 | .132 | .346 | .1178 | .690 | 43 |
| 18 | .436 | .588 | .8548 | .953 | 42 | 18 | .160 | .376 | .1066 | .686 | 42 |
| 19 | .464 | .618 | .8408 | .948 | 41 | 19 | .189 | .405 | .0955 | .681 | 41 |
| 20 | .14493 | .14648 | 6.8269 | .98944 | 40 | 20 | .16218 | .16435 | 6.0844 | .98676 | 40 |
| 21 | .522 | .678 | .8131 | .940 | 39 | 21 | .246 | .465 | .0734 | .671 | 39 |
| 22 | .551 | .707 | .7994 | .936 | 38 | 22 | .275 | .495 | .0624 | .667 | 38 |
| 23 | .580 | .737 | .7856 | .931 | 37 | 23 | .304 | .525 | .0514 | .662 | 37 |
| 24 | .608 | .767 | .7720 | .927 | 36 | 24 | .333 | .555 | .0405 | .657 | 36 |
| 25 | .14637 | .14796 | 6.7584 | .98923 | 35 | 25 | .16361 | .16585 | 6.0296 | .98652 | 35 |
| 26 | .666 | .826 | .7448 | .919 | 34 | 26 | .390 | .615 | .0188 | .648 | 34 |
| 27 | .695 | .856 | .7313 | .914 | 33 | 27 | .419 | .645 | 6.0080 | .643 | 33 |
| 28 | .723 | .886 | .7179 | .910 | 32 | 28 | .447 | .674 | 5.9972 | .738 | 32 |
| 29 | .752 | .915 | .7045 | .906 | 31 | 29 | .476 | .704 | .9865 | .633 | 31 |
| 30 | .14781 | .14945 | 6.6912 | .98902 | 30 | 30 | .16505 | .16734 | 5.9758 | .98629 | 30 |
| 31 | .810 | .14975 | .6779 | .897 | 29 | 31 | .533 | .764 | .9651 | .624 | 29 |
| 32 | .838 | .15005 | .6646 | .893 | 28 | 32 | .562 | .794 | .9545 | .619 | 28 |
| 33 | .867 | .034 | .6511 | .889 | 27 | 33 | .591 | .824 | .9439 | .614 | 27 |
| 34 | .896 | .064 | .6363 | .884 | 26 | 34 | .620 | .854 | .9333 | .609 | 26 |
| 35 | .14925 | .15094 | 6.6252 | .98880 | 25 | 35 | .16648 | .16884 | 5.9228 | .98604 | 25 |
| 36 | .954 | .124 | .6122 | .876 | 24 | 36 | .677 | .914 | .9124 | .600 | 24 |
| 37 | .14982 | .153 | .5992 | .871 | 23 | 37 | .706 | .944 | .9019 | .595 | 23 |
| 38 | .15011 | .183 | .5863 | .867 | 22 | 38 | .734 | .16974 | .8915 | .590 | 22 |
| 39 | .040 | .213 | .5734 | .863 | 21 | 39 | .763 | .17004 | .8811 | .585 | 21 |
| 40 | .15069 | .15243 | 6.5606 | .98858 | 20 | 40 | .16792 | .17033 | 5.8708 | .98580 | 20 |
| 41 | .097 | .272 | .5478 | .854 | 19 | 41 | .820 | .063 | .8605 | .575 | 19 |
| 42 | .120 | .302 | .5350 | .849 | 18 | 42 | .849 | .093 | .8502 | .570 | 18 |
| 43 | .155 | .332 | .5223 | .845 | 17 | 43 | .878 | .123 | .8400 | .565 | 17 |
| 44 | .184 | .362 | .5097 | .841 | 16 | 44 | .906 | .153 | .8298 | .561 | 16 |
| 45 | .15212 | .15391 | 6.4971 | .98836 | 15 | 45 | .16935 | .17183 | 5.8197 | .98556 | 15 |
| 46 | .241 | .421 | .4846 | .832 | 14 | 46 | .964 | .213 | .8095 | .551 | 14 |
| 47 | .270 | .451 | .4721 | .827 | 13 | 47 | .16992 | .243 | .7994 | .546 | 13 |
| 48 | .299 | .481 | .4596 | .823 | 12 | 48 | .17021 | .273 | .7894 | .541 | 12 |
| 49 | .327 | .511 | .4472 | .818 | 11 | 49 | .050 | .303 | .7794 | .536 | 11 |
| 50 | .15356 | .15540 | 6.4348 | .98814 | 10 | 50 | .17078 | .17333 | 5.7694 | .98531 | 10 |
| 51 | .385 | .570 | .4225 | .809 | 9 | 51 | .107 | .363 | .7594 | .526 | 9 |
| 52 | .414 | .600 | .4103 | .805 | 8 | 52 | .136 | .393 | .7495 | .521 | 8 |
| 53 | .442 | .630 | .3980 | .800 | 7 | 53 | .164 | .423 | .7396 | .516 | 7 |
| 54 | .471 | .660 | .3859 | .796 | 6 | 54 | .193 | .453 | .7297 | .511 | 6 |
| 55 | .15500 | .15689 | 6.3737 | .98791 | 5 | 55 | .17222 | .17483 | 5.7199 | .98506 | 5 |
| 56 | .529 | .719 | .3617 | .787 | 4 | 56 | .250 | .513 | .7101 | .501 | 4 |
| 57 | .557 | .749 | .3496 | .782 | 3 | 57 | .279 | .543 | .7004 | .496 | 3 |
| 58 | .586 | .779 | .3376 | .778 | 2 | 58 | .308 | .573 | .6906 | .491 | 2 |
| 59 | .615 | .809 | .3257 | .773 | 1 | 59 | .336 | .603 | .6809 | .486 | 1 |
| 60 | .15643 | .15838 | 6.3138 | .98769 | 0 | 60 | .17365 | .17633 | 5.6713 | .98481 | 0 |
| | Cos | Ctn | Tan | Sin | ' | | Cos | Ctn | Tan | Sin | ' |

| ' | Sin | Tan | Ctn | Cos | ' | Sin | Tan | Ctn | Cos | ' | |
|----|--------|--------|--------|--------|----|-----|--------|--------|--------|--------|----|
| 0 | .17365 | .17633 | 5.6713 | .98481 | 60 | 0 | .19081 | .19438 | 5.1446 | .98163 | 60 |
| 1 | 393 | 663 | .6617 | 476 | 59 | 1 | 109 | 468 | .1366 | 157 | 59 |
| 2 | 422 | 693 | .6521 | 471 | 58 | 2 | 138 | 498 | .1286 | 152 | 58 |
| 3 | 451 | 723 | .6425 | 466 | 57 | 3 | 167 | 529 | .1207 | 146 | 57 |
| 4 | 479 | 753 | .6329 | 461 | 56 | 4 | 195 | 559 | .1128 | 110 | 56 |
| 5 | .17508 | .17783 | 5.6234 | .98455 | 55 | 5 | .19224 | .19589 | 5.1049 | .98135 | 55 |
| 6 | 537 | 813 | .6140 | 450 | 51 | 6 | 252 | 619 | .0970 | 129 | 51 |
| 7 | 565 | 843 | .6045 | 445 | 53 | 7 | 281 | 649 | .0892 | 124 | 53 |
| 8 | 591 | 873 | .5951 | 440 | 52 | 8 | 309 | 680 | .0811 | 118 | 52 |
| 9 | 623 | 903 | .5857 | 435 | 51 | 9 | 338 | 710 | .0736 | 112 | 51 |
| 10 | .17651 | .17933 | 5.5764 | .98430 | 50 | 10 | .19366 | .19740 | 5.0658 | .98107 | 50 |
| 11 | 680 | 963 | .5671 | 425 | 49 | 11 | 395 | 770 | .0581 | 101 | 49 |
| 12 | 708 | .17993 | .5578 | 420 | 48 | 12 | 423 | 801 | .0501 | 096 | 48 |
| 13 | 737 | .18023 | .5485 | 414 | 47 | 13 | 452 | 831 | .0427 | 090 | 47 |
| 14 | 766 | 053 | .5393 | 409 | 46 | 14 | 481 | 861 | .0350 | 084 | 46 |
| 15 | .17794 | .18083 | 5.5301 | .98404 | 45 | 15 | .19509 | .19891 | 5.0273 | .98079 | 45 |
| 16 | 823 | 113 | .5209 | 399 | 11 | 16 | 538 | 921 | .0197 | 073 | 44 |
| 17 | 852 | 143 | .5118 | 394 | 13 | 17 | 566 | 952 | .0121 | 067 | 43 |
| 18 | 880 | 173 | .5026 | 389 | 12 | 18 | 595 | .19982 | 5.0015 | 061 | 42 |
| 19 | 909 | 203 | .4936 | 383 | 11 | 19 | 623 | .20012 | 4.9969 | 056 | 41 |
| 20 | .17937 | .18233 | 5.4845 | .98378 | 40 | 20 | .19652 | .20012 | 4.9894 | .98050 | 40 |
| 21 | 966 | 263 | .4755 | 373 | 39 | 21 | 680 | 073 | .9819 | 044 | 39 |
| 22 | .17995 | 293 | .4665 | 368 | 38 | 22 | 709 | 103 | .9744 | 039 | 38 |
| 23 | .18023 | 323 | .4575 | 362 | 37 | 23 | 737 | 133 | .9669 | 033 | 37 |
| 24 | 052 | 353 | .4486 | 357 | 36 | 24 | 766 | 161 | .9594 | 027 | 36 |
| 25 | .18081 | .18381 | 5.4397 | .98352 | 35 | 25 | .19794 | .20194 | 4.9520 | .98021 | 35 |
| 26 | 109 | 411 | .4308 | 317 | 31 | 26 | 823 | 224 | .9446 | 016 | 31 |
| 27 | 138 | 444 | .4219 | 311 | 33 | 27 | 851 | 251 | .9372 | 010 | 33 |
| 28 | 166 | 471 | .4131 | 316 | 32 | 28 | 880 | 285 | .9298 | .98001 | 32 |
| 29 | 195 | 504 | .4043 | 311 | 31 | 29 | 908 | 315 | .9225 | .97998 | 31 |
| 30 | .18224 | .18534 | 5.3955 | .98325 | 30 | 30 | .19937 | .20315 | 4.9152 | .97992 | 30 |
| 31 | 252 | 561 | .3868 | 320 | 29 | 31 | 965 | 376 | .9078 | 987 | 29 |
| 32 | 281 | 591 | .3781 | 315 | 28 | 32 | .19991 | 406 | .9006 | 981 | 28 |
| 33 | 309 | 624 | .3694 | 310 | 27 | 33 | .20022 | 436 | .8933 | 975 | 27 |
| 34 | 338 | 651 | .3607 | 304 | 26 | 34 | 051 | 466 | .8860 | 969 | 26 |
| 35 | .18367 | .18684 | 5.3521 | .98299 | 25 | 35 | .20079 | .20497 | 4.8788 | .97963 | 25 |
| 36 | 395 | 714 | .3435 | 294 | 21 | 36 | 108 | 527 | .8716 | 958 | 24 |
| 37 | 424 | 745 | .3349 | 288 | 23 | 37 | 136 | 557 | .8644 | 952 | 23 |
| 38 | 452 | 775 | .3263 | 283 | 22 | 38 | 165 | 588 | .8573 | 946 | 22 |
| 39 | 481 | 805 | .3178 | 277 | 21 | 39 | 193 | 618 | .8501 | 940 | 21 |
| 40 | .18509 | .18835 | 5.3093 | .98272 | 20 | 40 | .20222 | .20618 | 4.8430 | .97934 | 20 |
| 41 | 538 | 865 | .3008 | 267 | 19 | 41 | 250 | 679 | .8359 | 928 | 19 |
| 42 | 567 | 895 | .2924 | 261 | 18 | 42 | 279 | 709 | .8288 | 922 | 18 |
| 43 | 595 | 925 | .2839 | 256 | 17 | 43 | 307 | 739 | .8218 | 916 | 17 |
| 44 | 621 | 955 | .2755 | 250 | 16 | 44 | 336 | 770 | .8147 | 910 | 16 |
| 45 | .18652 | .18986 | 5.2672 | .98245 | 15 | 45 | .20364 | .20800 | 4.8077 | .97905 | 15 |
| 46 | 681 | .19016 | .2588 | 240 | 14 | 46 | 393 | 830 | .8007 | 899 | 14 |
| 47 | 710 | 046 | .2505 | 234 | 13 | 47 | 421 | 861 | .7937 | 893 | 13 |
| 48 | 738 | 076 | .2422 | 229 | 12 | 48 | 450 | 891 | .7867 | 887 | 12 |
| 49 | 767 | 106 | .2339 | 223 | 11 | 49 | 478 | 921 | .7798 | 881 | 11 |
| 50 | .18795 | .19136 | 5.2257 | .98218 | 10 | 50 | .20507 | .20952 | 4.7729 | .97875 | 10 |
| 51 | 824 | 166 | .2174 | 212 | 9 | 51 | 535 | .20982 | .7659 | 869 | 9 |
| 52 | 852 | 197 | .2092 | 207 | 8 | 52 | 563 | .21013 | .7591 | 863 | 8 |
| 53 | 881 | 227 | .2011 | 201 | 7 | 53 | 592 | 043 | .7522 | 857 | 7 |
| 54 | 910 | 257 | .1929 | 196 | 6 | 54 | 620 | 073 | .7453 | 851 | 6 |
| 55 | .18938 | .19287 | 5.1848 | .98190 | 5 | 55 | .20649 | .21104 | 4.7385 | .97845 | 5 |
| 56 | 967 | 317 | .1767 | 185 | 4 | 56 | 677 | 134 | .7317 | 839 | 4 |
| 57 | .18995 | 347 | .1686 | 179 | 3 | 57 | 706 | 164 | .7249 | 833 | 3 |
| 58 | .19024 | 378 | .1606 | 174 | 2 | 58 | 734 | 195 | .7181 | 827 | 2 |
| 59 | 052 | 408 | .1526 | 168 | 1 | 59 | 763 | 225 | .7114 | 821 | 1 |
| 60 | .19081 | .19438 | 5.1446 | .98163 | 0 | 60 | .20791 | .21256 | 4.7046 | .97815 | 0 |
| | Cos | Ctn | Tan | Sin | ' | | Cos | Ctn | Tan | Sin | ' |

79°

78°

| ' | Sin | Tan | Ctn | Cos | ' | Sin | Tan | Ctn | Cos |
|----|--------|--------|--------|--------|----|--------|--------|--------|--------|
| 0 | .20791 | .21256 | 4.7046 | .97815 | 60 | .22495 | .23087 | 4.3315 | .97437 |
| 1 | 820 | 286 | .6979 | 809 | 59 | 523 | 117 | .3257 | 430 |
| 2 | 848 | 316 | .6912 | 803 | 58 | 552 | 148 | .3200 | 424 |
| 3 | 877 | 347 | .6845 | 797 | 57 | 580 | 179 | .3143 | 417 |
| 4 | 905 | 377 | .6779 | 791 | 56 | 608 | 209 | .3086 | 411 |
| 5 | .20933 | .21408 | 4.6712 | .97784 | 55 | .22637 | .23240 | 4.3029 | .97404 |
| 6 | 962 | 438 | .6616 | 778 | 54 | 665 | 271 | .2972 | 398 |
| 7 | .20990 | 469 | .6550 | 772 | 53 | 693 | 301 | .2916 | 391 |
| 8 | .21019 | 499 | .6514 | 766 | 52 | 722 | 332 | .2859 | 384 |
| 9 | 047 | 529 | .6445 | 760 | 51 | 750 | 363 | .2803 | 378 |
| 10 | .21076 | .21560 | 4.6382 | .97754 | 50 | .22778 | .23393 | 4.2747 | .97371 |
| 11 | 104 | 590 | .6317 | 718 | 49 | 807 | 421 | .2691 | 365 |
| 12 | 132 | 621 | .6252 | 712 | 48 | 835 | 455 | .2635 | 358 |
| 13 | 161 | 651 | .6187 | 705 | 47 | 863 | 485 | .2580 | 351 |
| 14 | 189 | 682 | .6122 | 700 | 46 | 892 | 516 | .2524 | 345 |
| 15 | .21218 | .21712 | 4.6057 | .97723 | 45 | .22920 | .23547 | 4.2468 | .97338 |
| 16 | 216 | 713 | .5993 | 717 | 44 | 918 | 578 | .2413 | 331 |
| 17 | 275 | 773 | .5928 | 711 | 43 | .22977 | 608 | .2358 | 325 |
| 18 | 303 | 804 | .5861 | 705 | 42 | .23005 | 639 | .2303 | 318 |
| 19 | 331 | 834 | .5800 | 698 | 41 | 033 | 670 | .2248 | 311 |
| 20 | .21360 | .21864 | 4.5736 | .97692 | 40 | .23062 | .23700 | 4.2193 | .97304 |
| 21 | 388 | 895 | .5673 | 686 | 39 | 090 | 731 | .2139 | 298 |
| 22 | 417 | 925 | .5609 | 680 | 38 | 118 | 762 | .2084 | 291 |
| 23 | 445 | 956 | .5546 | 673 | 37 | 146 | 793 | .2030 | 284 |
| 24 | 474 | .21986 | .5483 | 667 | 36 | 175 | 823 | .1976 | 278 |
| 25 | .21502 | .22017 | 4.5420 | .97661 | 35 | .23203 | .23854 | 4.1922 | .97271 |
| 26 | 530 | 017 | .5357 | 655 | 34 | 231 | 885 | .1868 | 264 |
| 27 | 559 | 078 | .5291 | 648 | 33 | 260 | 916 | .1814 | 257 |
| 28 | 587 | 108 | .5222 | 642 | 32 | 288 | 946 | .1760 | 251 |
| 29 | 616 | 139 | .5169 | 636 | 31 | 316 | .23977 | .1706 | 244 |
| 30 | .21644 | .22169 | 4.5107 | .97630 | 30 | .23345 | .24008 | 4.1653 | .97237 |
| 31 | 672 | 200 | .5045 | 623 | 29 | 343 | 039 | .1600 | 230 |
| 32 | 701 | 231 | .4983 | 617 | 28 | 401 | 069 | .1547 | 223 |
| 33 | 729 | 261 | .4922 | 611 | 27 | 429 | 100 | .1493 | 217 |
| 34 | 758 | 292 | .4860 | 604 | 26 | 458 | 131 | .1441 | 210 |
| 35 | .21786 | .22322 | 4.4799 | .97598 | 25 | .23486 | .24162 | 4.1388 | .97203 |
| 36 | 811 | 353 | .4737 | 592 | 24 | 514 | 193 | .1335 | 196 |
| 37 | 843 | 383 | .4676 | 585 | 23 | 542 | 223 | .1282 | 189 |
| 38 | 871 | 414 | .4615 | 579 | 22 | 571 | 251 | .1230 | 182 |
| 39 | 899 | 444 | .4555 | 573 | 21 | 599 | 285 | .1178 | 176 |
| 40 | .21928 | .22475 | 4.4494 | .97566 | 20 | .23627 | .24316 | 4.1126 | .97169 |
| 41 | 956 | 505 | .4434 | 560 | 19 | 656 | 317 | .1074 | 162 |
| 42 | .21985 | 536 | .4373 | 553 | 18 | 684 | 377 | .1022 | 155 |
| 43 | .22013 | 567 | .4313 | 547 | 17 | 712 | 408 | .0970 | 148 |
| 44 | 041 | 597 | .4253 | 541 | 16 | 740 | 439 | .0918 | 141 |
| 45 | .22070 | .22628 | 4.4194 | .97534 | 15 | .23769 | .24470 | 4.0867 | .97134 |
| 46 | 098 | 658 | .4131 | 528 | 14 | 797 | 501 | .0815 | 127 |
| 47 | 126 | 689 | .4073 | 521 | 13 | 825 | 532 | .0764 | 120 |
| 48 | 155 | 719 | .4015 | 515 | 12 | 853 | 562 | .0713 | 113 |
| 49 | 183 | 750 | .3956 | 508 | 11 | 882 | 593 | .0662 | 106 |
| 50 | .22212 | .22781 | 4.3897 | .97502 | 10 | .23910 | .24621 | 4.0611 | .97100 |
| 51 | 210 | 811 | .3838 | 496 | 9 | 938 | 655 | .0560 | 093 |
| 52 | 268 | 842 | .3779 | 489 | 8 | 966 | 686 | .0509 | 086 |
| 53 | 297 | 872 | .3721 | 483 | 7 | .23995 | 717 | .0459 | 079 |
| 54 | 325 | 903 | .3662 | 476 | 6 | .24023 | 747 | .0408 | 072 |
| 55 | .22353 | .22934 | 4.3604 | .97470 | 5 | .24051 | .24778 | 4.0358 | .97065 |
| 56 | 382 | 964 | .3546 | 463 | 4 | 079 | 809 | .0308 | 058 |
| 57 | 410 | .22995 | .3488 | 457 | 3 | 108 | 840 | .0257 | 051 |
| 58 | 438 | .23026 | .3430 | 450 | 2 | 136 | 871 | .0207 | 044 |
| 59 | 467 | 056 | .3372 | 444 | 1 | 164 | 902 | .0158 | 037 |
| 60 | .22495 | .23087 | 4.3315 | .97437 | 0 | .24192 | .24933 | 4.0108 | .97030 |
| | Cos | Ctn | Tan | Sin | | Cos | Ctn | Tan | Sin |

| | Sin | Tan | Ctn | Cos | | | Sin | Tan | Ctn | Cos | |
|----|--------|--------|--------|--------|----|----|--------|--------|--------|--------|----|
| 0 | .24192 | .24933 | 4.0108 | .97030 | 60 | 0 | .25882 | .26795 | 3.7321 | .96593 | 60 |
| 1 | 220 | 964 | .0058 | 023 | 59 | 1 | 910 | 826 | .7277 | 585 | 59 |
| 2 | 249 | .24995 | 4.0009 | 015 | 58 | 2 | 938 | 857 | .7234 | 578 | 58 |
| 3 | 277 | .25026 | 3.9959 | 008 | 57 | 3 | 966 | 888 | .7191 | 570 | 57 |
| 4 | 305 | 056 | .9910 | .97001 | 56 | 4 | .25994 | 920 | .7148 | 562 | 56 |
| 5 | .24333 | .25087 | 3.9861 | .96994 | 55 | 5 | .26022 | .26951 | 3.7105 | .96555 | 55 |
| 6 | 362 | 118 | .9812 | 987 | 54 | 6 | 050 | .26982 | .7062 | 547 | 54 |
| 7 | 390 | 149 | .9763 | 980 | 53 | 7 | 079 | .27013 | .7019 | 540 | 53 |
| 8 | 418 | 180 | .9714 | 973 | 52 | 8 | 107 | 044 | .6976 | 532 | 52 |
| 9 | 446 | 211 | .9665 | 966 | 51 | 9 | 135 | 076 | .6933 | 524 | 51 |
| 10 | .24474 | .25242 | 3.9617 | .96959 | 50 | 10 | .26163 | .27107 | 3.6891 | .96517 | 50 |
| 11 | 503 | 273 | .9568 | 952 | 49 | 11 | 191 | 138 | .6848 | 509 | 49 |
| 12 | 531 | 304 | .9520 | 945 | 48 | 12 | 219 | 169 | .6806 | 502 | 48 |
| 13 | 559 | 335 | .9471 | 937 | 47 | 13 | 247 | 201 | .6764 | 494 | 47 |
| 14 | 587 | 366 | .9423 | 930 | 46 | 14 | 275 | 232 | .6722 | 486 | 46 |
| 15 | .24615 | .25397 | 3.9375 | .96923 | 45 | 15 | .26303 | .27263 | 3.6680 | .96479 | 45 |
| 16 | 644 | 428 | .9327 | 916 | 44 | 16 | 331 | 294 | .6638 | 471 | 44 |
| 17 | 672 | 459 | .9279 | 909 | 43 | 17 | 359 | 326 | .6596 | 463 | 43 |
| 18 | 700 | 490 | .9232 | 902 | 42 | 18 | 387 | 357 | .6554 | 456 | 42 |
| 19 | 728 | 521 | .9184 | 894 | 41 | 19 | 415 | 388 | .6512 | 448 | 41 |
| 20 | .24756 | .25552 | 3.9136 | .96887 | 40 | 20 | .26443 | .27419 | 3.6470 | .96440 | 40 |
| 21 | 784 | 583 | .9089 | 880 | 39 | 21 | 471 | 451 | .6429 | 433 | 39 |
| 22 | 813 | 614 | .9042 | 873 | 38 | 22 | 500 | 482 | .6387 | 425 | 38 |
| 23 | 841 | 645 | .8995 | 866 | 37 | 23 | 528 | 513 | .6346 | 417 | 37 |
| 24 | 869 | 676 | .8947 | 858 | 36 | 24 | 556 | 545 | .6305 | 410 | 36 |
| 25 | .24897 | .25707 | 3.8900 | .96851 | 35 | 25 | .26584 | .27576 | 3.6264 | .96402 | 35 |
| 26 | 925 | 738 | .8854 | 844 | 34 | 26 | 612 | 607 | .6222 | 394 | 34 |
| 27 | 954 | 769 | .8807 | 837 | 33 | 27 | 640 | 638 | .6181 | 386 | 33 |
| 28 | .24982 | 800 | .8760 | 829 | 32 | 28 | 668 | 670 | .6140 | 379 | 32 |
| 29 | .25010 | 831 | .8714 | 822 | 31 | 29 | 696 | 701 | .6100 | 371 | 31 |
| 30 | .25038 | .25862 | 3.8667 | .96815 | 30 | 30 | .26724 | .27732 | 3.6059 | .96363 | 30 |
| 31 | 066 | 893 | .8621 | 807 | 29 | 31 | 752 | 761 | .6018 | 355 | 29 |
| 32 | 094 | 924 | .8575 | 800 | 28 | 32 | 780 | 795 | .5978 | 347 | 28 |
| 33 | 122 | 955 | .8528 | 793 | 27 | 33 | 808 | 826 | .5937 | 340 | 27 |
| 34 | 151 | .25986 | .8482 | 786 | 26 | 34 | 836 | 858 | .5897 | 332 | 26 |
| 35 | .25179 | .26017 | 3.8436 | .96778 | 25 | 35 | .26864 | .27889 | 3.5856 | .96324 | 25 |
| 36 | 207 | 048 | .8391 | 771 | 24 | 36 | 892 | 921 | .5816 | 316 | 24 |
| 37 | 235 | 079 | .8345 | 764 | 23 | 37 | 920 | 952 | .5776 | 308 | 23 |
| 38 | 263 | 110 | .8299 | 756 | 22 | 38 | 948 | .27983 | .5736 | 301 | 22 |
| 39 | 291 | 141 | .8254 | 749 | 21 | 39 | .26976 | .28015 | .5696 | 293 | 21 |
| 40 | .25320 | .26172 | 3.8208 | .96742 | 20 | 40 | .27004 | .28046 | 3.5656 | .96285 | 20 |
| 41 | 348 | 203 | .8163 | 734 | 19 | 41 | 032 | 077 | .5616 | 277 | 19 |
| 42 | 376 | 235 | .8118 | 727 | 18 | 42 | 060 | 109 | .5576 | 269 | 18 |
| 43 | 404 | 266 | .8073 | 719 | 17 | 43 | 088 | 140 | .5536 | 261 | 17 |
| 44 | 432 | 297 | .8028 | 712 | 16 | 44 | 116 | 172 | .5497 | 253 | 16 |
| 45 | .25460 | .26328 | 3.7983 | .96705 | 15 | 45 | .27144 | .28203 | 3.5457 | .96246 | 15 |
| 46 | 488 | 359 | .7938 | 697 | 14 | 46 | 172 | 234 | .5418 | 238 | 14 |
| 47 | 516 | 390 | .7893 | 690 | 13 | 47 | 200 | 266 | .5379 | 230 | 13 |
| 48 | 545 | 421 | .7848 | 682 | 12 | 48 | 228 | 297 | .5339 | 222 | 12 |
| 49 | 573 | 452 | .7804 | 675 | 11 | 49 | 256 | 329 | .5300 | 214 | 11 |
| 50 | .25601 | .26483 | 3.7760 | .96667 | 10 | 50 | .27284 | .28360 | 3.5261 | .96206 | 10 |
| 51 | 629 | 515 | .7715 | 660 | 9 | 51 | 312 | 391 | .5222 | 198 | 9 |
| 52 | 657 | 546 | .7671 | 653 | 8 | 52 | 340 | 423 | .5183 | 190 | 8 |
| 53 | 685 | 577 | .7627 | 645 | 7 | 53 | 368 | 454 | .5144 | 182 | 7 |
| 54 | 713 | 608 | .7583 | 638 | 6 | 54 | 396 | 486 | .5105 | 174 | 6 |
| 55 | .25741 | .26639 | 3.7539 | .96630 | 5 | 55 | .27424 | .28517 | 3.5067 | .96166 | 5 |
| 56 | 769 | 670 | .7495 | 623 | 4 | 56 | 452 | 549 | .5028 | 158 | 4 |
| 57 | 798 | 701 | .7451 | 615 | 3 | 57 | 480 | 580 | .4989 | 150 | 3 |
| 58 | 826 | 733 | .7408 | 608 | 2 | 58 | 508 | 612 | .4951 | 142 | 2 |
| 59 | 854 | 764 | .7364 | 600 | 1 | 59 | 536 | 643 | .4912 | 134 | 1 |
| 60 | .25882 | .26795 | 3.7321 | .96593 | 0 | 60 | .27564 | .28675 | 3.4874 | .96126 | 0 |
| | Cos | Ctn | Tan | Sin | | | Cos | Ctn | Tan | Sin | |

75°

74°

| ' | Sin | Tan | Ctn | Cos | ' | Sin | Tan | Ctn | Cos | ' | |
|----|--------|--------|--------|--------|----|--------|--------|--------|--------|--------|----|
| 0 | .27564 | .28675 | 3.4874 | .96126 | 60 | .29237 | .30573 | 3.2709 | .95630 | 60 | |
| 1 | 592 | 706 | .4836 | 118 | 59 | 1 | 265 | .605 | .2675 | 59 | |
| 2 | 620 | 738 | .4798 | 110 | 58 | 2 | 293 | .637 | .2641 | 58 | |
| 3 | 648 | 769 | .4760 | 102 | 57 | 3 | 321 | .669 | .2607 | 57 | |
| 4 | 676 | 801 | .4722 | 94 | 56 | 4 | 348 | .700 | .2573 | 56 | |
| 5 | .27704 | .28832 | 3.4684 | .96086 | 55 | 5 | .29376 | .30732 | 3.2539 | .95588 | 55 |
| 6 | 731 | 864 | .4646 | 078 | 54 | 6 | 404 | .764 | .2506 | 579 | 54 |
| 7 | 759 | 895 | .4608 | 070 | 53 | 7 | 432 | .796 | .2472 | 571 | 53 |
| 8 | 787 | 927 | .4570 | 062 | 52 | 8 | 460 | .828 | .2438 | 562 | 52 |
| 9 | 815 | 958 | .4533 | 054 | 51 | 9 | 487 | .860 | .2405 | 554 | 51 |
| 10 | .27843 | .28990 | 3.4495 | .96046 | 50 | 10 | .29515 | .30891 | 3.2371 | .95545 | 50 |
| 11 | 871 | .29021 | .4458 | 037 | 49 | 11 | 543 | .923 | .2338 | 536 | 49 |
| 12 | 899 | 053 | .4420 | 029 | 48 | 12 | 571 | .955 | .2305 | 528 | 48 |
| 13 | 927 | 084 | .4383 | 021 | 47 | 13 | 599 | .30987 | .2272 | 519 | 47 |
| 14 | 955 | 116 | .4346 | 013 | 46 | 14 | 626 | .31019 | .2238 | 511 | 46 |
| 15 | .27983 | .29147 | 3.4308 | .96005 | 45 | 15 | .29654 | .31051 | 3.2205 | .95502 | 45 |
| 16 | .28011 | 179 | .4271 | .95997 | 44 | 16 | 682 | .083 | .2172 | 493 | 44 |
| 17 | 039 | 210 | .4234 | .989 | 43 | 17 | 710 | 115 | .2139 | 485 | 43 |
| 18 | 067 | 242 | .4197 | .981 | 42 | 18 | 737 | 147 | .2106 | 476 | 42 |
| 19 | 095 | 274 | .4160 | .972 | 41 | 19 | 765 | 178 | .2073 | 467 | 41 |
| 20 | .28123 | .29305 | 3.4124 | .95964 | 40 | 20 | .29793 | .31210 | 3.2041 | .95459 | 40 |
| 21 | 150 | 337 | .4087 | .956 | 39 | 21 | 821 | 212 | .2008 | 450 | 39 |
| 22 | 178 | 368 | .4050 | .948 | 38 | 22 | 849 | 274 | .1975 | 441 | 38 |
| 23 | 206 | 400 | .4014 | .940 | 37 | 23 | 876 | 306 | .1943 | 433 | 37 |
| 24 | 234 | 432 | .3977 | .931 | 36 | 24 | 904 | 338 | .1910 | 424 | 36 |
| 25 | .28262 | .29463 | 3.3941 | .95923 | 35 | 25 | .29932 | .31370 | 3.1978 | .95415 | 35 |
| 26 | 290 | 495 | .3904 | .915 | 34 | 26 | 960 | 402 | .1845 | 407 | 34 |
| 27 | 318 | 526 | .3868 | .907 | 33 | 27 | .29987 | 434 | .1813 | 398 | 33 |
| 28 | 346 | 558 | .3832 | .898 | 32 | 28 | .30015 | 466 | .1780 | 389 | 32 |
| 29 | 374 | 590 | .3796 | .890 | 31 | 29 | 04 | 498 | .1748 | 380 | 31 |
| 30 | .28402 | .29621 | 3.3759 | .95882 | 30 | 30 | .30071 | .31530 | 3.1716 | .95372 | 30 |
| 31 | 429 | 653 | .3723 | .874 | 29 | 31 | 098 | 562 | .1684 | 363 | 29 |
| 32 | 457 | 685 | .3687 | .865 | 28 | 32 | 126 | 594 | .1652 | 354 | 28 |
| 33 | 485 | 716 | .3652 | .857 | 27 | 33 | 154 | 626 | .1620 | 345 | 27 |
| 34 | 513 | 748 | .3616 | .849 | 26 | 34 | 182 | 658 | .1588 | 337 | 26 |
| 35 | .28541 | .29780 | 3.3580 | .95841 | 25 | 35 | .30209 | .31690 | 3.1556 | .95328 | 25 |
| 36 | 569 | 811 | .3544 | .832 | 24 | 36 | 237 | 722 | .1524 | 319 | 24 |
| 37 | 597 | 843 | .3509 | .824 | 23 | 37 | 265 | 754 | .1492 | 310 | 23 |
| 38 | 625 | 875 | .3473 | .816 | 22 | 38 | 292 | 786 | .1460 | 301 | 22 |
| 39 | 652 | 906 | .3438 | .807 | 21 | 39 | 320 | 818 | .1429 | 293 | 21 |
| 40 | .28680 | .29938 | 3.3402 | .95799 | 20 | 40 | .30348 | .31850 | 3.1397 | .95284 | 20 |
| 41 | 708 | .29970 | .3367 | .791 | 19 | 41 | 376 | 882 | .1366 | 275 | 19 |
| 42 | 736 | .30001 | .3332 | .782 | 18 | 42 | 403 | 914 | .1334 | 266 | 18 |
| 43 | 764 | 033 | .3297 | .774 | 17 | 43 | 431 | 946 | .1303 | 257 | 17 |
| 44 | 792 | 065 | .3261 | .766 | 16 | 44 | 459 | .31978 | .1271 | 248 | 16 |
| 45 | .28820 | .30097 | 3.3226 | .95757 | 15 | 45 | .30486 | .32010 | 3.1240 | .95240 | 15 |
| 46 | 847 | 128 | .3191 | .749 | 14 | 46 | 514 | 042 | .1209 | 231 | 14 |
| 47 | 875 | 160 | .3156 | .740 | 13 | 47 | 512 | 074 | .1178 | 222 | 13 |
| 48 | 903 | 192 | .3122 | .732 | 12 | 48 | 570 | 106 | .1146 | 213 | 12 |
| 49 | 931 | 224 | .3087 | .724 | 11 | 49 | 597 | 139 | .1115 | 204 | 11 |
| 50 | .28959 | .30255 | 3.3052 | .95715 | 10 | 50 | .30625 | .32171 | 3.1084 | .95195 | 10 |
| 51 | .28987 | 287 | .3017 | .707 | 9 | 51 | 653 | 203 | .1053 | 186 | 9 |
| 52 | .29015 | 319 | .2983 | .698 | 8 | 52 | 680 | 235 | .1022 | 177 | 8 |
| 53 | 042 | 351 | .2948 | .690 | 7 | 53 | 708 | 267 | .0991 | 168 | 7 |
| 54 | 070 | 382 | .2914 | .681 | 6 | 54 | 736 | 299 | .0961 | 159 | 6 |
| 55 | .29098 | .30414 | 3.2879 | .95673 | 5 | 55 | .30763 | .32331 | 3.0930 | .95150 | 5 |
| 56 | 126 | 446 | .2845 | .664 | 4 | 56 | 791 | 363 | .0899 | 142 | 4 |
| 57 | 154 | 478 | .2811 | .656 | 3 | 57 | 819 | 396 | .0868 | 133 | 3 |
| 58 | 182 | 509 | .2777 | .647 | 2 | 58 | 846 | 428 | .0838 | 124 | 2 |
| 59 | 209 | 541 | .2743 | .639 | 1 | 59 | 874 | 460 | .0807 | 115 | 1 |
| 60 | .29237 | .30573 | 3.2709 | .95630 | 0 | 60 | .30902 | .32492 | 3.0777 | .95106 | 0 |
| | Cos | Ctn | Tan | Sin | | | Cos | Ctn | Tan | Sin | |

| | Sin | Tan | Ctn | Cos | | | Sin | Tan | Ctn | Cos | |
|-----------|--------|--------|--------|--------|-----------|-----------|--------|--------|--------|--------|-----------|
| 0 | .30302 | .32492 | 3.0777 | .95106 | 60 | 0 | .32557 | .34433 | 2.9042 | .94552 | 60 |
| 1 | 929 | 524 | .0746 | .097 | 59 | 1 | 584 | 465 | .9015 | 542 | 59 |
| 2 | 957 | 556 | .0716 | .088 | 58 | 2 | 612 | 498 | .8987 | 533 | 58 |
| 3 | .30985 | 588 | .0386 | .079 | 57 | 3 | 639 | 530 | .8960 | 523 | 57 |
| 4 | .31012 | 621 | .0655 | .070 | 56 | 4 | 667 | 563 | .8933 | 514 | 56 |
| 5 | .31040 | .32653 | 3.0625 | .95061 | 55 | 5 | .32694 | .34596 | 2.8905 | .94504 | 55 |
| 6 | 068 | 685 | .0595 | .052 | 54 | 6 | 722 | 628 | .8878 | 495 | 54 |
| 7 | 095 | 717 | .0565 | .043 | 53 | 7 | 749 | 661 | .8851 | 485 | 53 |
| 8 | 123 | 749 | .0535 | .033 | 52 | 8 | 777 | 693 | .8824 | 476 | 52 |
| 9 | 151 | 782 | .0505 | .024 | 51 | 9 | 804 | 726 | .8797 | 466 | 51 |
| 10 | .31178 | .32814 | 3.0475 | .95015 | 50 | 10 | .32832 | .34758 | 2.8770 | .94457 | 50 |
| 11 | 206 | 846 | .0445 | .95006 | 49 | 11 | 859 | 791 | .8743 | 447 | 49 |
| 12 | 233 | 878 | .0415 | .94997 | 48 | 12 | 887 | 824 | .8716 | 438 | 48 |
| 13 | 261 | 911 | .0385 | .988 | 47 | 13 | 914 | 856 | .8689 | 428 | 47 |
| 14 | 289 | 943 | .0356 | .979 | 46 | 14 | 942 | 889 | .8662 | 418 | 46 |
| 15 | .31316 | .32975 | 3.0326 | .94970 | 45 | 15 | .32969 | .34922 | 2.8636 | .94409 | 45 |
| 16 | 344 | .33007 | .0296 | .961 | 44 | 16 | .32997 | .954 | .8609 | 399 | 44 |
| 17 | 372 | 040 | .0267 | .952 | 43 | 17 | .33024 | .34987 | .8582 | 390 | 43 |
| 18 | 399 | 072 | .0237 | .943 | 42 | 18 | 051 | .35020 | .8556 | 380 | 42 |
| 19 | 427 | 104 | .0208 | .933 | 41 | 19 | 079 | 052 | .8529 | 370 | 41 |
| 20 | .31454 | .33136 | 3.0178 | .94924 | 40 | 20 | .33106 | .35085 | 2.8502 | .94361 | 40 |
| 21 | 482 | 169 | .0149 | .915 | 39 | 21 | 134 | 118 | .8476 | 351 | 39 |
| 22 | 510 | 201 | .0120 | .906 | 38 | 22 | 161 | 150 | .8449 | 342 | 38 |
| 23 | 537 | 233 | .0090 | .897 | 37 | 23 | 189 | 183 | .8423 | 332 | 37 |
| 24 | 565 | 266 | .0061 | .888 | 36 | 24 | 216 | 216 | .8397 | 322 | 36 |
| 25 | .31593 | .33298 | 3.0032 | .94878 | 35 | 25 | .33244 | .35248 | 2.8370 | .94313 | 35 |
| 26 | 620 | 330 | 3.0003 | .869 | 34 | 26 | 271 | 281 | .8344 | 303 | 34 |
| 27 | 648 | 363 | 2.9974 | .860 | 33 | 27 | 298 | 314 | .8318 | 293 | 33 |
| 28 | 675 | 395 | .9915 | .851 | 32 | 28 | 326 | 346 | .8291 | 284 | 32 |
| 29 | 703 | 427 | .9916 | .842 | 31 | 29 | 353 | 379 | .8265 | 274 | 31 |
| 30 | .31730 | .33460 | 2.9887 | .94832 | 30 | 30 | .33381 | .35412 | 2.8239 | .94264 | 30 |
| 31 | 758 | 492 | .9858 | .823 | 29 | 31 | 408 | 415 | .8213 | 251 | 29 |
| 32 | 786 | 524 | .9829 | .814 | 28 | 32 | 436 | 477 | .8187 | 245 | 28 |
| 33 | 813 | 557 | .9800 | .805 | 27 | 33 | 463 | 510 | .8161 | 235 | 27 |
| 34 | 841 | 589 | .9772 | .795 | 26 | 34 | 490 | 543 | .8135 | 225 | 26 |
| 35 | .31868 | .33621 | 2.9743 | .94786 | 25 | 35 | .33518 | .35576 | 2.8109 | .94215 | 25 |
| 36 | 896 | 654 | .9714 | .777 | 24 | 36 | 545 | 608 | .8083 | 206 | 24 |
| 37 | 923 | 686 | .9686 | .768 | 23 | 37 | 573 | 641 | .8057 | 196 | 23 |
| 38 | 951 | 718 | .9657 | .758 | 22 | 38 | 600 | 674 | .8032 | 186 | 22 |
| 39 | .31979 | 751 | .9629 | .749 | 21 | 39 | 627 | 707 | .8006 | 176 | 21 |
| 40 | .32006 | .33783 | 2.9600 | .94740 | 20 | 40 | .33655 | .35740 | 2.7980 | .94167 | 20 |
| 41 | 034 | 816 | .9572 | .730 | 19 | 41 | 682 | 772 | .7955 | 157 | 19 |
| 42 | 061 | 848 | .9544 | .721 | 18 | 42 | 710 | 805 | .7929 | 147 | 18 |
| 43 | 089 | 881 | .9515 | .712 | 17 | 43 | 737 | 838 | .7903 | 137 | 17 |
| 44 | 116 | 913 | .9487 | .702 | 16 | 44 | 764 | 871 | .7878 | 127 | 16 |
| 45 | .32144 | .33945 | 2.9459 | .94693 | 15 | 45 | .33792 | .35904 | 2.7852 | .94118 | 15 |
| 46 | 171 | .33978 | .9431 | .684 | 14 | 46 | 819 | 937 | .7827 | 108 | 14 |
| 47 | 199 | .34010 | .9403 | .674 | 13 | 47 | 846 | .35969 | .7801 | 098 | 13 |
| 48 | 227 | 043 | .9375 | .665 | 12 | 48 | 874 | .36002 | .7776 | 088 | 12 |
| 49 | 254 | 075 | .9347 | .656 | 11 | 49 | 901 | 035 | .7751 | 078 | 11 |
| 50 | .32282 | .34108 | 2.9319 | .94646 | 10 | 50 | .33929 | .36068 | 2.7725 | .94068 | 10 |
| 51 | 309 | 140 | .9291 | .637 | 9 | 51 | 956 | 101 | .7700 | 058 | 9 |
| 52 | 337 | 173 | .9263 | .627 | 8 | 52 | .33983 | 134 | .7675 | 049 | 8 |
| 53 | 364 | 205 | .9235 | .618 | 7 | 53 | .34011 | 167 | .7650 | 039 | 7 |
| 54 | 392 | 238 | .9208 | .609 | 6 | 54 | 038 | 199 | .7625 | 029 | 6 |
| 55 | .32419 | .34270 | 2.9180 | .94599 | 5 | 55 | .34065 | .36232 | 2.7600 | .94019 | 5 |
| 56 | 447 | 303 | .9152 | .590 | 4 | 56 | 093 | 265 | .7575 | .94009 | 4 |
| 57 | 474 | 335 | .9125 | .580 | 3 | 57 | 120 | 298 | .7550 | .93999 | 3 |
| 58 | 502 | 368 | .9097 | .571 | 2 | 58 | 147 | 331 | .7525 | 989 | 2 |
| 59 | 529 | 400 | .9070 | .561 | 1 | 59 | 175 | 364 | .7500 | 979 | 1 |
| 60 | .32557 | .34433 | 2.9042 | .94552 | 0 | 60 | .34202 | .36397 | 2.7475 | .93969 | 0 |
| | Cos | Ctn | Tan | Sin | | | Cos | Ctn | Tan | Sin | |

71°

70°

| | Sin | Tan | Ctn | Cos | | | Sin | Tan | Ctn | Cos | |
|----|--------|--------|--------|--------|----|----|--------|--------|--------|--------|----|
| 0 | .34202 | .36397 | 2.7475 | .93969 | 60 | 0 | .35837 | .38386 | 2.6051 | .93358 | 60 |
| 1 | 229 | 430 | .7450 | 959 | 59 | 1 | 864 | 420 | .6028 | 348 | 59 |
| 2 | 257 | 463 | .7425 | 949 | 58 | 2 | 891 | 453 | .6006 | 337 | 58 |
| 3 | 284 | 496 | .7400 | 939 | 57 | 3 | 918 | 487 | .5983 | 327 | 57 |
| 4 | 311 | 529 | .7376 | 929 | 56 | 4 | 945 | 520 | .5961 | 316 | 56 |
| 5 | .34339 | .36562 | 2.7351 | .93919 | 55 | 5 | .35973 | .38553 | 2.5938 | .93306 | 55 |
| 6 | 366 | 595 | .7326 | 909 | 54 | 6 | .36000 | 587 | .5916 | 295 | 54 |
| 7 | 393 | 628 | .7302 | 899 | 53 | 7 | 027 | 626 | .5893 | 285 | 53 |
| 8 | 421 | 661 | .7277 | 889 | 52 | 8 | 054 | 654 | .5871 | 274 | 52 |
| 9 | 448 | 694 | .7253 | 879 | 51 | 9 | 081 | 687 | .5848 | 264 | 51 |
| 10 | .34475 | .36727 | 2.7228 | .93869 | 50 | 10 | .36108 | .38721 | 2.5826 | .93253 | 50 |
| 11 | 503 | 760 | .7204 | 859 | 49 | 11 | 135 | 754 | .5804 | 243 | 49 |
| 12 | 530 | 793 | .7179 | 849 | 48 | 12 | 162 | 787 | .5782 | 232 | 48 |
| 13 | 557 | 826 | .7155 | 839 | 47 | 13 | 190 | 821 | .5759 | 222 | 47 |
| 14 | 584 | 859 | .7130 | 829 | 46 | 14 | 217 | 854 | .5737 | 211 | 46 |
| 15 | .34612 | .36892 | 2.7106 | .93819 | 45 | 15 | .36244 | .38888 | 2.5715 | .93201 | 45 |
| 16 | 639 | 925 | .7082 | 809 | 44 | 16 | 271 | 921 | .5693 | 190 | 44 |
| 17 | 666 | 958 | .7058 | 799 | 43 | 17 | 298 | 955 | .5671 | 180 | 43 |
| 18 | 694 | .36991 | .7034 | 789 | 42 | 18 | 325 | .38988 | .5649 | 169 | 42 |
| 19 | 721 | .37024 | .7009 | 779 | 41 | 19 | 352 | .39022 | .5627 | 159 | 41 |
| 20 | .34748 | .37057 | 2.6985 | .93769 | 40 | 20 | .36379 | .39055 | 2.5605 | .93148 | 40 |
| 21 | 775 | 090 | .6961 | 759 | 39 | 21 | 406 | 089 | .5583 | 137 | 39 |
| 22 | 803 | 123 | .6937 | 748 | 38 | 22 | 434 | 122 | .5561 | 127 | 38 |
| 23 | 830 | 157 | .6913 | 738 | 37 | 23 | 461 | 156 | .5539 | 116 | 37 |
| 24 | 857 | 190 | .6889 | 728 | 36 | 24 | 488 | 190 | .5517 | 106 | 36 |
| 25 | .34884 | .37223 | 2.6865 | .93718 | 35 | 25 | .36515 | .39223 | 2.5495 | .93095 | 35 |
| 26 | 912 | 256 | .6841 | 708 | 34 | 26 | 542 | 257 | .5473 | 084 | 34 |
| 27 | 939 | 289 | .6818 | 698 | 33 | 27 | 569 | 290 | .5452 | 074 | 33 |
| 28 | 966 | 322 | .6794 | 688 | 32 | 28 | 596 | 324 | .5430 | 063 | 32 |
| 29 | .34993 | 355 | .6770 | 677 | 31 | 29 | 6 | 357 | .5408 | 052 | 31 |
| 30 | .35021 | .37388 | 2.6746 | .93667 | 30 | 30 | .36650 | .39391 | 2.5386 | .93042 | 30 |
| 31 | 018 | 422 | .6723 | 657 | 29 | 31 | 677 | 425 | .5365 | 031 | 29 |
| 32 | 075 | 455 | .6699 | 647 | 28 | 32 | 704 | 458 | .5343 | 020 | 28 |
| 33 | 102 | 488 | .6675 | 637 | 27 | 33 | 731 | 492 | .5322 | .93010 | 27 |
| 34 | 130 | 521 | .6652 | 626 | 26 | 34 | 758 | 526 | .5300 | .92999 | 26 |
| 35 | .35157 | .37554 | 2.6628 | .93616 | 25 | 35 | .36785 | .39559 | 2.5279 | .92988 | 25 |
| 36 | 181 | 588 | .6605 | 606 | 24 | 36 | 812 | 593 | .5257 | 978 | 24 |
| 37 | 211 | 621 | .6581 | 596 | 23 | 37 | 839 | 626 | .5236 | 967 | 23 |
| 38 | 239 | 654 | .6558 | 585 | 22 | 38 | 867 | 660 | .5214 | 956 | 22 |
| 39 | 266 | 687 | .6534 | 575 | 21 | 39 | 894 | 694 | .5193 | 945 | 21 |
| 40 | .35293 | .37720 | 2.6511 | .93565 | 20 | 40 | .36921 | .39727 | 2.5172 | .92935 | 20 |
| 41 | 320 | 754 | .6488 | 555 | 19 | 41 | 948 | 761 | .5150 | 924 | 19 |
| 42 | 347 | 787 | .6464 | 544 | 18 | 42 | .36975 | 795 | .5129 | 913 | 18 |
| 43 | 375 | 820 | .6441 | 534 | 17 | 43 | .37002 | 829 | .5108 | 902 | 17 |
| 44 | 402 | 853 | .6418 | 524 | 16 | 44 | 029 | 862 | .5086 | 892 | 16 |
| 45 | .35429 | .37887 | 2.6395 | .93514 | 15 | 45 | .37056 | .39896 | 2.5065 | .92881 | 15 |
| 46 | 456 | 920 | .6371 | 503 | 14 | 46 | 083 | 930 | .5044 | 870 | 14 |
| 47 | 484 | 953 | .6348 | 493 | 13 | 47 | 110 | 963 | .5023 | 859 | 13 |
| 48 | 511 | .37986 | .6325 | 483 | 12 | 48 | 137 | .39997 | .5002 | 849 | 12 |
| 49 | 538 | .38020 | .6302 | 472 | 11 | 49 | 164 | .40031 | .4981 | 838 | 11 |
| 50 | .35565 | .38053 | 2.6279 | .93462 | 10 | 50 | .37191 | .40065 | 2.4960 | .92827 | 10 |
| 51 | 592 | 086 | .6256 | 452 | 9 | 51 | 218 | 098 | .4939 | 816 | 9 |
| 52 | 619 | 120 | .6233 | 441 | 8 | 52 | 245 | 132 | .4918 | 805 | 8 |
| 53 | 647 | 153 | .6210 | 431 | 7 | 53 | 272 | 166 | .4897 | 794 | 7 |
| 54 | 674 | 186 | .6187 | 420 | 6 | 54 | 299 | 200 | .4876 | 784 | 6 |
| 55 | .35701 | .38220 | 2.6165 | .93410 | 5 | 55 | .37326 | .40234 | 2.4855 | .92773 | 5 |
| 56 | 728 | 253 | .6142 | 400 | 4 | 56 | 353 | 267 | .4834 | 762 | 4 |
| 57 | 755 | 286 | .6119 | 389 | 3 | 57 | 380 | 301 | .4813 | 751 | 3 |
| 58 | 782 | 320 | .6096 | 379 | 2 | 58 | 407 | 335 | .4792 | 740 | 2 |
| 59 | 810 | 353 | .6074 | 368 | 1 | 59 | 434 | 369 | .4772 | 729 | 1 |
| 60 | .35837 | .38386 | 2.6051 | .93358 | 0 | 60 | .37461 | .40403 | 2.4751 | .92718 | 0 |
| | Cos | Ctn | Tan | Sin | | | Cos | Ctn | Tan | Sin | |

| ° | Sin | Tan | Ctn | Cos | ' | ° | Sin | Tan | Ctn | Cos | ' |
|----|--------|--------|--------|--------|----|----|--------|--------|--------|--------|----|
| 0 | .37461 | .40403 | 2.4751 | .92718 | 60 | 0 | .39073 | .42447 | 2.3559 | .92050 | 60 |
| 1 | 488 | 436 | .4730 | 707 | 59 | 1 | 100 | 482 | .3539 | 039 | 59 |
| 2 | 515 | 470 | .4709 | 697 | 58 | 2 | 127 | 516 | .3520 | 028 | 58 |
| 3 | 542 | 504 | .4689 | 686 | 57 | 3 | 153 | 551 | .3501 | 016 | 57 |
| 4 | 569 | 538 | .4668 | 675 | 56 | 4 | 180 | 585 | .3483 | .02005 | 56 |
| 5 | .37595 | .40572 | 2.4648 | .92664 | 55 | 5 | .39207 | .42619 | 2.3464 | .91994 | 55 |
| 6 | 622 | 606 | .4627 | 653 | 54 | 6 | 234 | 654 | .3445 | 982 | 54 |
| 7 | 649 | 640 | .4606 | 642 | 53 | 7 | 260 | 688 | .3426 | 971 | 53 |
| 8 | 676 | 674 | .4586 | 631 | 52 | 8 | 287 | 722 | .3407 | 959 | 52 |
| 9 | 703 | 707 | .4566 | 620 | 51 | 9 | 314 | 757 | .3388 | 948 | 51 |
| 10 | .37730 | .40741 | 2.4545 | .92609 | 50 | 10 | .39341 | .42791 | 2.3369 | .91936 | 50 |
| 11 | 757 | 775 | .4525 | 598 | 49 | 11 | 367 | 826 | .3351 | 925 | 49 |
| 12 | 784 | 809 | .4504 | 587 | 48 | 12 | 394 | 860 | .3332 | 914 | 48 |
| 13 | 811 | 843 | .4484 | 576 | 47 | 13 | 421 | 894 | .3313 | 902 | 47 |
| 14 | 838 | 877 | .4464 | 565 | 46 | 14 | 448 | 929 | .3294 | 891 | 46 |
| 15 | .37865 | .40911 | 2.4443 | .92554 | 45 | 15 | .39474 | .42963 | 2.3276 | .91879 | 45 |
| 16 | 892 | 945 | .4423 | 543 | 44 | 16 | 501 | .42998 | .3257 | 868 | 44 |
| 17 | 919 | .40979 | .4403 | 532 | 43 | 17 | 528 | .43032 | .3238 | 856 | 43 |
| 18 | 946 | .41013 | .4383 | 521 | 42 | 18 | 555 | 067 | .3220 | 845 | 42 |
| 19 | 973 | 047 | .4362 | 510 | 41 | 19 | 581 | 101 | .3201 | 833 | 41 |
| 20 | .37999 | .41081 | 2.4342 | .92499 | 40 | 20 | .39608 | .43136 | 2.3183 | .91822 | 40 |
| 21 | .38026 | 115 | .4322 | 488 | 39 | 21 | 635 | 170 | .3164 | 810 | 39 |
| 22 | 053 | 149 | .4302 | 477 | 38 | 22 | 661 | 205 | .3146 | 799 | 38 |
| 23 | 080 | 183 | .4282 | 466 | 37 | 23 | 688 | 239 | .3127 | 787 | 37 |
| 24 | 107 | 217 | .4262 | 455 | 36 | 24 | 715 | 274 | .3109 | 775 | 36 |
| 25 | .38134 | .41251 | 2.4242 | .92444 | 35 | 25 | .39741 | .43308 | 2.3090 | .91764 | 35 |
| 26 | 161 | 285 | .4222 | 432 | 34 | 26 | 768 | 343 | .3072 | 752 | 34 |
| 27 | 188 | 319 | .4202 | 421 | 33 | 27 | 795 | 378 | .3053 | 741 | 33 |
| 28 | 215 | 353 | .4182 | 410 | 32 | 28 | 822 | 412 | .3035 | 729 | 32 |
| 29 | 241 | 387 | .4162 | 399 | 31 | 29 | 848 | 447 | .3017 | 718 | 31 |
| 30 | .38268 | .41421 | 2.4142 | .92388 | 30 | 30 | .39875 | .43481 | 2.2998 | .91706 | 30 |
| 31 | 295 | 455 | .4122 | 377 | 29 | 31 | 902 | 516 | .2980 | 694 | 29 |
| 32 | 322 | 490 | .4102 | 366 | 28 | 32 | 928 | 550 | .2962 | 683 | 28 |
| 33 | 349 | 524 | .4083 | 355 | 27 | 33 | 955 | 585 | .2944 | 671 | 27 |
| 34 | 376 | 558 | .4063 | 343 | 26 | 34 | .39982 | 620 | .2925 | 660 | 26 |
| 35 | .38403 | .41592 | 2.4043 | .92332 | 25 | 35 | .40008 | .43654 | 2.2907 | .91648 | 25 |
| 36 | 430 | 626 | .4023 | 321 | 24 | 36 | 035 | 689 | .2889 | 636 | 24 |
| 37 | 456 | 660 | .4004 | 310 | 23 | 37 | 062 | 724 | .2871 | 625 | 23 |
| 38 | 483 | 694 | .3984 | 299 | 22 | 38 | 088 | 758 | .2853 | 613 | 22 |
| 39 | 510 | 728 | .3964 | 287 | 21 | 39 | 115 | 793 | .2835 | 601 | 21 |
| 40 | .38537 | .41763 | 2.3945 | .92276 | 20 | 40 | .40141 | .43828 | 2.2817 | .91590 | 20 |
| 41 | 564 | 797 | .3925 | 265 | 19 | 41 | 168 | 862 | .2799 | 578 | 19 |
| 42 | 591 | 831 | .3906 | 254 | 18 | 42 | 195 | 897 | .2781 | 566 | 18 |
| 43 | 617 | 865 | .3886 | 243 | 17 | 43 | 221 | 932 | .2763 | 555 | 17 |
| 44 | 644 | 899 | .3867 | 231 | 16 | 44 | 248 | .43966 | .2745 | 543 | 16 |
| 45 | .38671 | .41933 | 2.3847 | .92220 | 15 | 45 | .40275 | .44001 | 2.2727 | .91531 | 15 |
| 46 | 698 | .41968 | .3828 | 209 | 14 | 46 | 301 | 036 | .2709 | 519 | 14 |
| 47 | 725 | .42002 | .3808 | 198 | 13 | 47 | 328 | 071 | .2691 | 508 | 13 |
| 48 | 752 | 036 | .3789 | 186 | 12 | 48 | 355 | 105 | .2673 | 496 | 12 |
| 49 | 778 | 070 | .3770 | 175 | 11 | 49 | 381 | 140 | .2655 | 484 | 11 |
| 50 | .38805 | .42105 | 2.3750 | .92164 | 10 | 50 | .40408 | .44175 | 2.2637 | .91472 | 10 |
| 51 | 832 | 139 | .3731 | 152 | 9 | 51 | 434 | 210 | .2620 | 461 | 9 |
| 52 | 859 | 173 | .3712 | 141 | 8 | 52 | 461 | 244 | .2602 | 449 | 8 |
| 53 | 886 | 207 | .3693 | 130 | 7 | 53 | 488 | 279 | .2584 | 437 | 7 |
| 54 | 912 | 242 | .3673 | 119 | 6 | 54 | 514 | 314 | .2566 | 425 | 6 |
| 55 | .38939 | .42276 | 2.3654 | .92107 | 5 | 55 | .40541 | .44349 | 2.2549 | .91414 | 5 |
| 56 | 966 | 310 | .3635 | 096 | 4 | 56 | 567 | 384 | .2531 | 402 | 4 |
| 57 | .38993 | 345 | .3616 | 085 | 3 | 57 | 594 | 418 | .2513 | 390 | 3 |
| 58 | .39020 | 379 | .3597 | 073 | 2 | 58 | 621 | 453 | .2496 | 378 | 2 |
| 59 | 046 | 413 | .3578 | 062 | 1 | 59 | 647 | 488 | .2478 | 366 | 1 |
| 60 | .39073 | .42447 | 2.3559 | .92050 | 0 | 60 | .40674 | .44523 | 2.2460 | .91355 | 0 |
| | Cos | Ctn | Tan | Sin | ' | | Cos | Ctn | Tan | Sin | ' |

| ' | Sin | Tan | Ctn | Cos | ' | Sin | Tan | Ctn | Cos | ' |
|----|--------|--------|--------|--------|----|-----|--------|--------|--------|--------|
| 0 | .40674 | .44523 | 2.2460 | .91355 | 60 | 0 | .42262 | .46631 | 2.1445 | .90631 |
| 1 | 700 | 558 | .2443 | 343 | 59 | 1 | 288 | 660 | .1429 | 618 |
| 2 | 727 | 593 | .2425 | 331 | 58 | 2 | 315 | 702 | .1413 | 606 |
| 3 | 753 | 627 | .2408 | 319 | 57 | 3 | 341 | 737 | .1396 | 594 |
| 4 | 780 | 662 | .2390 | 307 | 56 | 4 | 367 | 772 | .1380 | 582 |
| 5 | .40806 | .44697 | 2.2373 | .91295 | 55 | 5 | .42394 | .46808 | 2.1364 | .90569 |
| 6 | 833 | 732 | .2355 | 283 | 54 | 6 | 420 | 843 | .1348 | 557 |
| 7 | 860 | 767 | .2338 | 272 | 53 | 7 | 446 | 879 | .1332 | 545 |
| 8 | 886 | 802 | .2320 | 260 | 52 | 8 | 473 | 914 | .1315 | 532 |
| 9 | 913 | 837 | .2303 | 248 | 51 | 9 | 499 | 950 | .1299 | 520 |
| 10 | .40939 | .44872 | 2.2286 | .91236 | 50 | 10 | .42525 | .46985 | 2.1283 | .90507 |
| 11 | 966 | 907 | .2268 | 224 | 49 | 11 | 552 | .47021 | .1267 | 495 |
| 12 | .40992 | 942 | .2251 | 212 | 48 | 12 | 578 | 056 | .1251 | 483 |
| 13 | .41019 | .44977 | 2.234 | 200 | 47 | 13 | 604 | 092 | .1235 | 470 |
| 14 | 045 | .45012 | .2216 | 188 | 46 | 14 | 631 | 128 | .1219 | 458 |
| 15 | .41072 | .45047 | 2.2199 | .91170 | 45 | 15 | 42657 | .47163 | 2.1203 | .90446 |
| 16 | 098 | 082 | .2182 | 164 | 44 | 16 | 683 | 199 | .1187 | 433 |
| 17 | 125 | 117 | .2165 | 152 | 43 | 17 | 709 | 234 | .1171 | 421 |
| 18 | 151 | 152 | .2148 | 140 | 42 | 18 | 736 | 270 | .1155 | 408 |
| 19 | 178 | 187 | .2130 | 128 | 41 | 19 | 762 | 305 | .1139 | 396 |
| 20 | .41204 | .45222 | 2.2113 | .91116 | 40 | 20 | 42788 | 47341 | 2.1123 | .90383 |
| 21 | 231 | 257 | .2096 | 101 | 39 | 21 | 815 | 377 | .1107 | 371 |
| 22 | 257 | 292 | .2079 | 092 | 38 | 22 | 841 | 412 | .1092 | 358 |
| 23 | 284 | 327 | .2062 | 080 | 37 | 23 | 867 | 448 | .1076 | 346 |
| 24 | 310 | 362 | .2045 | 068 | 36 | 24 | 894 | 483 | .1060 | 334 |
| 25 | .41337 | .45397 | 2.2028 | .91056 | 35 | 25 | .42920 | .47519 | 2.1044 | .90321 |
| 26 | 363 | 432 | .2011 | 044 | 34 | 26 | 916 | 555 | .1028 | 309 |
| 27 | 390 | 467 | .1994 | 032 | 33 | 27 | 972 | 590 | .1013 | 296 |
| 28 | 416 | 502 | .1977 | 020 | 32 | 28 | 42999 | 626 | .0997 | 284 |
| 29 | 443 | 538 | .1960 | .91008 | 31 | 29 | 43025 | 662 | .0981 | 271 |
| 30 | .41469 | .45573 | 2.1943 | .90996 | 30 | 30 | .43051 | .47698 | 2.0965 | .90259 |
| 31 | 496 | 608 | .1926 | 984 | 29 | 31 | 077 | 733 | .0950 | 246 |
| 32 | 522 | 643 | .1909 | 972 | 28 | 32 | 104 | 769 | .0934 | 233 |
| 33 | 549 | 678 | .1892 | 960 | 27 | 33 | 130 | 805 | .0918 | 221 |
| 34 | 575 | 713 | .1876 | 948 | 26 | 34 | 156 | 840 | .0903 | 208 |
| 35 | .41602 | .45748 | 2.1859 | .90936 | 25 | 35 | 43182 | .47876 | 2.0887 | .90196 |
| 36 | 628 | 781 | .1842 | 924 | 24 | 36 | 209 | 912 | .0872 | 183 |
| 37 | 655 | 819 | .1825 | 911 | 23 | 37 | 235 | 948 | .0856 | 171 |
| 38 | 681 | 854 | .1808 | 899 | 22 | 38 | 261 | 47984 | .0840 | 158 |
| 39 | 707 | 889 | .1792 | 887 | 21 | 39 | 287 | 48019 | .0825 | 146 |
| 40 | .41734 | .45924 | 2.1775 | .90875 | 20 | 40 | 43313 | .48055 | 2.0809 | .90133 |
| 41 | 760 | 960 | .1758 | 863 | 19 | 41 | 340 | 091 | .0794 | 120 |
| 42 | 787 | 45995 | .1742 | 851 | 18 | 42 | 366 | 127 | .0778 | 108 |
| 43 | 813 | .46030 | .1725 | 839 | 17 | 43 | 392 | 163 | .0763 | 095 |
| 44 | 840 | 065 | .1708 | 826 | 16 | 44 | 418 | 198 | .0748 | 082 |
| 45 | .41866 | .46101 | 2.1692 | .90814 | 15 | 45 | 43445 | 48234 | 2.0732 | .90070 |
| 46 | 892 | 136 | .1675 | 802 | 14 | 46 | 471 | 270 | .0717 | 057 |
| 47 | 919 | 171 | .1659 | 790 | 13 | 47 | 497 | 306 | .0701 | 045 |
| 48 | 945 | 206 | .1642 | 778 | 12 | 48 | 523 | 342 | .0686 | 032 |
| 49 | 972 | 242 | .1625 | 766 | 11 | 49 | 549 | 378 | .0671 | 019 |
| 50 | .41998 | .46277 | 2.1609 | .90753 | 10 | 50 | 43575 | .48414 | 2.0655 | .90007 |
| 51 | .42024 | 312 | .1592 | 741 | 9 | 51 | 602 | 450 | .0640 | .89994 |
| 52 | 051 | 348 | .1576 | 729 | 8 | 52 | 628 | 486 | .0625 | 981 |
| 53 | 077 | 383 | .1560 | 717 | 7 | 53 | 654 | 521 | .0609 | 968 |
| 54 | 104 | 418 | .1543 | 704 | 6 | 54 | 680 | 557 | .0594 | 956 |
| 55 | .42130 | .46454 | 2.1527 | .90692 | 5 | 55 | 43706 | .48593 | 2.0579 | .89943 |
| 56 | 156 | 489 | .1510 | 680 | 4 | 56 | 733 | 629 | .0564 | 930 |
| 57 | 183 | 525 | .1494 | 668 | 3 | 57 | 759 | 665 | .0549 | 918 |
| 58 | 209 | 560 | .1478 | 655 | 2 | 58 | 785 | 701 | .0533 | 905 |
| 59 | 235 | 595 | .1461 | 643 | 1 | 59 | 811 | 737 | .0518 | 892 |
| 60 | .42262 | .46631 | 2.1445 | .90631 | 0 | 60 | .43837 | 48773 | 2.0503 | .89879 |
| | Cos | Ctn | Tan | Sin | | | Cos | Ctn | Tan | Sin |

| | Sin | Tan | Ctn | Cos | | | Sin | Tan | Ctn | Cos | |
|-----------|--------|--------|--------|--------|-----------|-----------|--------|--------|--------|--------|-----------|
| 0 | .43837 | .48773 | 2.0503 | .89879 | 60 | 0 | .45399 | .50953 | 1.9626 | .89101 | 60 |
| 1 | 863 | 809 | .0188 | 867 | 59 | 1 | 425 | .50989 | .9612 | 087 | 59 |
| 2 | 889 | 845 | .0473 | 854 | 58 | 2 | 451 | .51026 | .9598 | 074 | 58 |
| 3 | 916 | 881 | .0458 | 841 | 57 | 3 | 477 | 063 | .9584 | 061 | 57 |
| 4 | 942 | 917 | .0443 | 828 | 56 | 4 | 503 | 099 | .9570 | 048 | 56 |
| 5 | .43968 | .48953 | 2.0428 | .89816 | 55 | 5 | .45529 | .51136 | 1.9556 | .89035 | 55 |
| 6 | .43994 | .48989 | .0413 | 803 | 54 | 6 | 554 | 173 | .9512 | 021 | 54 |
| 7 | .44020 | .49026 | .0398 | 790 | 53 | 7 | 580 | 209 | .9528 | .89008 | 53 |
| 8 | 016 | 062 | .0383 | 777 | 52 | 8 | 606 | 246 | .9514 | .88995 | 52 |
| 9 | 072 | 098 | .0368 | 764 | 51 | 9 | 632 | 283 | .9500 | 981 | 51 |
| 10 | .44098 | .49134 | 2.0353 | .89752 | 50 | 10 | .45658 | .51319 | 1.9486 | .88968 | 50 |
| 11 | 124 | 170 | .0338 | 739 | 49 | 11 | 684 | 356 | .9472 | 955 | 49 |
| 12 | 151 | 206 | .0323 | 726 | 48 | 12 | 710 | 393 | .9458 | 942 | 48 |
| 13 | 177 | 212 | .0308 | 713 | 47 | 13 | 736 | 430 | .9444 | 928 | 47 |
| 14 | 203 | 278 | .0293 | 700 | 46 | 14 | 762 | 467 | .9430 | 915 | 46 |
| 15 | .44229 | .49315 | 2.0278 | .89687 | 45 | 15 | .45787 | .51503 | 1.9416 | .88902 | 45 |
| 16 | 255 | 351 | .0263 | 674 | 44 | 16 | 813 | 519 | .9402 | 888 | 44 |
| 17 | 281 | 387 | .0248 | 662 | 43 | 17 | 839 | 577 | .9388 | 875 | 43 |
| 18 | 307 | 423 | .0233 | 649 | 42 | 18 | 865 | 614 | .9375 | 862 | 42 |
| 19 | 333 | 459 | .0219 | 636 | 41 | 19 | 891 | 651 | .9361 | 848 | 41 |
| 20 | .44359 | .49495 | 2.0204 | .89623 | 40 | 20 | .45917 | .51688 | 1.9347 | .88835 | 40 |
| 21 | 385 | 532 | .0189 | 610 | 39 | 21 | 912 | 721 | .9333 | 822 | 39 |
| 22 | 411 | 568 | .0174 | 597 | 38 | 22 | 968 | 761 | .9319 | 808 | 38 |
| 23 | 437 | 604 | .0160 | 584 | 37 | 23 | .45994 | 798 | .9306 | 795 | 37 |
| 24 | 464 | 640 | .0145 | 571 | 36 | 24 | .46020 | 835 | .9292 | 782 | 36 |
| 25 | .44490 | .49677 | 2.0130 | .89558 | 35 | 25 | .46046 | .51872 | 1.9278 | .88768 | 35 |
| 26 | 516 | 713 | .0115 | 545 | 34 | 26 | 072 | 909 | .9265 | 755 | 34 |
| 27 | 542 | 749 | .0101 | 532 | 33 | 27 | 097 | 946 | .9251 | 741 | 33 |
| 28 | 568 | 786 | .0086 | 519 | 32 | 28 | 123 | .51983 | .9237 | 728 | 32 |
| 29 | 594 | 822 | .0072 | 506 | 31 | 29 | 149 | .52020 | .9223 | 715 | 31 |
| 30 | .44620 | .49858 | 2.0057 | .89493 | 30 | 30 | .46175 | .52057 | 1.9210 | .88701 | 30 |
| 31 | 616 | 894 | .0042 | 480 | 29 | 31 | 201 | 094 | .9196 | 688 | 29 |
| 32 | 672 | 931 | .0028 | 467 | 28 | 32 | 226 | 131 | .9183 | 674 | 28 |
| 33 | 698 | .49967 | 2.0013 | 454 | 27 | 33 | 252 | 168 | .9169 | 661 | 27 |
| 34 | 724 | .50001 | 1.9999 | 441 | 26 | 34 | 278 | 205 | .9155 | 647 | 26 |
| 35 | .44750 | .50040 | 1.9984 | .89428 | 25 | 35 | .46304 | .52242 | 1.9142 | .88634 | 25 |
| 36 | 776 | 076 | .9970 | 415 | 24 | 36 | 330 | 279 | .9128 | 620 | 24 |
| 37 | 802 | 113 | .9955 | 402 | 23 | 37 | 355 | 316 | .9115 | 607 | 23 |
| 38 | 828 | 149 | .9941 | 389 | 22 | 38 | 381 | 353 | .9101 | 593 | 22 |
| 39 | 854 | 185 | .9926 | 376 | 21 | 39 | 407 | 390 | .9088 | 580 | 21 |
| 40 | .44880 | .50222 | 1.9912 | .89363 | 20 | 40 | .46433 | .52427 | 1.9074 | .88566 | 20 |
| 41 | 906 | 258 | .9897 | 350 | 19 | 41 | 458 | 464 | .9061 | 553 | 19 |
| 42 | 932 | 295 | .9883 | 337 | 18 | 42 | 484 | 501 | .9047 | 539 | 18 |
| 43 | 958 | 331 | .9868 | 324 | 17 | 43 | 510 | 538 | .9034 | 526 | 17 |
| 44 | .44984 | 368 | .9854 | 311 | 16 | 44 | 536 | 575 | .9020 | 512 | 16 |
| 45 | .45010 | .50404 | 1.9840 | .89298 | 15 | 45 | .46561 | .52613 | 1.9007 | .88499 | 15 |
| 46 | 036 | 411 | .9825 | 285 | 14 | 46 | 587 | 650 | .8993 | 485 | 14 |
| 47 | 062 | 477 | .9811 | 272 | 13 | 47 | 613 | 687 | .8980 | 472 | 13 |
| 48 | 088 | 514 | .9797 | 259 | 12 | 48 | 639 | 724 | .8967 | 458 | 12 |
| 49 | 114 | 550 | .9782 | 245 | 11 | 49 | 664 | 761 | .8953 | 445 | 11 |
| 50 | .45140 | .50587 | 1.9768 | .89232 | 10 | 50 | .46690 | .52798 | 1.8940 | .88431 | 10 |
| 51 | 166 | 623 | .9754 | 219 | 9 | 51 | 716 | 836 | .8927 | 417 | 9 |
| 52 | 192 | 660 | .9740 | 206 | 8 | 52 | 742 | 873 | .8913 | 404 | 8 |
| 53 | 218 | 696 | .9725 | 193 | 7 | 53 | 767 | 910 | .8900 | 390 | 7 |
| 54 | 243 | 733 | .9711 | 180 | 6 | 54 | 793 | 947 | .8887 | 377 | 6 |
| 55 | .45269 | .50769 | 1.9697 | .89167 | 5 | 55 | .46819 | .52985 | 1.8873 | .88363 | 5 |
| 56 | 295 | 806 | .9683 | 153 | 4 | 56 | 844 | .53022 | .8860 | 349 | 4 |
| 57 | 321 | 843 | .9669 | 140 | 3 | 57 | 870 | 059 | .8847 | 336 | 3 |
| 58 | 347 | 879 | .9654 | 127 | 2 | 58 | 896 | 096 | .8834 | 322 | 2 |
| 59 | 373 | 916 | .9640 | 114 | 1 | 59 | 921 | 134 | .8820 | 308 | 1 |
| 60 | .45399 | .50953 | 1.9626 | .89101 | 0 | 60 | .46947 | .53171 | 1.8807 | .88295 | 0 |
| | Cos | Ctn | Tan | Sin | | | Cos | Ctn | Tan | Sin | |

63°

62°

| | Sin | Tan | Ctn | Cos | | | Sin | Tan | Ctn | Cos | |
|----|--------|--------|--------|--------|----|----|--------|--------|--------|--------|----|
| 0 | .46947 | .53171 | 1.8807 | .88295 | 60 | 0 | .48481 | .55431 | 1.8040 | .87462 | 60 |
| 1 | .973 | 208 | .8794 | 281 | 59 | 1 | .506 | .469 | .8028 | 448 | 59 |
| 2 | .46999 | 246 | .8781 | 267 | 58 | 2 | .532 | .507 | .8016 | 434 | 58 |
| 3 | .47024 | 283 | .8768 | 254 | 57 | 3 | .557 | .515 | .8003 | 420 | 57 |
| 4 | .050 | 320 | .8755 | 240 | 56 | 4 | .583 | .583 | .7991 | 406 | 56 |
| 5 | .47076 | .53358 | 1.8741 | .88226 | 55 | 5 | .48608 | .55621 | 1.7979 | .87391 | 55 |
| 6 | .101 | 395 | .8728 | 213 | 51 | 6 | .634 | .659 | .7966 | 377 | 54 |
| 7 | .127 | 432 | .8715 | 199 | 53 | 7 | .659 | .697 | .7954 | 363 | 53 |
| 8 | .153 | 470 | .8702 | 185 | 52 | 8 | .684 | .736 | .7942 | 349 | 52 |
| 9 | .178 | 507 | .8689 | 172 | 51 | 9 | .710 | .774 | .7930 | 335 | 51 |
| 10 | .47204 | .53545 | 1.8676 | .88158 | 50 | 10 | .48735 | .55812 | 1.7917 | .87321 | 50 |
| 11 | .229 | 582 | .8663 | 144 | 49 | 11 | .761 | .850 | .7905 | 306 | 49 |
| 12 | .255 | 620 | .8650 | 130 | 48 | 12 | .786 | .888 | .7893 | 292 | 48 |
| 13 | .281 | 657 | .8637 | 117 | 47 | 13 | .811 | .926 | .7881 | 278 | 47 |
| 14 | .306 | 694 | .8624 | 103 | 46 | 14 | .837 | .55964 | .7868 | 264 | 46 |
| 15 | .47332 | .53732 | 1.8611 | .88089 | 45 | 15 | .48862 | .56003 | 1.7856 | .87250 | 45 |
| 16 | .358 | 769 | .8598 | 075 | 44 | 16 | .888 | .011 | .7844 | 235 | 44 |
| 17 | .383 | 807 | .8585 | 062 | 43 | 17 | .915 | .079 | .7832 | 221 | 43 |
| 18 | .409 | 844 | .8572 | 048 | 42 | 18 | .938 | .117 | .7820 | 207 | 42 |
| 19 | .434 | 882 | .8559 | 034 | 41 | 19 | .964 | .156 | .7808 | 193 | 41 |
| 20 | .47460 | .53920 | 1.8546 | .88020 | 40 | 20 | .48989 | .56194 | 1.7796 | .87178 | 40 |
| 21 | .486 | .957 | .8533 | .88009 | 39 | 21 | .49014 | 2.2 | .7783 | 164 | 39 |
| 22 | .511 | .53995 | .8520 | .87993 | 38 | 22 | .040 | 270 | .7771 | 150 | 38 |
| 23 | .537 | .54032 | .8507 | .87979 | 37 | 23 | .065 | 309 | .7759 | 136 | 37 |
| 24 | .562 | .070 | .8495 | .87965 | 36 | 24 | .090 | 347 | .7747 | 121 | 36 |
| 25 | .47588 | .54107 | 1.8482 | .87951 | 35 | 25 | .49116 | .56385 | 1.7735 | .87107 | 35 |
| 26 | .614 | 145 | .8469 | .87937 | 34 | 26 | .141 | .424 | .7723 | 093 | 34 |
| 27 | .639 | 183 | .8456 | .87923 | 33 | 27 | .166 | .462 | .7711 | 079 | 33 |
| 28 | .665 | 220 | .8443 | .87909 | 32 | 28 | .192 | .501 | .7699 | 064 | 32 |
| 29 | .690 | 258 | .8430 | .87896 | 31 | 29 | .217 | .539 | .7687 | 050 | 31 |
| 30 | .47716 | .54296 | 1.8418 | .87882 | 30 | 30 | .49242 | .56577 | 1.7675 | .87036 | 30 |
| 31 | .741 | 343 | .8405 | .87868 | 29 | 31 | .268 | .616 | .7663 | 021 | 29 |
| 32 | .767 | 371 | .8392 | .87854 | 28 | 32 | .293 | .651 | .7651 | .87007 | 28 |
| 33 | .793 | 409 | .8379 | .87840 | 27 | 33 | .318 | .693 | .7639 | .86993 | 27 |
| 34 | .818 | 446 | .8367 | .87826 | 26 | 34 | .344 | .731 | .7627 | .86978 | 26 |
| 35 | .47844 | .54484 | 1.8354 | .87812 | 25 | 35 | .49369 | .56769 | 1.7615 | .86964 | 25 |
| 36 | .869 | 522 | .8341 | .87798 | 24 | 36 | .394 | .808 | .7603 | .86949 | 24 |
| 37 | .895 | 560 | .8329 | .87784 | 23 | 37 | .419 | .846 | .7591 | .86935 | 23 |
| 38 | .920 | 597 | .8316 | .87770 | 22 | 38 | .445 | .885 | .7579 | .86921 | 22 |
| 39 | .946 | 635 | .8303 | .87756 | 21 | 39 | .470 | .923 | .7567 | .86906 | 21 |
| 40 | .47971 | .54673 | 1.8291 | .87743 | 20 | 40 | .49495 | .56962 | 1.7556 | .86892 | 20 |
| 41 | .47997 | .711 | .8278 | .87729 | 19 | 41 | .521 | .57000 | .7544 | .86878 | 19 |
| 42 | .48022 | .748 | .8265 | .87715 | 18 | 42 | .546 | .039 | .7532 | .86863 | 18 |
| 43 | .048 | .786 | .8253 | .87701 | 17 | 43 | .571 | .078 | .7520 | .86849 | 17 |
| 44 | .073 | .824 | .8240 | .87687 | 16 | 44 | .596 | .116 | .7508 | .86834 | 16 |
| 45 | .48099 | .54862 | 1.8228 | .87673 | 15 | 45 | .49622 | .57155 | 1.7496 | .86820 | 15 |
| 46 | .124 | .900 | .8215 | .87659 | 14 | 46 | .647 | .193 | .7485 | .86805 | 14 |
| 47 | .150 | .938 | .8202 | .87645 | 13 | 47 | .672 | .232 | .7473 | .86791 | 13 |
| 48 | .175 | .54975 | .8190 | .87631 | 12 | 48 | .697 | .271 | .7461 | .86777 | 12 |
| 49 | .201 | .55013 | .8177 | .87617 | 11 | 49 | .723 | .309 | .7449 | .86762 | 11 |
| 50 | .48226 | .55051 | 1.8165 | .87603 | 10 | 50 | .49748 | .57348 | 1.7437 | .86748 | 10 |
| 51 | .252 | .089 | .8152 | .87589 | 9 | 51 | .773 | .386 | .7426 | .86733 | 9 |
| 52 | .277 | .127 | .8140 | .87575 | 8 | 52 | .798 | .425 | .7414 | .86719 | 8 |
| 53 | .303 | .165 | .8127 | .87561 | 7 | 53 | .824 | .464 | .7402 | .86704 | 7 |
| 54 | .328 | .203 | .8115 | .87546 | 6 | 54 | .849 | .503 | .7391 | .86690 | 6 |
| 55 | .48354 | .55241 | 1.8103 | .87532 | 5 | 55 | .49874 | .57541 | 1.7379 | .86675 | 5 |
| 56 | .379 | .279 | .8090 | .87518 | 4 | 56 | .899 | .580 | .7367 | .86661 | 4 |
| 57 | .405 | .317 | .8078 | .87504 | 3 | 57 | .924 | .619 | .7355 | .86646 | 3 |
| 58 | .430 | .355 | .8065 | .87490 | 2 | 58 | .950 | .657 | .7344 | .86632 | 2 |
| 59 | .456 | .393 | .8053 | .87476 | 1 | 59 | .49975 | .696 | .7332 | .86617 | 1 |
| 60 | .48481 | .55431 | 1.8040 | .87462 | 0 | 60 | .50000 | .57735 | 1.7321 | .86603 | 0 |
| | Cos | Ctn | Tan | Sin | | | Cos | Ctn | Tan | Sin | |

61°

60°

| | Sin | Tan | Ctn | Cos | | Sin | Tan | Ctn | Cos | | |
|----|--------|--------|--------|--------|----|-----|--------|--------|--------|--------|----|
| 0 | .50000 | .57735 | 1.7321 | .86603 | 60 | 0 | .51504 | .60086 | 1.6643 | .85717 | 60 |
| 1 | .025 | .774 | .7309 | .588 | 59 | 1 | .529 | .628 | .6632 | .702 | 59 |
| 2 | .050 | .813 | .7297 | .573 | 58 | 2 | .554 | .665 | .6621 | .687 | 58 |
| 3 | .076 | .851 | .7286 | .559 | 57 | 3 | .579 | .695 | .6610 | .672 | 57 |
| 4 | .101 | .890 | .7274 | .544 | 56 | 4 | .604 | .725 | .6599 | .657 | 56 |
| 5 | .50126 | .57929 | 1.7262 | .86530 | 55 | 5 | .51628 | .60284 | 1.6588 | .85642 | 55 |
| 6 | .151 | .57968 | .7251 | .515 | 54 | 6 | .653 | .324 | .6577 | .627 | 54 |
| 7 | .176 | .58007 | .7239 | .501 | 53 | 7 | .678 | .364 | .6566 | .612 | 53 |
| 8 | .201 | .046 | .7228 | .486 | 52 | 8 | .703 | .403 | .6555 | .597 | 52 |
| 9 | .227 | .085 | .7216 | .471 | 51 | 9 | .728 | .443 | .6545 | .582 | 51 |
| 10 | .50252 | .58124 | 1.7205 | .86457 | 50 | 10 | .51753 | .60483 | 1.6534 | .85567 | 50 |
| 11 | .277 | .162 | .7193 | .442 | 49 | 11 | .778 | .522 | .6523 | .551 | 49 |
| 12 | .302 | .201 | .7182 | .427 | 48 | 12 | .803 | .562 | .6512 | .536 | 48 |
| 13 | .327 | .240 | .7170 | .413 | 47 | 13 | .828 | .602 | .6501 | .521 | 47 |
| 14 | .352 | .279 | .7159 | .398 | 46 | 14 | .852 | .642 | .6490 | .506 | 46 |
| 15 | .50377 | .58318 | 1.7147 | .86384 | 45 | 15 | .51877 | .60681 | 1.6479 | .85491 | 45 |
| 16 | .403 | .357 | .7136 | .369 | 44 | 16 | .902 | .721 | .6469 | .476 | 44 |
| 17 | .428 | .396 | .7124 | .354 | 43 | 17 | .927 | .761 | .6458 | .461 | 43 |
| 18 | .453 | .435 | .7113 | .340 | 42 | 18 | .952 | .801 | .6447 | .446 | 42 |
| 19 | .478 | .474 | .7102 | .325 | 41 | 19 | .51977 | .841 | .6436 | .431 | 41 |
| 20 | .50503 | .58513 | 1.7090 | .86310 | 40 | 20 | .52002 | .60881 | 1.6426 | .85416 | 40 |
| 21 | .528 | .552 | .7079 | .295 | 39 | 21 | .026 | .921 | .6415 | .401 | 39 |
| 22 | .553 | .591 | .7067 | .281 | 38 | 22 | .051 | .60960 | .6404 | .385 | 38 |
| 23 | .578 | .631 | .7056 | .266 | 37 | 23 | .076 | .61000 | .6393 | .370 | 37 |
| 24 | .603 | .670 | .7045 | .251 | 36 | 24 | .101 | .040 | .6383 | .355 | 36 |
| 25 | .50628 | .58709 | 1.7033 | .86237 | 35 | 25 | .52126 | .61080 | 1.6372 | .85340 | 35 |
| 26 | .654 | .748 | .7022 | .222 | 34 | 26 | .151 | .120 | .6361 | .325 | 34 |
| 27 | .679 | .787 | .7011 | .207 | 33 | 27 | .175 | .160 | .6351 | .310 | 33 |
| 28 | .704 | .826 | .6999 | .192 | 32 | 28 | .200 | .200 | .6340 | .294 | 32 |
| 29 | .729 | .865 | .6988 | .178 | 31 | 29 | .225 | .240 | .6329 | .279 | 31 |
| 30 | .50754 | .58905 | 1.6977 | .86163 | 30 | 30 | .52250 | .61280 | 1.6319 | .85264 | 30 |
| 31 | .779 | .944 | .6965 | .148 | 29 | 31 | .275 | .320 | .6308 | .249 | 29 |
| 32 | .804 | .58983 | .6954 | .133 | 28 | 32 | .299 | .360 | .6297 | .234 | 28 |
| 33 | .829 | .59022 | .6943 | .119 | 27 | 33 | .324 | .400 | .6287 | .218 | 27 |
| 34 | .854 | .061 | .6932 | .104 | 26 | 34 | .349 | .440 | .6276 | .203 | 26 |
| 35 | .50879 | .59101 | 1.6920 | .86089 | 25 | 35 | .52374 | .61480 | 1.6265 | .85188 | 25 |
| 36 | .904 | .140 | .6909 | .074 | 24 | 36 | .399 | .520 | .6255 | .173 | 24 |
| 37 | .929 | .179 | .6898 | .059 | 23 | 37 | .423 | .561 | .6244 | .157 | 23 |
| 38 | .954 | .218 | .6887 | .045 | 22 | 38 | .447 | .601 | .6234 | .142 | 22 |
| 39 | .50979 | .258 | .6875 | .030 | 21 | 39 | .473 | .641 | .6223 | .127 | 21 |
| 40 | .51004 | .59297 | 1.6864 | .86015 | 20 | 40 | .52498 | .61681 | 1.6212 | .85112 | 20 |
| 41 | .029 | .336 | .6853 | .86000 | 19 | 41 | .522 | .721 | .6202 | .096 | 19 |
| 42 | .054 | .376 | .6842 | .85985 | 18 | 42 | .547 | .761 | .6191 | .081 | 18 |
| 43 | .079 | .415 | .6831 | .970 | 17 | 43 | .572 | .801 | .6181 | .066 | 17 |
| 44 | .104 | .454 | .6820 | .956 | 16 | 44 | .597 | .842 | .6170 | .051 | 16 |
| 45 | .51129 | .59494 | 1.6808 | .85941 | 15 | 45 | .52621 | .61882 | 1.6160 | .85035 | 15 |
| 46 | .154 | .533 | .6797 | .926 | 14 | 46 | .616 | .922 | .6149 | .020 | 14 |
| 47 | .179 | .573 | .6786 | .911 | 13 | 47 | .671 | .61962 | .6139 | .85005 | 13 |
| 48 | .204 | .612 | .6775 | .896 | 12 | 48 | .696 | .62003 | .6128 | .84989 | 12 |
| 49 | .229 | .651 | .6764 | .881 | 11 | 49 | .720 | .043 | .6118 | .974 | 11 |
| 50 | .51254 | .59691 | 1.6753 | .85866 | 10 | 50 | .52745 | .62083 | 1.6107 | .84959 | 10 |
| 51 | .279 | .730 | .6742 | .851 | 9 | 51 | .770 | .124 | .6097 | .943 | 9 |
| 52 | .304 | .770 | .6731 | .836 | 8 | 52 | .794 | .164 | .6087 | .928 | 8 |
| 53 | .329 | .809 | .6720 | .821 | 7 | 53 | .819 | .204 | .6076 | .913 | 7 |
| 54 | .354 | .849 | .6709 | .806 | 6 | 54 | .844 | .245 | .6066 | .897 | 6 |
| 55 | .51379 | .59888 | 1.6698 | .85792 | 5 | 55 | .52869 | .62285 | 1.6055 | .84882 | 5 |
| 56 | .404 | .928 | .6687 | .777 | 4 | 56 | .893 | .325 | .6045 | .866 | 4 |
| 57 | .429 | .59967 | .6676 | .762 | 3 | 57 | .918 | .366 | .6034 | .851 | 3 |
| 58 | .454 | .60007 | .6665 | .747 | 2 | 58 | .943 | .406 | .6024 | .836 | 2 |
| 59 | .479 | .046 | .6654 | .732 | 1 | 59 | .967 | .446 | .6014 | .820 | 1 |
| 60 | .51504 | .60086 | 1.6643 | .85717 | 0 | 60 | .52992 | .62487 | 1.6003 | .84805 | 0 |
| | Cos | Ctn | Tan | Sin | | | Cos | Ctn | Tan | Sin | |

59°

58°

| | Sin | Tan | Ctn | Cos | | | Sin | Tan | Ctn | Cos | |
|----|--------|--------|--------|--------|----|----|--------|--------|--------|--------|----|
| 0 | .52992 | .62487 | 1.6003 | .84805 | 60 | 0 | .54464 | .64941 | 1.5399 | .83867 | 60 |
| 1 | .53017 | .527 | .5993 | 789 | 59 | 1 | .488 | .64982 | .5389 | .851 | 59 |
| 2 | .041 | .568 | .5983 | 774 | 58 | 2 | .513 | .65024 | .5370 | .835 | 58 |
| 3 | .066 | .608 | .5972 | 759 | 57 | 3 | .537 | .605 | .5369 | .819 | 57 |
| 4 | .091 | .649 | .5962 | 743 | 56 | 4 | .561 | .106 | .5359 | .804 | 56 |
| 5 | .53115 | .62689 | 1.5952 | .84728 | 55 | 5 | .54586 | .65148 | 1.5350 | .83788 | 55 |
| 6 | .140 | .730 | .5911 | 712 | 54 | 6 | .610 | .189 | .5340 | .772 | 54 |
| 7 | .164 | .770 | .5931 | 697 | 53 | 7 | .635 | .231 | .5330 | .756 | 53 |
| 8 | .189 | .811 | .5921 | 681 | 52 | 8 | .659 | .272 | .5320 | .740 | 52 |
| 9 | .214 | .852 | .5911 | 666 | 51 | 9 | .683 | .311 | .5311 | .724 | 51 |
| 10 | .53238 | .62892 | 1.5900 | .84650 | 50 | 10 | .54708 | .65375 | 1.5301 | .83708 | 50 |
| 11 | .263 | .933 | .5890 | 635 | 49 | 11 | .732 | .397 | .5291 | .692 | 49 |
| 12 | .288 | .62973 | .5880 | 619 | 48 | 12 | .756 | .438 | .5282 | .676 | 48 |
| 13 | .312 | .63014 | .5869 | 604 | 47 | 13 | .781 | .480 | .5272 | .660 | 47 |
| 14 | .337 | .055 | .5859 | 588 | 46 | 14 | .805 | .521 | .5262 | .645 | 46 |
| 15 | .53361 | .63095 | 1.5849 | .84573 | 45 | 15 | .54829 | .65563 | 1.5253 | .83629 | 45 |
| 16 | .386 | .136 | .5839 | 557 | 44 | 16 | .854 | .604 | .5243 | .613 | 44 |
| 17 | .411 | .177 | .5829 | 542 | 43 | 17 | .878 | .646 | .5233 | .597 | 43 |
| 18 | .435 | .217 | .5818 | 526 | 42 | 18 | .902 | .688 | .5224 | .581 | 42 |
| 19 | .460 | .258 | .5808 | 511 | 41 | 19 | .927 | .729 | .5214 | .565 | 41 |
| 20 | .53484 | .63299 | 1.5798 | .84495 | 40 | 20 | .54951 | .65771 | 1.5204 | .83549 | 40 |
| 21 | .509 | .340 | .5788 | 480 | 39 | 21 | .975 | .813 | .5195 | .533 | 39 |
| 22 | .534 | .380 | .5778 | 464 | 38 | 22 | .54999 | .854 | .5185 | .517 | 38 |
| 23 | .558 | .421 | .5768 | 448 | 37 | 23 | .55024 | .896 | .5175 | .501 | 37 |
| 24 | .583 | .462 | .5757 | 433 | 36 | 24 | .048 | .938 | .5166 | .485 | 36 |
| 25 | .53607 | .63503 | 1.5747 | .84417 | 35 | 25 | .55072 | .65980 | 1.5156 | .83469 | 35 |
| 26 | .632 | .544 | .5737 | 402 | 34 | 26 | .097 | .66021 | .5147 | .453 | 34 |
| 27 | .656 | .584 | .5727 | 386 | 33 | 27 | .121 | .063 | .5137 | .437 | 33 |
| 28 | .681 | .625 | .5717 | 370 | 32 | 28 | .145 | .105 | .5127 | .421 | 32 |
| 29 | .705 | .666 | .5707 | 355 | 31 | 29 | .169 | .147 | .5118 | .405 | 31 |
| 30 | .53730 | .63707 | 1.5697 | .84339 | 30 | 30 | .55144 | .66189 | 1.5108 | .83389 | 30 |
| 31 | .754 | .718 | .5687 | 324 | 29 | 31 | .218 | .240 | .5099 | .373 | 29 |
| 32 | .779 | .759 | .5677 | 308 | 28 | 32 | .242 | .272 | .5089 | .356 | 28 |
| 33 | .804 | .830 | .5667 | 292 | 27 | 33 | .266 | .314 | .5080 | .340 | 27 |
| 34 | .828 | .871 | .5657 | 277 | 26 | 34 | .291 | .356 | .5070 | .324 | 26 |
| 35 | .53853 | .63912 | 1.5647 | .84261 | 25 | 35 | .55315 | .66398 | 1.5061 | .83308 | 25 |
| 36 | .877 | .953 | .5637 | 245 | 24 | 36 | .339 | .440 | .5051 | .292 | 24 |
| 37 | .902 | .63994 | .5627 | 230 | 23 | 37 | .363 | .482 | .5042 | .276 | 23 |
| 38 | .926 | .64035 | .5617 | 214 | 22 | 38 | .388 | .524 | .5032 | .260 | 22 |
| 39 | .951 | .076 | .5607 | 198 | 21 | 39 | .412 | .566 | .5023 | .244 | 21 |
| 40 | .53975 | .64117 | 1.5597 | .84182 | 20 | 40 | .55436 | .66608 | 1.5013 | .83228 | 20 |
| 41 | .54000 | .158 | .5587 | 167 | 19 | 41 | .460 | .630 | .5004 | .212 | 19 |
| 42 | .024 | .199 | .5577 | 151 | 18 | 42 | .484 | .692 | .4994 | .195 | 18 |
| 43 | .049 | .240 | .5567 | 135 | 17 | 43 | .509 | .734 | .4985 | .179 | 17 |
| 44 | .073 | .281 | .5557 | 120 | 16 | 44 | .533 | .776 | .4975 | .163 | 16 |
| 45 | .54097 | .64322 | 1.5547 | .84104 | 15 | 45 | .55557 | .66818 | 1.4966 | .83147 | 15 |
| 46 | .122 | .363 | .5537 | .088 | 14 | 46 | .581 | .860 | .4957 | .131 | 14 |
| 47 | .146 | .404 | .5527 | .072 | 13 | 47 | .605 | .902 | .4947 | .115 | 13 |
| 48 | .171 | .446 | .5517 | .057 | 12 | 48 | .630 | .944 | .4938 | .098 | 12 |
| 49 | .195 | .487 | .5507 | .041 | 11 | 49 | .654 | .66986 | .4928 | .082 | 11 |
| 50 | .54220 | .64528 | 1.5497 | .84025 | 10 | 50 | .55678 | .67028 | 1.4919 | .83066 | 10 |
| 51 | .244 | .569 | .5487 | .84009 | 9 | 51 | .702 | .071 | .4910 | .050 | 9 |
| 52 | .269 | .610 | .5477 | .83994 | 8 | 52 | .726 | .113 | .4900 | .034 | 8 |
| 53 | .293 | .652 | .5468 | .978 | 7 | 53 | .750 | .155 | .4891 | .017 | 7 |
| 54 | .317 | .693 | .5458 | .962 | 6 | 54 | .775 | .197 | .4882 | .83001 | 6 |
| 55 | .54342 | .64734 | 1.5448 | .83946 | 5 | 55 | .55799 | .67239 | 1.4872 | .82985 | 5 |
| 56 | .366 | .775 | .5438 | .930 | 4 | 56 | .823 | .282 | .4863 | .969 | 4 |
| 57 | .391 | .817 | .5428 | .915 | 3 | 57 | .847 | .324 | .4854 | .953 | 3 |
| 58 | .415 | .858 | .5418 | .899 | 2 | 58 | .871 | .366 | .4844 | .936 | 2 |
| 59 | .440 | .899 | .5408 | .883 | 1 | 59 | .895 | .409 | .4835 | .920 | 1 |
| 60 | .54464 | .64941 | 1.5399 | .83867 | 0 | 60 | .55919 | .67451 | 1.4826 | .82904 | 0 |
| | Cos | Ctn | Tan | Sin | | | Cos | Ctn | Tan | Sin | |

| ' | Sin | Tan | Ctn | Cos | ' | Sin | Tan | Ctn | Cos | ' | |
|----|-------|-------|--------|-------|----|-----|-------|--------|--------|-------|---|
| 0 | 53919 | 67451 | 1 4828 | 82904 | 60 | 0 | 57358 | 70021 | 1 4281 | 81915 | |
| 1 | 943 | 493 | .4816 | 887 | 59 | 1 | 351 | 004 | 4273 | 899 | |
| 2 | 968 | 536 | .4807 | 871 | 58 | 2 | 405 | 107 | 4264 | 882 | |
| 3 | 53992 | 578 | .4798 | 855 | 57 | 3 | 429 | 151 | 4255 | 865 | |
| 4 | 56016 | 620 | 4788 | 839 | 56 | 4 | 453 | 191 | 4246 | 848 | |
| 5 | 56040 | 67663 | 1 4779 | 82822 | 55 | 5 | 57477 | 70238 | 1 4237 | 81832 | |
| 6 | 061 | 705 | 4770 | 806 | 54 | 6 | 501 | 281 | 4229 | 815 | |
| 7 | 088 | 748 | .4761 | 790 | 53 | 7 | 521 | 323 | 4220 | 798 | |
| 8 | 112 | 790 | 4751 | 773 | 52 | 8 | 548 | 368 | 4211 | 782 | |
| 9 | 136 | 832 | 4742 | 757 | 51 | 9 | 572 | 412 | 4202 | 765 | |
| 10 | 56160 | 67875 | 1 4733 | 82741 | 50 | 10 | 57596 | 70455 | 1 4193 | 81748 | |
| 11 | 184 | 917 | .4724 | 721 | 49 | 11 | 619 | 499 | 4185 | 731 | |
| 12 | 208 | 67960 | 1715 | 708 | 48 | 12 | 613 | 542 | 4176 | 711 | |
| 13 | 232 | 68002 | .4705 | 692 | 47 | 13 | 667 | 586 | 4167 | 698 | |
| 14 | 256 | 045 | 4696 | 675 | 46 | 14 | 691 | 629 | 4158 | 681 | |
| 15 | 56280 | 68088 | 1 4687 | 82659 | 45 | 15 | 57715 | 70673 | 1 4150 | 81664 | |
| 16 | 305 | 130 | 4678 | 643 | 44 | 16 | 738 | 717 | 4141 | 617 | |
| 17 | 329 | 173 | 4669 | 626 | 43 | 17 | 762 | 760 | 4132 | 631 | |
| 18 | 353 | 215 | 4659 | 610 | 42 | 18 | 786 | 801 | 4121 | 614 | |
| 19 | 377 | 258 | 4650 | 593 | 41 | 19 | 810 | 848 | 4115 | 597 | |
| 20 | 56401 | 68301 | 1 4641 | 82577 | 40 | 20 | 57833 | 70891 | 1 4106 | 81580 | |
| 21 | 425 | 313 | 4632 | 561 | 39 | 21 | 857 | 935 | 4097 | 563 | |
| 22 | 449 | 386 | 4623 | 541 | 38 | 22 | 881 | 70379 | 4089 | 546 | |
| 23 | 473 | 429 | 4614 | 528 | 37 | 23 | 901 | 71023 | 4080 | 530 | |
| 24 | 497 | 471 | 4605 | 511 | 36 | 24 | 928 | 066 | 4071 | 513 | |
| 25 | 56521 | 68514 | 1 4596 | 82495 | 35 | 25 | 57952 | 71110 | 1 4063 | 81496 | |
| 26 | 515 | 537 | 4588 | 478 | 34 | 26 | 976 | 131 | 4054 | 479 | |
| 27 | 569 | 600 | 4577 | 462 | 33 | 27 | 57999 | 198 | 4045 | 462 | |
| 28 | 593 | 612 | 4568 | 446 | 32 | 28 | 58023 | 212 | 4037 | 445 | |
| 29 | 617 | 683 | 4559 | 429 | 31 | 29 | 017 | 283 | 4028 | 428 | |
| 30 | 56641 | 68728 | 1 4550 | 82413 | 30 | 30 | 58070 | .7129 | 1 4019 | 81412 | |
| 31 | 665 | 771 | 4541 | 396 | 29 | 31 | 094 | 37 | 4011 | 395 | |
| 32 | 689 | 814 | 4532 | 380 | 28 | 32 | 118 | 417 | 4002 | 378 | |
| 33 | 713 | 857 | 4523 | 363 | 27 | 33 | 141 | 461 | 3994 | 361 | |
| 34 | 736 | 900 | 4514 | 347 | 26 | 34 | 165 | 509 | 3985 | 344 | |
| 35 | 56760 | 68942 | 1 4505 | 82330 | 25 | 35 | 58189 | .71519 | 1 3976 | 81327 | |
| 36 | 784 | 68985 | 4496 | 331 | 24 | 6 | 212 | 51 | 3968 | 310 | |
| 37 | 808 | 69028 | 4487 | 277 | 23 | 7 | 236 | 63 | 3959 | 293 | |
| 38 | 832 | 071 | 4478 | 261 | 22 | 8 | 260 | 681 | 3951 | 276 | |
| 39 | 856 | 114 | 4469 | 244 | 21 | 9 | 283 | 723 | 3942 | 259 | |
| 40 | 56880 | 69157 | 1 4460 | 82248 | 20 | 40 | 58307 | .71769 | 1 3934 | 81242 | |
| 41 | 904 | 200 | 4451 | 231 | 19 | 11 | 330 | 813 | 3925 | 225 | |
| 42 | 928 | 243 | 4442 | 214 | 18 | 12 | 354 | 857 | 3916 | 208 | |
| 43 | 952 | 285 | 4433 | 198 | 17 | 13 | 378 | 901 | 3908 | 191 | |
| 44 | 56976 | 329 | 4424 | 181 | 16 | 44 | 401 | 946 | 3899 | 174 | |
| 45 | 57000 | 69372 | 1 4415 | 82165 | 15 | 45 | 58425 | .71990 | 1 3891 | 81157 | |
| 46 | 021 | 416 | 4406 | 165 | 14 | 16 | 419 | .72034 | 3882 | 110 | |
| 47 | 047 | 459 | 4397 | 152 | 13 | 47 | 472 | 078 | 3874 | 123 | |
| 48 | 071 | 502 | 4388 | 135 | 12 | 18 | 496 | 122 | 3865 | 106 | |
| 49 | 095 | 545 | 4379 | 098 | 11 | 49 | 519 | 167 | 3857 | 089 | |
| 50 | 57119 | 69588 | 1 4370 | 82082 | 10 | 50 | 58543 | .72211 | 1 3845 | 81072 | |
| 51 | 143 | 631 | 4361 | 069 | 9 | 51 | 567 | 233 | 3830 | 055 | |
| 52 | 167 | 675 | 4352 | 048 | 8 | 52 | 590 | 299 | 3821 | 038 | |
| 53 | 191 | 718 | 4344 | 032 | 7 | 53 | 614 | 344 | 3813 | 021 | |
| 54 | 215 | 761 | 4335 | 02015 | 6 | 54 | 637 | 388 | 3804 | 81004 | |
| 55 | 57238 | 69804 | 1 4326 | 81999 | 5 | 55 | 58661 | .72432 | 1 3806 | 80987 | |
| 56 | 262 | 847 | 4317 | 982 | 4 | 56 | 684 | 477 | 3798 | 970 | |
| 57 | 286 | 891 | 4308 | 965 | 3 | 57 | 708 | 521 | 3789 | 953 | |
| 58 | 310 | 934 | 4299 | 949 | 2 | 58 | 731 | 565 | 3781 | 936 | |
| 59 | 334 | 69977 | 4290 | 932 | 1 | 59 | 755 | 610 | 3772 | 919 | |
| 60 | 57358 | 70021 | 1 4281 | 81915 | 0 | 60 | 58779 | 72654 | 1 3764 | 80902 | |
| | Cos | Ctn | Tan | Sin | ' | | Cos | Ctn | Tan | Sin | ' |

| ' | Sin | Tan | Ctn | Cos | ' | Sin | Tan | Ctn | Cos | ' |
|----|-------|--------|--------|--------|----|-------|--------|--------|--------|--------|
| 0 | 58779 | .72654 | 1 3761 | .80902 | 60 | 60182 | .75355 | 1 3270 | .79864 | 60 |
| 1 | 802 | 699 | 3755 | 885 | 59 | 1 | 205 | 4 11 | .3262 | 846 |
| 2 | 826 | 743 | 3747 | 867 | 58 | 2 | 228 | 447 | 3254 | 829 |
| 3 | 849 | 788 | 3739 | 850 | 57 | 3 | 251 | 492 | 3246 | 811 |
| 4 | 873 | 832 | 3730 | 833 | 56 | 4 | 274 | 538 | 3238 | 793 |
| 5 | 58896 | .72677 | 1 3722 | .80816 | 55 | 5 | 60298 | .75584 | 1 3230 | .79776 |
| 6 | 920 | 921 | 3713 | 799 | 54 | 6 | 321 | 629 | 3222 | 758 |
| 7 | 943 | .72966 | 3705 | 782 | 53 | 7 | 344 | 675 | 3214 | 741 |
| 8 | 967 | .73010 | 3697 | 765 | 52 | 8 | 367 | 721 | 3206 | 723 |
| 9 | 58990 | 055 | 3688 | 748 | 51 | 9 | 390 | 767 | 3198 | 706 |
| 10 | 59014 | .73100 | 1 3680 | .80730 | 50 | 10 | .60414 | .75812 | 1 3190 | .79688 |
| 11 | 037 | 144 | 3672 | 713 | 49 | 11 | 4 | 7 | 838 | 3182 |
| 12 | 061 | 189 | 3663 | 696 | 48 | 12 | 460 | 901 | 3175 | 653 |
| 13 | 084 | 231 | 3655 | 679 | 47 | 13 | 483 | 950 | 3167 | 635 |
| 14 | 108 | 278 | 3647 | 662 | 46 | 14 | 506 | 996 | 3159 | 618 |
| 15 | 59131 | .73123 | 1 3638 | .80644 | 45 | 15 | 60529 | .76042 | 1 3151 | .79600 |
| 16 | 151 | 368 | 3630 | 627 | 44 | 16 | 533 | 088 | 3143 | 583 |
| 17 | 178 | 413 | 3622 | 610 | 43 | 17 | 576 | 134 | 3135 | 565 |
| 18 | 201 | 457 | 3613 | 593 | 42 | 18 | 599 | 180 | 3127 | 547 |
| 19 | 225 | 502 | 3605 | 576 | 41 | 19 | 622 | 226 | 3119 | 530 |
| 20 | 59248 | .73517 | 1 3597 | .80558 | 40 | 20 | .60645 | .76272 | 1 3111 | .79512 |
| 21 | 272 | 592 | 3588 | 541 | 39 | 21 | 668 | 318 | 3103 | 494 |
| 22 | 295 | 637 | 3580 | 524 | 38 | 22 | 691 | 364 | 3095 | 477 |
| 23 | 318 | 681 | 3572 | 507 | 37 | 23 | 714 | 410 | 3087 | 459 |
| 24 | 342 | 726 | 3564 | 489 | 36 | 24 | 738 | 456 | 3079 | 441 |
| 25 | 59365 | .73771 | 1 3555 | .80472 | 35 | 25 | 60761 | .76502 | 1 3072 | .79424 |
| 26 | 389 | 810 | 3547 | 473 | 34 | 26 | 781 | 518 | 3064 | 406 |
| 27 | 412 | 861 | 3539 | 458 | 33 | 27 | 807 | 564 | 3056 | 388 |
| 28 | 436 | 906 | 3531 | 440 | 32 | 28 | 830 | 610 | 3048 | 371 |
| 29 | 459 | 951 | 3522 | 424 | 31 | 29 | 85 | 686 | 3040 | 353 |
| 30 | 59482 | .73996 | 1 3514 | .80386 | 30 | 30 | 6087 | .76733 | 1 3032 | .79335 |
| 31 | 506 | .74041 | 3506 | 408 | 29 | 1 | 899 | 779 | 3024 | 318 |
| 32 | 529 | 086 | 3498 | 391 | 28 | 32 | 922 | 825 | 3017 | 300 |
| 33 | 552 | 131 | 3490 | 374 | 27 | 33 | 945 | 871 | 3009 | 282 |
| 34 | 576 | 176 | 3481 | 356 | 26 | 34 | 968 | 918 | 3001 | 264 |
| 35 | 59599 | .74221 | 1 3473 | .80299 | 25 | 35 | 60991 | .76964 | 1 2995 | .79247 |
| 36 | 622 | 264 | 3465 | 342 | 24 | 36 | 61015 | .77010 | 2987 | 229 |
| 37 | 646 | 312 | 3457 | 324 | 23 | 37 | 038 | 037 | 2977 | 211 |
| 38 | 669 | 357 | 3449 | 307 | 22 | 38 | 061 | 10 | 2970 | 193 |
| 39 | 693 | 402 | 3440 | 290 | 21 | 39 | 084 | 149 | 2962 | 176 |
| 40 | 59716 | .74447 | 1 3432 | .80212 | 20 | 40 | 61107 | .77196 | 1 2954 | .79158 |
| 41 | 739 | 492 | 3424 | 195 | 19 | 41 | 130 | 242 | 2946 | 140 |
| 42 | 763 | 538 | 3416 | 178 | 18 | 42 | 153 | 289 | 2938 | 122 |
| 43 | 786 | 583 | 3408 | 160 | 17 | 43 | 176 | 335 | 2931 | 105 |
| 44 | 809 | 628 | 3400 | 143 | 16 | 44 | 199 | 382 | 2923 | 87 |
| 45 | 59832 | .74674 | 1 3392 | .80125 | 15 | 45 | 61222 | .77428 | 1 2915 | .79069 |
| 46 | 856 | 719 | 3384 | 108 | 14 | 46 | 245 | 475 | 2907 | 051 |
| 47 | 879 | 764 | 3375 | 091 | 13 | 47 | 268 | 521 | 2900 | 033 |
| 48 | 902 | 810 | 3367 | 073 | 12 | 48 | 291 | 568 | 2892 | 79016 |
| 49 | 926 | 855 | 3359 | 056 | 11 | 49 | 314 | 615 | 2884 | 78908 |
| 50 | 59949 | .74900 | 1 3351 | .80038 | 10 | 50 | 61357 | .77661 | 1 2876 | .78980 |
| 51 | 972 | 946 | 3343 | 021 | 9 | 51 | 360 | 708 | 2869 | 962 |
| 52 | 59995 | .74991 | 3335 | 80003 | 8 | 52 | 383 | 754 | 2861 | 944 |
| 53 | 60019 | .75037 | 3327 | 79986 | 7 | 53 | 406 | 801 | 2853 | 926 |
| 54 | 042 | 082 | 3319 | 968 | 6 | 54 | 429 | 848 | 2846 | 908 |
| 55 | 60065 | .75128 | 1 3311 | .79951 | 5 | 55 | 61451 | .77895 | 1 2838 | .78891 |
| 56 | 089 | 173 | 3303 | 934 | 4 | 56 | 471 | 911 | 2830 | 873 |
| 57 | 112 | 219 | 3295 | 916 | 3 | 57 | 497 | 77988 | 2822 | 855 |
| 58 | 135 | 264 | 3287 | 899 | 2 | 58 | 520 | 78035 | 2815 | 837 |
| 59 | 158 | 310 | 3278 | 881 | 1 | 59 | 543 | 082 | 2807 | 819 |
| 60 | 60182 | .75355 | 1 3270 | .79864 | 0 | 60 | 61566 | .78129 | 1 2799 | .78801 |
| | Cos | Ctn | Tan | Sin | | Cos | Ctn | Tan | Sin | |

| ' | Sin | Tan | Ctn | Cos | ' | Sin | Tan | Ctn | Cos | ' | |
|----|--------|--------|--------|--------|----|-----|--------|--------|--------|--------|----|
| 0 | .61566 | .78129 | 1.2799 | .78801 | 60 | 0 | .62932 | .80978 | 1.2349 | .77715 | 60 |
| 1 | 589 | 175 | .2792 | 783 | 59 | 1 | 955 | .81027 | .2342 | 696 | 59 |
| 2 | 612 | 222 | .2784 | 765 | 58 | 2 | .62977 | 075 | .2334 | 678 | 58 |
| 3 | 635 | 269 | .2776 | 747 | 57 | 3 | .63000 | 123 | .2327 | 660 | 57 |
| 4 | 658 | 316 | .2769 | 729 | 56 | 4 | 022 | 171 | .2320 | 641 | 56 |
| 5 | .61681 | .78363 | 1.2761 | .78711 | 55 | 5 | .63045 | .81220 | 1.2312 | .77623 | 55 |
| 6 | 704 | 410 | .2753 | 694 | 54 | 6 | 068 | 268 | .2305 | 605 | 54 |
| 7 | 726 | 457 | .2746 | 676 | 53 | 7 | 090 | 316 | .2298 | 586 | 53 |
| 8 | 749 | 504 | .2738 | 658 | 52 | 8 | 113 | 364 | .2290 | 568 | 52 |
| 9 | 772 | 551 | .2731 | 640 | 51 | 9 | 135 | 413 | .2283 | 550 | 51 |
| 10 | .61795 | .78598 | 1.2723 | .78622 | 50 | 10 | .63158 | .81461 | 1.2276 | .77531 | 50 |
| 11 | 818 | 645 | .2715 | 604 | 49 | 11 | 180 | 510 | .2268 | 513 | 49 |
| 12 | 841 | 692 | .2708 | 586 | 48 | 12 | 203 | 558 | .2261 | 494 | 48 |
| 13 | 864 | 739 | .2700 | 568 | 47 | 13 | 225 | 606 | .2254 | 476 | 47 |
| 14 | 887 | 786 | .2693 | 550 | 46 | 14 | 248 | 655 | .2247 | 458 | 46 |
| 15 | .61909 | .78834 | 1.2685 | .78532 | 45 | 15 | .63271 | .81703 | 1.2239 | .77439 | 45 |
| 16 | 932 | 881 | .2677 | 514 | 44 | 16 | 293 | 752 | .2232 | 421 | 44 |
| 17 | 955 | 928 | .2670 | 496 | 43 | 17 | 316 | 800 | .2225 | 402 | 43 |
| 18 | .61978 | .78975 | .2662 | 478 | 42 | 18 | 338 | 849 | .2218 | 384 | 42 |
| 19 | .62001 | .79022 | .2655 | 460 | 41 | 19 | 361 | 898 | .2210 | 366 | 41 |
| 20 | .62024 | .79070 | 1.2647 | .78442 | 40 | 20 | .63383 | .81946 | 1.2203 | .77347 | 40 |
| 21 | 046 | 117 | .2640 | 424 | 39 | 21 | 406 | .81995 | .2196 | 329 | 39 |
| 22 | 069 | 164 | .2632 | 405 | 38 | 22 | 428 | .82044 | .2189 | 310 | 38 |
| 23 | 092 | 212 | .2624 | 387 | 37 | 23 | 451 | 092 | .2181 | 292 | 37 |
| 24 | 115 | 259 | .2617 | 369 | 36 | 24 | 473 | 141 | .2174 | 273 | 36 |
| 25 | .62138 | .79306 | 1.2609 | .78351 | 35 | 25 | .63496 | .82190 | 1.2167 | .77255 | 35 |
| 26 | 160 | 354 | .2602 | 333 | 34 | 26 | 518 | 238 | .2160 | 236 | 34 |
| 27 | 183 | 401 | .2594 | 315 | 33 | 27 | 540 | 287 | .2153 | 218 | 33 |
| 28 | 206 | 449 | .2587 | 297 | 32 | 28 | 563 | 336 | .2145 | 199 | 32 |
| 29 | 229 | 496 | .2579 | 279 | 31 | 29 | 585 | 385 | .2138 | 181 | 31 |
| 30 | .62251 | .79544 | 1.2572 | .78261 | 30 | 30 | .63608 | .82434 | 1.2131 | .77162 | 30 |
| 31 | 274 | 591 | .2564 | 243 | 29 | 31 | 630 | 483 | .2124 | 144 | 29 |
| 32 | 297 | 639 | .2557 | 225 | 28 | 32 | 653 | 531 | .2117 | 125 | 28 |
| 33 | 320 | 686 | .2549 | 206 | 27 | 33 | 675 | 580 | .2109 | 107 | 27 |
| 34 | 342 | 734 | .2542 | 188 | 26 | 34 | 698 | 629 | .2102 | 088 | 26 |
| 35 | .62365 | .79781 | 1.2534 | .78170 | 25 | 35 | .63720 | .82678 | 1.2095 | .77070 | 25 |
| 36 | 388 | 829 | .2527 | 152 | 24 | 36 | 742 | 727 | .2088 | 051 | 24 |
| 37 | 411 | 877 | .2519 | 134 | 23 | 37 | 765 | 776 | .2081 | 033 | 23 |
| 38 | 433 | 924 | .2512 | 116 | 22 | 38 | 787 | 825 | .2074 | .77014 | 22 |
| 39 | 456 | .79972 | .2504 | 098 | 21 | 39 | 810 | 874 | .2066 | .76996 | 21 |
| 40 | .62479 | .80020 | 1.2497 | .78079 | 20 | 40 | .63832 | .82923 | 1.2059 | .76977 | 20 |
| 41 | 502 | 067 | .2489 | 061 | 19 | 41 | 854 | .82972 | .2052 | 959 | 19 |
| 42 | 524 | 115 | .2482 | 043 | 18 | 42 | 877 | .83022 | .2045 | 940 | 18 |
| 43 | 547 | 163 | .2475 | 025 | 17 | 43 | 899 | 071 | .2038 | 921 | 17 |
| 44 | 570 | 211 | .2467 | .78007 | 16 | 44 | 922 | 120 | .2031 | 903 | 16 |
| 45 | .62592 | .80258 | 1.2460 | .77988 | 15 | 45 | .63944 | .83169 | 1.2024 | .76884 | 15 |
| 46 | 615 | 306 | .2452 | 970 | 14 | 46 | 966 | 218 | .2017 | 866 | 14 |
| 47 | 638 | 354 | .2445 | 952 | 13 | 47 | .63989 | 268 | .2009 | 847 | 13 |
| 48 | 660 | 402 | .2437 | 934 | 12 | 48 | .64011 | 317 | .2002 | 828 | 12 |
| 49 | 683 | 450 | .2430 | 916 | 11 | 49 | 033 | 366 | .1995 | 810 | 11 |
| 50 | .62706 | .80498 | 1.2423 | .77897 | 10 | 50 | .64056 | .83415 | 1.1988 | .76791 | 10 |
| 51 | 728 | 546 | .2415 | 879 | 9 | 51 | 078 | 465 | .1981 | 772 | 9 |
| 52 | 751 | 594 | .2408 | 861 | 8 | 52 | 100 | 514 | .1974 | 754 | 8 |
| 53 | 774 | 642 | .2401 | 843 | 7 | 53 | 123 | 564 | .1967 | 735 | 7 |
| 54 | 796 | 690 | .2393 | 824 | 6 | 54 | 145 | 613 | .1960 | 717 | 6 |
| 55 | .62819 | .80738 | 1.2386 | .77806 | 5 | 55 | .64167 | .83662 | 1.1953 | .76698 | 5 |
| 56 | 842 | 786 | .2378 | 788 | 4 | 56 | 190 | 712 | .1946 | 679 | 4 |
| 57 | 864 | 834 | .2371 | 769 | 3 | 57 | 212 | 761 | .1939 | 661 | 3 |
| 58 | 887 | 882 | .2364 | 751 | 2 | 58 | 234 | 811 | .1932 | 642 | 2 |
| 59 | 909 | 930 | .2356 | 733 | 1 | 59 | 256 | 860 | .1925 | 623 | 1 |
| 60 | .62932 | .80978 | 1.2349 | .77715 | 0 | 60 | .64279 | .83910 | 1.1918 | .76604 | 0 |
| | Cos | Ctn | Tan | Sin | ' | | Cos | Ctn | Tan | Sin | ' |

51°

50°

| ' | Sin | Tan | Ctn | Cos | ' |
|----|--------|--------|--------|--------|----|
| 0 | .64279 | .83910 | 1.1918 | .76604 | 60 |
| 1 | 301 | .83960 | .1910 | 586 | 59 |
| 2 | 323 | .84009 | .1903 | 567 | 58 |
| 3 | 346 | 059 | .1896 | 548 | 57 |
| 4 | 368 | 108 | .1889 | 530 | 56 |
| 5 | .64390 | .84158 | 1.1882 | .76511 | 55 |
| 6 | 412 | 208 | .1875 | 492 | 54 |
| 7 | 435 | 258 | .1868 | 473 | 53 |
| 8 | 457 | 307 | .1861 | 455 | 52 |
| 9 | 479 | 357 | .1854 | 436 | 51 |
| 10 | .64501 | .84407 | 1.1847 | .76417 | 50 |
| 11 | 524 | 457 | .1840 | 398 | 49 |
| 12 | 546 | 507 | .1833 | 380 | 48 |
| 13 | 568 | 556 | .1826 | 361 | 47 |
| 14 | 590 | 606 | .1819 | 342 | 46 |
| 15 | .64612 | .84656 | 1.1812 | .76323 | 45 |
| 16 | 635 | 706 | .1806 | 304 | 44 |
| 17 | 657 | 756 | .1799 | 286 | 43 |
| 18 | 679 | 806 | .1792 | 267 | 42 |
| 19 | 701 | 856 | .1785 | 248 | 41 |
| 20 | .64723 | .84906 | 1.1778 | .76229 | 40 |
| 21 | 746 | .84956 | .1771 | 210 | 39 |
| 22 | 768 | .85006 | .1764 | 192 | 38 |
| 23 | 790 | 057 | .1757 | 173 | 37 |
| 24 | 812 | 107 | .1750 | 154 | 36 |
| 25 | .64834 | .85157 | 1.1743 | .76135 | 35 |
| 26 | 856 | 207 | .1736 | 116 | 34 |
| 27 | 878 | 257 | .1729 | 097 | 33 |
| 28 | 901 | 308 | .1722 | 078 | 32 |
| 29 | 923 | 358 | .1715 | 059 | 31 |
| 30 | .64945 | .85408 | 1.1708 | .76041 | 30 |
| 31 | 967 | 458 | .1702 | 022 | 29 |
| 32 | .64989 | 509 | .1695 | .76003 | 28 |
| 33 | .65011 | 559 | .1688 | .75984 | 27 |
| 34 | 033 | 609 | .1681 | 965 | 26 |
| 35 | .65055 | .85660 | 1.1674 | .75946 | 25 |
| 36 | 077 | 710 | .1667 | 927 | 24 |
| 37 | 100 | 761 | .1660 | 908 | 23 |
| 38 | 122 | 811 | .1653 | 889 | 22 |
| 39 | 144 | 862 | .1647 | 870 | 21 |
| 40 | .65166 | .85912 | 1.1640 | .75851 | 20 |
| 41 | 188 | .85963 | .1633 | 832 | 19 |
| 42 | 210 | .86014 | .1626 | 813 | 18 |
| 43 | 232 | 064 | .1619 | 794 | 17 |
| 44 | 254 | 115 | .1612 | 775 | 16 |
| 45 | .65276 | .86166 | 1.1606 | .75756 | 15 |
| 46 | 298 | 216 | .1599 | 738 | 14 |
| 47 | 320 | 267 | .1592 | 719 | 13 |
| 48 | 342 | 318 | .1585 | 700 | 12 |
| 49 | 364 | 368 | .1578 | 680 | 11 |
| 50 | .65386 | .86419 | 1.1571 | .75661 | 10 |
| 51 | 408 | 470 | .1565 | 642 | 9 |
| 52 | 430 | 521 | .1558 | 623 | 8 |
| 53 | 452 | 572 | .1551 | 604 | 7 |
| 54 | 474 | 623 | .1544 | 585 | 6 |
| 55 | .65496 | .86674 | 1.1538 | .75566 | 5 |
| 56 | 518 | 725 | .1531 | 547 | 4 |
| 57 | 540 | 776 | .1524 | 528 | 3 |
| 58 | 562 | 827 | .1517 | 509 | 2 |
| 59 | 584 | 878 | .1510 | 490 | 1 |
| 60 | .65606 | .86929 | 1.1504 | .75471 | 0 |
| | Cos | Ctn | Tan | Sin | ' |

49°

| ' | Sin | Tan | Ctn | Cos | ' |
|----|--------|--------|--------|--------|----|
| 0 | .65606 | .86929 | 1.1504 | .75471 | 60 |
| 1 | 628 | .86980 | .1497 | 452 | 59 |
| 2 | 650 | .87031 | .1490 | 433 | 58 |
| 3 | 672 | 082 | .1483 | 414 | 57 |
| 4 | 694 | 133 | .1477 | 395 | 56 |
| 5 | .65716 | .87184 | 1.1470 | .75375 | 55 |
| 6 | 738 | 236 | .1463 | 356 | 54 |
| 7 | 759 | 287 | .1456 | 337 | 53 |
| 8 | 781 | 338 | .1450 | 318 | 52 |
| 9 | 803 | 389 | .1443 | 299 | 51 |
| 10 | .65825 | .87441 | 1.1436 | .75280 | 50 |
| 11 | 847 | 492 | .1430 | 261 | 49 |
| 12 | 869 | 543 | .1423 | 241 | 48 |
| 13 | 891 | 595 | .1416 | 222 | 47 |
| 14 | 913 | 646 | .1410 | 203 | 46 |
| 15 | .65935 | .87698 | 1.1403 | .75184 | 45 |
| 16 | 956 | 749 | .1396 | 165 | 44 |
| 17 | .65978 | 801 | .1389 | 146 | 43 |
| 18 | .66000 | 852 | .1383 | 126 | 42 |
| 19 | 022 | 904 | .1376 | 107 | 41 |
| 20 | .66044 | .87955 | 1.1369 | .75088 | 40 |
| 21 | 066 | .88007 | .1363 | 069 | 39 |
| 22 | 088 | 059 | .1356 | 050 | 38 |
| 23 | 109 | 110 | .1349 | 030 | 37 |
| 24 | 131 | 162 | .1343 | .75011 | 36 |
| 25 | .66153 | .88214 | 1.1336 | .74992 | 35 |
| 26 | 175 | 265 | .1329 | 973 | 34 |
| 27 | 197 | 317 | .1323 | 953 | 33 |
| 28 | 218 | 369 | .1316 | 934 | 32 |
| 29 | 240 | 421 | .1310 | 915 | 31 |
| 30 | .66262 | .88473 | 1.1303 | .74896 | 30 |
| 31 | 284 | 524 | .1296 | 876 | 29 |
| 32 | 306 | 576 | .1290 | 857 | 28 |
| 33 | 327 | 628 | .1283 | 838 | 27 |
| 34 | 349 | 680 | .1276 | 818 | 26 |
| 35 | .66371 | .88732 | 1.1270 | .74799 | 25 |
| 36 | 393 | 784 | .1263 | 780 | 24 |
| 37 | 414 | 836 | .1257 | 760 | 23 |
| 38 | 436 | 888 | .1250 | 741 | 22 |
| 39 | 458 | 940 | .1243 | 722 | 21 |
| 40 | .66480 | .88992 | 1.1237 | .74703 | 20 |
| 41 | 501 | .89045 | .1230 | 683 | 19 |
| 42 | 523 | 097 | .1224 | 664 | 18 |
| 43 | 545 | 149 | .1217 | 644 | 17 |
| 44 | 566 | 201 | .1211 | 625 | 16 |
| 45 | .66588 | .89253 | 1.1204 | .74606 | 15 |
| 46 | 610 | 306 | .1197 | 586 | 14 |
| 47 | 632 | 358 | .1191 | 567 | 13 |
| 48 | 653 | 410 | .1184 | 548 | 12 |
| 49 | 675 | 463 | .1178 | 528 | 11 |
| 50 | .66697 | .89515 | 1.1171 | .74509 | 10 |
| 51 | 718 | 567 | .1165 | 489 | 9 |
| 52 | 740 | 620 | .1158 | 470 | 8 |
| 53 | 762 | 672 | .1152 | 451 | 7 |
| 54 | 783 | 725 | .1145 | 431 | 6 |
| 55 | .66805 | .89777 | 1.1139 | .74412 | 5 |
| 56 | 827 | 830 | .1132 | 392 | 4 |
| 57 | 848 | 883 | .1126 | 373 | 3 |
| 58 | 870 | 935 | .1119 | 353 | 2 |
| 59 | 891 | .89988 | .1113 | 334 | 1 |
| 60 | .66913 | .90040 | 1.1106 | .74314 | 0 |
| | Cos | Ctn | Tan | Sin | ' |

48°

| ' | Sin | Tan | Ctn | Cos | ' |
|----|--------|--------|--------|--------|----|
| 0 | .66913 | .90040 | 1.1106 | .74314 | 60 |
| 1 | .935 | .093 | .1100 | .295 | 59 |
| 2 | .956 | .146 | .1093 | .276 | 58 |
| 3 | .978 | .199 | .1087 | .256 | 57 |
| 4 | .66999 | .251 | .1080 | .237 | 56 |
| 5 | .67021 | .90304 | 1.1074 | .74217 | 55 |
| 6 | .043 | .357 | .1067 | .198 | 54 |
| 7 | .064 | .410 | .1061 | .178 | 53 |
| 8 | .086 | .463 | .1054 | .159 | 52 |
| 9 | .107 | .516 | .1048 | .139 | 51 |
| 10 | .67129 | .90569 | 1.1041 | .74120 | 50 |
| 11 | .151 | .621 | .1035 | .100 | 49 |
| 12 | .172 | .674 | .1028 | .080 | 48 |
| 13 | .194 | .727 | .1022 | .061 | 47 |
| 14 | .215 | .781 | .1016 | .041 | 46 |
| 15 | .67237 | .90834 | 1.1009 | .74022 | 45 |
| 16 | .258 | .887 | .1003 | .74002 | 44 |
| 17 | .280 | .940 | .0996 | .73983 | 43 |
| 18 | .301 | .90093 | .0990 | .963 | 42 |
| 19 | .323 | .91046 | .0983 | .944 | 41 |
| 20 | .67344 | .91099 | 1.0977 | .73924 | 40 |
| 21 | .366 | .153 | .0971 | .904 | 39 |
| 22 | .387 | .206 | .0964 | .885 | 38 |
| 23 | .409 | .259 | .0958 | .865 | 37 |
| 24 | .430 | .313 | .0951 | .846 | 36 |
| 25 | .67452 | .91366 | 1.0945 | .73826 | 35 |
| 26 | .473 | .419 | .0939 | .806 | 34 |
| 27 | .495 | .473 | .0932 | .787 | 33 |
| 28 | .516 | .526 | .0926 | .767 | 32 |
| 29 | .538 | .580 | .0919 | .747 | 31 |
| 30 | .67559 | .91633 | 1.0913 | .73728 | 30 |
| 31 | .580 | .687 | .0907 | .708 | 29 |
| 32 | .602 | .740 | .0900 | .688 | 28 |
| 33 | .623 | .794 | .0894 | .669 | 27 |
| 34 | .645 | .847 | .0888 | .649 | 26 |
| 35 | .67666 | .91901 | 1.0881 | .73629 | 25 |
| 36 | .688 | .91955 | .0875 | .610 | 24 |
| 37 | .709 | .92008 | .0869 | .590 | 23 |
| 38 | .730 | .062 | .0862 | .570 | 22 |
| 39 | .752 | .116 | .0856 | .551 | 21 |
| 40 | .67773 | .92170 | 1.0850 | .73531 | 20 |
| 41 | .795 | .224 | .0843 | .511 | 19 |
| 42 | .816 | .277 | .0837 | .491 | 18 |
| 43 | .837 | .331 | .0831 | .472 | 17 |
| 44 | .859 | .385 | .0824 | .452 | 16 |
| 45 | .67880 | .92439 | 1.0818 | .73432 | 15 |
| 46 | .901 | .493 | .0812 | .413 | 14 |
| 47 | .923 | .547 | .0805 | .393 | 13 |
| 48 | .944 | .601 | .0799 | .373 | 12 |
| 49 | .965 | .655 | .0793 | .353 | 11 |
| 50 | .67987 | .92709 | 1.0786 | .73333 | 10 |
| 51 | .68008 | .763 | .0780 | .314 | 9 |
| 52 | .029 | .817 | .0774 | .294 | 8 |
| 53 | .051 | .872 | .0768 | .274 | 7 |
| 54 | .072 | .926 | .0761 | .254 | 6 |
| 55 | .68093 | .92980 | 1.0755 | .73234 | 5 |
| 56 | .115 | .93034 | .0749 | .215 | 4 |
| 57 | .136 | .088 | .0742 | .195 | 3 |
| 58 | .157 | .143 | .0736 | .175 | 2 |
| 59 | .179 | .197 | .0730 | .155 | 1 |
| 60 | .68200 | .93252 | 1.0724 | .73135 | 0 |
| | Cos | Ctn | Tan | Sin | ' |

47°

| ' | Sin | Tan | Ctn | Cos | ' |
|----|--------|--------|--------|--------|----|
| 0 | .68200 | .93252 | 1.0724 | .73135 | 60 |
| 1 | .221 | .306 | .0717 | .116 | 59 |
| 2 | .242 | .360 | .0711 | .096 | 58 |
| 3 | .264 | .415 | .0705 | .076 | 57 |
| 4 | .285 | .469 | .0699 | .056 | 56 |
| 5 | .68306 | .93524 | 1.0692 | .73036 | 55 |
| 6 | .327 | .578 | .0686 | .73016 | 54 |
| 7 | .349 | .633 | .0680 | .72996 | 53 |
| 8 | .370 | .688 | .0674 | .976 | 52 |
| 9 | .391 | .742 | .0668 | .957 | 51 |
| 10 | .68412 | .93797 | 1.0661 | .72937 | 50 |
| 11 | .434 | .852 | .0655 | .917 | 49 |
| 12 | .455 | .906 | .0649 | .897 | 48 |
| 13 | .476 | .93961 | .0643 | .877 | 47 |
| 14 | .497 | .94016 | .0637 | .857 | 46 |
| 15 | .68518 | .94071 | 1.0630 | .72837 | 45 |
| 16 | .539 | .125 | .0624 | .817 | 44 |
| 17 | .561 | .180 | .0618 | .797 | 43 |
| 18 | .582 | .235 | .0612 | .777 | 42 |
| 19 | .603 | .290 | .0606 | .757 | 41 |
| 20 | .68624 | .94345 | 1.0599 | .72737 | 40 |
| 21 | .645 | .400 | .0593 | .717 | 39 |
| 22 | .666 | .455 | .0587 | .697 | 38 |
| 23 | .688 | .510 | .0581 | .677 | 37 |
| 24 | .709 | .565 | .0575 | .657 | 36 |
| 25 | .68730 | .94620 | 1.0569 | .72637 | 35 |
| 26 | .751 | .676 | .0562 | .617 | 34 |
| 27 | .772 | .731 | .0556 | .597 | 33 |
| 28 | .793 | .786 | .0550 | .577 | 32 |
| 29 | .814 | .841 | .0544 | .557 | 31 |
| 30 | .68835 | .94896 | 1.0538 | .72537 | 30 |
| 31 | .857 | .94952 | .0532 | .517 | 29 |
| 32 | .878 | .95007 | .0526 | .497 | 28 |
| 33 | .899 | .062 | .0519 | .477 | 27 |
| 34 | .920 | .118 | .0513 | .457 | 26 |
| 35 | .68941 | .95173 | 1.0507 | .72437 | 25 |
| 36 | .962 | .229 | .0501 | .417 | 24 |
| 37 | .68983 | .284 | .0495 | .397 | 23 |
| 38 | .69004 | .340 | .0489 | .377 | 22 |
| 39 | .025 | .395 | .0483 | .357 | 21 |
| 40 | .69046 | .95451 | 1.0477 | .72337 | 20 |
| 41 | .067 | .506 | .0470 | .317 | 19 |
| 42 | .088 | .562 | .0464 | .297 | 18 |
| 43 | .109 | .618 | .0458 | .277 | 17 |
| 44 | .130 | .673 | .0452 | .257 | 16 |
| 45 | .69151 | .95729 | 1.0446 | .72236 | 15 |
| 46 | .172 | .785 | .0440 | .216 | 14 |
| 47 | .193 | .841 | .0434 | .196 | 13 |
| 48 | .214 | .897 | .0428 | .176 | 12 |
| 49 | .235 | .95952 | .0422 | .156 | 11 |
| 50 | .69256 | .96008 | 1.0416 | .72136 | 10 |
| 51 | .277 | .064 | .0410 | .116 | 9 |
| 52 | .298 | .120 | .0404 | .095 | 8 |
| 53 | .319 | .176 | .0398 | .075 | 7 |
| 54 | .340 | .232 | .0392 | .055 | 6 |
| 55 | .69361 | .96288 | 1.0385 | .72035 | 5 |
| 56 | .382 | .344 | .0379 | .72015 | 4 |
| 57 | .403 | .400 | .0373 | .71995 | 3 |
| 58 | .424 | .457 | .0367 | .974 | 2 |
| 59 | .445 | .513 | .0361 | .954 | 1 |
| 60 | .69466 | .96569 | 1.0355 | .71934 | 0 |
| | Cos | Ctn | Tan | Sin | ' |

46°

44° — Values of Trigonometric Functions

| ' | Sin | Tan | Ctn | Cos | ' |
|----|--------|--------|--------|--------|----|
| 0 | .69466 | .96569 | 1.0355 | .71934 | 60 |
| 1 | 487 | 625 | .0349 | 914 | 59 |
| 2 | 508 | 681 | .0343 | 894 | 58 |
| 3 | 529 | 738 | .0337 | 873 | 57 |
| 4 | 549 | 794 | .0331 | 853 | 56 |
| 5 | .69570 | .96850 | 1.0325 | .71833 | 55 |
| 6 | 591 | 907 | .0319 | 813 | 54 |
| 7 | 612 | .96963 | .0313 | 792 | 53 |
| 8 | 633 | .97020 | .0307 | 772 | 52 |
| 9 | 654 | 076 | .0301 | 752 | 51 |
| 10 | .69675 | .97133 | 1.0295 | .71732 | 50 |
| 11 | 696 | 189 | .0289 | 711 | 49 |
| 12 | 717 | 246 | .0283 | 691 | 48 |
| 13 | 737 | 302 | .0277 | 671 | 47 |
| 14 | 758 | 359 | .0271 | 650 | 46 |
| 15 | .69779 | .97416 | 1.0265 | .71630 | 45 |
| 16 | 800 | 472 | .0259 | 610 | 44 |
| 17 | 821 | 529 | .0253 | 590 | 43 |
| 18 | 842 | 586 | .0247 | 569 | 42 |
| 19 | 862 | 643 | .0241 | 549 | 41 |
| 20 | .69883 | .97700 | 1.0235 | .71529 | 40 |
| 21 | 904 | 756 | .0230 | 508 | 39 |
| 22 | 925 | 813 | .0224 | 488 | 38 |
| 23 | 946 | 870 | .0218 | 468 | 37 |
| 24 | 966 | 927 | .0212 | 447 | 36 |
| 25 | .69987 | .97984 | 1.0206 | .71427 | 35 |
| 26 | .70008 | .98041 | .0200 | 407 | 34 |
| 27 | 029 | 098 | .0194 | 386 | 33 |
| 28 | 049 | 155 | .0188 | 366 | 32 |
| 29 | 070 | 213 | .0182 | 345 | 31 |
| 30 | .70091 | .98270 | 1.0176 | .71325 | 30 |
| 31 | 112 | 327 | .0170 | 305 | 29 |
| 32 | 132 | 384 | .0164 | 284 | 28 |
| 33 | 153 | 441 | .0158 | 264 | 27 |
| 34 | 174 | 499 | .0152 | 243 | 26 |
| 35 | .70195 | .98556 | 1.0147 | .71223 | 25 |
| 36 | 215 | 613 | .0141 | 203 | 24 |
| 37 | 236 | 671 | .0135 | 182 | 23 |
| 38 | 257 | 728 | .0129 | 162 | 22 |
| 39 | 277 | 786 | .0123 | 141 | 21 |
| 40 | .70298 | .98843 | 1.0117 | .71121 | 20 |
| 41 | 319 | 901 | .0111 | 100 | 19 |
| 42 | 339 | .98958 | .0105 | 080 | 18 |
| 43 | 360 | .99016 | .0099 | 059 | 17 |
| 44 | 381 | 073 | .0094 | 039 | 16 |
| 45 | .70401 | .99131 | 1.0088 | .71019 | 15 |
| 46 | 422 | 189 | .0082 | .70998 | 14 |
| 47 | 443 | 247 | .0076 | 978 | 13 |
| 48 | 463 | 304 | .0070 | 957 | 12 |
| 49 | 484 | 362 | .0064 | 937 | 11 |
| 50 | .70505 | .99420 | 1.0058 | .70916 | 10 |
| 51 | 525 | 478 | .0052 | 896 | 9 |
| 52 | 546 | 536 | .0047 | 875 | 8 |
| 53 | 567 | 594 | .0041 | 855 | 7 |
| 54 | 587 | 652 | .0035 | 834 | 6 |
| 55 | .70608 | .99710 | 1.0029 | .70813 | 5 |
| 56 | 628 | 768 | .0023 | 793 | 4 |
| 57 | 649 | 826 | .0017 | 772 | 3 |
| 58 | 670 | 884 | .0012 | 752 | 2 |
| 59 | 690 | .99942 | .0006 | 731 | 1 |
| 60 | .70711 | 1.0000 | 1.0000 | .70711 | 0 |
| | Cos | Ctn | Tan | Sin | ' |

570 Table III — Degrees, Minutes, and Seconds to Radians [III]

| Degrees | | | | Minutes | | | | Seconds | | | |
|---------|------------|-----|------------|---------|------------|----|------------|---------|------------|----|------------|
| 0° | 0 00000 00 | 60° | 1 04719 76 | 120° | 2 09439 51 | 0 | 0 00000 00 | 0 | 0 00000 00 | 0 | 0 00000 00 |
| 1 | 0 01745 33 | 61 | 1 06165 08 | 121 | 2 11184 84 | 1 | 0 00009 09 | 1 | 0 00000 48 | 1 | 0 00000 48 |
| 2 | 0 03490 66 | 62 | 1 08210 41 | 122 | 2 12930 17 | 2 | 0 00058 18 | 2 | 0 00000 97 | 2 | 0 00000 97 |
| 3 | 0 05235 99 | 63 | 1 09955 71 | 123 | 2 14675 50 | 3 | 0 00087 27 | 3 | 0 00001 45 | 3 | 0 00001 45 |
| 4 | 0 06981 32 | 64 | 1 11701 07 | 124 | 2 16420 83 | 4 | 0 00116 36 | 4 | 0 00001 94 | 4 | 0 00001 94 |
| 5 | 0 08726 65 | 65 | 1 13446 40 | 125 | 2 18166 16 | 5 | 0 00145 44 | 5 | 0 00002 42 | 5 | 0 00002 42 |
| 6 | 0 10471 98 | 66 | 1 15191 73 | 126 | 2 19911 49 | 6 | 0 00174 53 | 6 | 0 00002 91 | 6 | 0 00002 91 |
| 7 | 0 12217 30 | 67 | 1 16937 06 | 127 | 2 21656 82 | 7 | 0 00203 62 | 7 | 0 00003 39 | 7 | 0 00003 39 |
| 8 | 0 13962 63 | 68 | 1 18682 39 | 128 | 2 23402 14 | 8 | 0 00232 71 | 8 | 0 00003 88 | 8 | 0 00003 88 |
| 9 | 0 15707 96 | 69 | 1 20427 72 | 129 | 2 25147 47 | 9 | 0 00261 80 | 9 | 0 00004 36 | 9 | 0 00004 36 |
| 10 | 0 17453 29 | 70 | 1 22173 05 | 130 | 2 26892 80 | 10 | 0 00290 89 | 10 | 0 00004 85 | 10 | 0 00004 85 |
| 11 | 0 19198 62 | 71 | 1 23918 38 | 131 | 2 28638 13 | 11 | 0 00319 98 | 11 | 0 00005 33 | 11 | 0 00005 33 |
| 12 | 0 20943 95 | 72 | 1 25663 71 | 132 | 2 30383 46 | 12 | 0 00349 07 | 12 | 0 00005 82 | 12 | 0 00005 82 |
| 13 | 0 22689 28 | 73 | 1 27409 04 | 133 | 2 32128 79 | 13 | 0 00378 15 | 13 | 0 00006 30 | 13 | 0 00006 30 |
| 14 | 0 24434 61 | 74 | 1 29154 36 | 134 | 2 33874 12 | 14 | 0 00407 24 | 14 | 0 00006 79 | 14 | 0 00006 79 |
| 15 | 0 26179 94 | 75 | 1 30900 69 | 135 | 2 35619 45 | 15 | 0 00436 33 | 15 | 0 00007 27 | 15 | 0 00007 27 |
| 16 | 0 27925 27 | 76 | 1 32645 02 | 136 | 2 37364 78 | 16 | 0 00465 42 | 16 | 0 00007 76 | 16 | 0 00007 76 |
| 17 | 0 29670 60 | 77 | 1 34390 35 | 137 | 2 39110 11 | 17 | 0 00494 51 | 17 | 0 00008 24 | 17 | 0 00008 24 |
| 18 | 0 31415 93 | 78 | 1 36135 68 | 138 | 2 40855 44 | 18 | 0 00523 60 | 18 | 0 00008 73 | 18 | 0 00008 73 |
| 19 | 0 33161 26 | 79 | 1 37881 01 | 139 | 2 42600 77 | 19 | 0 00552 69 | 19 | 0 00009 21 | 19 | 0 00009 21 |
| 20 | 0 34906 59 | 80 | 1 39626 34 | 140 | 2 44346 10 | 20 | 0 00581 78 | 20 | 0 00009 70 | 20 | 0 00009 70 |
| 21 | 0 36651 91 | 81 | 1 41371 67 | 141 | 2 46091 42 | 21 | 0 00610 87 | 21 | 0 00010 18 | 21 | 0 00010 18 |
| 22 | 0 38397 24 | 82 | 1 43117 00 | 142 | 2 47836 75 | 22 | 0 00639 95 | 22 | 0 00010 67 | 22 | 0 00010 67 |
| 23 | 0 40142 57 | 83 | 1 44862 33 | 143 | 2 49582 08 | 23 | 0 00669 04 | 23 | 0 00011 15 | 23 | 0 00011 15 |
| 24 | 0 41887 90 | 84 | 1 46607 66 | 144 | 2 51327 41 | 24 | 0 00698 13 | 24 | 0 00011 64 | 24 | 0 00011 64 |
| 25 | 0 43633 23 | 85 | 1 48352 99 | 145 | 2 53072 74 | 25 | 0 00727 22 | 25 | 0 00012 12 | 25 | 0 00012 12 |
| 26 | 0 45378 56 | 86 | 1 50098 32 | 146 | 2 54818 07 | 26 | 0 00756 31 | 26 | 0 00012 61 | 26 | 0 00012 61 |
| 27 | 0 47123 89 | 87 | 1 51843 64 | 147 | 2 56563 40 | 27 | 0 00785 40 | 27 | 0 00013 09 | 27 | 0 00013 09 |
| 28 | 0 48869 22 | 88 | 1 53588 97 | 148 | 2 58308 73 | 28 | 0 00814 49 | 28 | 0 00013 57 | 28 | 0 00013 57 |
| 29 | 0 50614 55 | 89 | 1 55334 30 | 149 | 2 60054 06 | 29 | 0 00843 58 | 29 | 0 00014 06 | 29 | 0 00014 06 |
| 30 | 0 52359 88 | 90 | 1 57079 63 | 150 | 2 61799 39 | 30 | 0 00872 66 | 30 | 0 00014 54 | 30 | 0 00014 54 |
| 31 | 0 54105 21 | 91 | 1 58824 96 | 151 | 2 63544 72 | 31 | 0 00901 75 | 31 | 0 00015 03 | 31 | 0 00015 03 |
| 32 | 0 55850 54 | 92 | 1 60570 29 | 152 | 2 65290 05 | 32 | 0 00930 84 | 32 | 0 00015 51 | 32 | 0 00015 51 |
| 33 | 0 57595 87 | 93 | 1 62315 62 | 153 | 2 67035 38 | 33 | 0 00959 93 | 33 | 0 00016 00 | 33 | 0 00016 00 |
| 34 | 0 59341 19 | 94 | 1 64060 95 | 154 | 2 68780 70 | 34 | 0 00989 02 | 34 | 0 00016 48 | 34 | 0 00016 48 |
| 35 | 0 61086 52 | 95 | 1 65806 28 | 155 | 2 70526 03 | 35 | 0 01018 11 | 35 | 0 00016 97 | 35 | 0 00016 97 |
| 36 | 0 62831 85 | 96 | 1 67551 61 | 156 | 2 72271 36 | 36 | 0 01047 20 | 36 | 0 00017 45 | 36 | 0 00017 45 |
| 37 | 0 64577 18 | 97 | 1 69296 94 | 157 | 2 74016 69 | 37 | 0 01076 29 | 37 | 0 00017 94 | 37 | 0 00017 94 |
| 38 | 0 66322 51 | 98 | 1 71042 27 | 158 | 2 75762 02 | 38 | 0 01105 38 | 38 | 0 00018 42 | 38 | 0 00018 42 |
| 39 | 0 68067 84 | 99 | 1 72787 60 | 159 | 2 77507 35 | 39 | 0 01134 46 | 39 | 0 00018 91 | 39 | 0 00018 91 |
| 40 | 0 69813 17 | 100 | 1 74532 93 | 160 | 2 79252 68 | 40 | 0 01163 55 | 40 | 0 00019 39 | 40 | 0 00019 39 |
| 41 | 0 71558 50 | 101 | 1 76278 25 | 161 | 2 80998 01 | 41 | 0 01192 64 | 41 | 0 00019 88 | 41 | 0 00019 88 |
| 42 | 0 73303 83 | 102 | 1 78023 58 | 162 | 2 82743 34 | 42 | 0 01221 73 | 42 | 0 00020 36 | 42 | 0 00020 36 |
| 43 | 0 75049 16 | 103 | 1 79768 91 | 163 | 2 84488 67 | 43 | 0 01250 82 | 43 | 0 00020 85 | 43 | 0 00020 85 |
| 44 | 0 76794 49 | 104 | 1 81514 24 | 164 | 2 86234 00 | 44 | 0 01279 91 | 44 | 0 00021 33 | 44 | 0 00021 33 |
| 45 | 0 78539 82 | 105 | 1 83259 57 | 165 | 2 87979 33 | 45 | 0 01309 00 | 45 | 0 00021 82 | 45 | 0 00021 82 |
| 46 | 0 80285 15 | 106 | 1 85004 90 | 166 | 2 89724 66 | 46 | 0 01338 09 | 46 | 0 00022 30 | 46 | 0 00022 30 |
| 47 | 0 82030 48 | 107 | 1 86750 23 | 167 | 2 91469 99 | 47 | 0 01367 17 | 47 | 0 00022 79 | 47 | 0 00022 79 |
| 48 | 0 83775 80 | 108 | 1 88495 56 | 168 | 2 93215 31 | 48 | 0 01396 26 | 48 | 0 00023 27 | 48 | 0 00023 27 |
| 49 | 0 85521 13 | 109 | 1 90240 89 | 169 | 2 94960 64 | 49 | 0 01425 35 | 49 | 0 00023 76 | 49 | 0 00023 76 |
| 50 | 0 87266 46 | 110 | 1 91986 22 | 170 | 2 96705 97 | 50 | 0 01454 44 | 50 | 0 00024 24 | 50 | 0 00024 24 |
| 51 | 0 89011 79 | 111 | 1 93731 55 | 171 | 2 98451 30 | 51 | 0 01483 53 | 51 | 0 00024 73 | 51 | 0 00024 73 |
| 52 | 0 90757 12 | 112 | 1 95476 88 | 172 | 3 00196 63 | 52 | 0 01512 62 | 52 | 0 00025 21 | 52 | 0 00025 21 |
| 53 | 0 92502 45 | 113 | 1 97222 21 | 173 | 3 01941 96 | 53 | 0 01541 71 | 53 | 0 00025 70 | 53 | 0 00025 70 |
| 54 | 0 94247 78 | 114 | 1 98967 53 | 174 | 3 03687 29 | 54 | 0 01570 80 | 54 | 0 00026 18 | 54 | 0 00026 18 |
| 55 | 0 95993 11 | 115 | 2 00712 86 | 175 | 3 05432 62 | 55 | 0 01599 89 | 55 | 0 00026 66 | 55 | 0 00026 66 |
| 56 | 0 97738 44 | 116 | 2 02458 19 | 176 | 3 07177 95 | 56 | 0 01628 97 | 56 | 0 00027 15 | 56 | 0 00027 15 |
| 57 | 0 99483 77 | 117 | 2 04203 52 | 177 | 3 08923 28 | 57 | 0 01658 06 | 57 | 0 00027 63 | 57 | 0 00027 63 |
| 58 | 1 01229 10 | 118 | 2 05948 85 | 178 | 3 10668 61 | 58 | 0 01687 15 | 58 | 0 00028 12 | 58 | 0 00028 12 |
| 59 | 1 02974 43 | 119 | 2 07694 18 | 179 | 3 12413 94 | 59 | 0 01716 24 | 59 | 0 00028 60 | 59 | 0 00028 60 |
| 60 | 1 04719 76 | 120 | 2 09439 51 | 180 | 3 14159 27 | 60 | 0 01745 33 | 60 | 0 00029 09 | 60 | 0 00029 09 |

IV] Table IV — Radian Measure — Trigonometric Functions 571

| x Radians | Sin x | Cos x | Tan x | Equivalent of x | x Radians | Sin x | Cos x | Tan x | Equivalent of x |
|-------------|---------|---------|---------|-------------------|-------------|---------|---------|---------|-------------------|
| .00 | 00000 | 1 0000 | 00000 | 0 00' 0 | 50 | 47943 | 87758 | 46 30 | 28° 38' 9 |
| .01 | 01000 | 99995 | 01000 | 0 34 4 | 51 | 48818 | 87274 | 4336 | 29° 13 3 |
| .02 | 02000 | 99980 | 02000 | 1 08 8 | 52 | 49688 | 86788 | 72 36 | 29° 47' 6 |
| .03 | 03000 | 99955 | 03001 | 1° 43 1 | 53 | 50553 | 86281 | 58592 | 30° 22' 0 |
| .04 | 03999 | 99920 | 04000 | 2° 17' 5 | 54 | 51414 | 85771 | 59945 | 30° 56' 4 |
| .05 | 04998 | 99875 | 05001 | 2° 51' 9 | 55 | 52271 | 85246 | 61511 | 31 0 8 |
| .06 | 05996 | 99820 | 06000 | 3° 26 3 | 56 | 53111 | 84707 | 63095 | 32° 03' 1 |
| .07 | 06994 | 99755 | 07011 | 4° 00 1 | 57 | 53933 | 84160 | 64697 | 32° 39' 5 |
| .08 | 07991 | 99680 | 08017 | 4° 33 0 | 58 | 54746 | 83606 | 66317 | 33° 13' 9 |
| .09 | 08988 | 99595 | 09024 | 5° 09 4 | 59 | 55551 | 83044 | 66956 | 33° 48 3 |
| 10 | 09983 | 99500 | 10033 | 5° 41 8 | 60 | 56348 | 82474 | 68114 | 34° 22' 6 |
| .11 | 10978 | 99396 | 11045 | 6 18 2 | 61 | 57137 | 81895 | 69812 | 34° 57' 0 |
| .12 | 11971 | 99281 | 12057 | 6 55 5 | 62 | 57918 | 81308 | 71331 | 35° 31' 4 |
| .13 | 12963 | 99156 | 13071 | 7° 26 3 | 63 | 58691 | 80714 | 72891 | 36° 03' 8 |
| .14 | 13954 | 99022 | 14092 | 8° 01' 3 | 64 | 59457 | 80112 | 74451 | 36° 40' 2 |
| .15 | 14944 | 98877 | 15114 | 8° 35' 7 | 65 | 60216 | 79608 | 76020 | 37° 14' 5 |
| .16 | 15932 | 98723 | 16138 | 9° 10' 0 | 66 | 60968 | 79099 | 77611 | 37° 48' 9 |
| .17 | 16918 | 98558 | 17166 | 9° 44' 4 | 67 | 61713 | 78582 | 79225 | 38° 23' 3 |
| .18 | 17903 | 98381 | 18197 | 10° 18' 8 | 68 | 62451 | 78057 | 80866 | 38° 57' 7 |
| .19 | 18886 | 98200 | 19232 | 10° 53' 2 | 69 | 63184 | 77524 | 82534 | 39° 32' 0 |
| 20 | 19867 | 98007 | 20271 | 11° 27' 5 | 70 | 63911 | 76984 | 84229 | 40° 06' 4 |
| .21 | 20846 | 97800 | 21314 | 12° 01' 9 | 71 | 64633 | 76436 | 85933 | 40° 40 8 |
| .22 | 21823 | 97590 | 22362 | 12° 36' 3 | 72 | 65348 | 75881 | 87707 | 41° 15' 2 |
| .23 | 22798 | 97367 | 23414 | 13° 10' 7 | 73 | 66057 | 75317 | 89492 | 41° 49' 6 |
| .24 | 23770 | 97134 | 24472 | 13° 45' 1 | 74 | 66761 | 74747 | 91309 | 42° 23' 9 |
| .25 | 24740 | 96891 | 25534 | 14° 19' 4 | 75 | 67461 | 74161 | 93160 | 42° 58' 3 |
| .26 | 25708 | 96639 | 26602 | 14° 53' 8 | 76 | 68157 | 73561 | 95045 | 43° 32' 7 |
| .27 | 26673 | 96377 | 27676 | 15° 28' 2 | 77 | 68848 | 72947 | 96967 | 44° 07' 1 |
| .28 | 27636 | 96106 | 28755 | 16° 02 6 | 78 | 69534 | 72321 | 98926 | 44° 41' 4 |
| .29 | 28595 | 95824 | 29841 | 16° 36' 9 | 79 | 70215 | 71684 | 1 00922 | 45° 15' 8 |
| 30 | 29552 | 95534 | 30934 | 17° 11' 3 | 80 | 70891 | 71037 | 1 0296 | 45° 50' 2 |
| .31 | 30506 | 95233 | 32033 | 17° 45' 7 | 81 | 71561 | 70380 | 1 0505 | 46° 24' 6 |
| .32 | 31457 | 94924 | 33139 | 18° 20' 1 | 82 | 72226 | 69714 | 1 0717 | 46° 59' 0 |
| .33 | 32404 | 94604 | 34252 | 18° 54' 5 | 83 | 72887 | 69038 | 1 0934 | 47° 33' 3 |
| .34 | 33349 | 94275 | 35374 | 19° 28' 8 | 84 | 73544 | 68353 | 1 1156 | 48° 07' 7 |
| .35 | 34290 | 93937 | 36503 | 20° 03' 2 | 85 | 74197 | 67659 | 1 1383 | 48° 42' 1 |
| .36 | 35227 | 93590 | 37640 | 20° 37' 6 | 86 | 74846 | 66957 | 1 1616 | 49° 16' 5 |
| .37 | 36162 | 93233 | 38786 | 21° 12' 0 | 87 | 75491 | 66247 | 1 1853 | 49° 50' 8 |
| .38 | 37092 | 92866 | 39941 | 21° 46' 3 | 88 | 76132 | 65529 | 1 2097 | 50° 25' 2 |
| .39 | 38019 | 92491 | 41105 | 22° 20' 7 | 89 | 76769 | 64804 | 1 2346 | 50° 59' 6 |
| 40 | 38942 | 92106 | 42279 | 22° 55' 1 | 90 | 77402 | 64074 | 1 2602 | 51° 34' 0 |
| .41 | 39861 | 91712 | 43463 | 23° 29' 5 | 91 | 78031 | 63339 | 1 2864 | 52° 08' 3 |
| .42 | 40776 | 91309 | 44657 | 24° 03' 9 | 92 | 78656 | 62599 | 1 3133 | 52° 42' 7 |
| .43 | 41687 | 90897 | 45862 | 24° 38' 2 | 93 | 79277 | 61854 | 1 3409 | 53° 17' 1 |
| .44 | 42594 | 90475 | 47075 | 25° 12' 6 | 94 | 79894 | 61105 | 1 3692 | 53° 51' 5 |
| .45 | 43497 | 90045 | 48306 | 25° 47' 0 | 95 | 80507 | 60352 | 1 3984 | 54° 25' 9 |
| .46 | 44395 | 89603 | 49545 | 26° 21' 4 | 96 | 81117 | 59600 | 1 4284 | 55° 00' 2 |
| .47 | 45289 | 89157 | 50797 | 26° 55' 7 | 97 | 81724 | 58844 | 1 4592 | 55° 34' 6 |
| .48 | 46178 | 88699 | 52061 | 27° 30' 1 | 98 | 82327 | 58084 | 1 4910 | 56° 09' 0 |
| .49 | 47063 | 88233 | 53339 | 28° 04' 5 | 99 | 82926 | 57321 | 1 5237 | 56° 43' 4 |
| 50 | 47943 | 87758 | 54630 | 28° 38' 9 | 100 | 83521 | 56556 | 1 5574 | 57° 17' 7 |

| x Radians | Sin x | Cos x | Tan x | Equivalent of x | x Radians | Sin x | Cos x | Tan x | Equivalent of x |
|-------------|---------|---------|---------|-------------------|-------------|---------|---------|---------|-------------------|
| 1.00 | .84147 | .54030 | 1.5574 | 57° 17'.7 | 1.30 | .96356 | .26750 | 3.6021 | 74° 29'.1 |
| 1.01 | .84683 | .53186 | 1.5922 | 57° 52'.1 | 1.31 | .96618 | .25785 | 3.7471 | 75° 03'.4 |
| 1.02 | .85211 | .52337 | 1.6281 | 58° 26'.5 | 1.32 | .96872 | .24818 | 3.9033 | 75° 37'.8 |
| 1.03 | .85730 | .51482 | 1.6652 | 59° 00'.9 | 1.33 | .97115 | .23848 | 4.0723 | 76° 12'.2 |
| 1.04 | .86240 | .50622 | 1.7036 | 59° 35'.3 | 1.34 | .97348 | .22875 | 4.2556 | 76° 46'.6 |
| 1.05 | .86742 | .49757 | 1.7433 | 60° 09'.6 | 1.35 | .97572 | .21901 | 4.4552 | 77° 21'.0 |
| 1.06 | .87236 | .48887 | 1.7844 | 60° 44'.0 | 1.36 | .97786 | .20924 | 4.6734 | 77° 55'.3 |
| 1.07 | .87720 | .48012 | 1.8270 | 61° 18'.4 | 1.37 | .97991 | .19945 | 4.9131 | 78° 29'.7 |
| 1.08 | .88196 | .47133 | 1.8712 | 61° 52'.8 | 1.38 | .98185 | .18964 | 5.1774 | 79° 04'.1 |
| 1.09 | .88663 | .46249 | 1.9171 | 62° 27'.1 | 1.39 | .98370 | .17981 | 5.4707 | 79° 38'.5 |
| 1.10 | .89121 | .45360 | 1.9648 | 63° 01'.5 | 1.40 | .98545 | .16997 | 5.7979 | 80° 12'.8 |
| 1.11 | .89570 | .44466 | 2.0143 | 63° 35'.9 | 1.41 | .98710 | .16010 | 6.1654 | 80° 47'.2 |
| 1.12 | .90010 | .43568 | 2.0660 | 64° 10'.3 | 1.42 | .98865 | .15023 | 6.5811 | 81° 21'.6 |
| 1.13 | .90441 | .42666 | 2.1198 | 64° 44'.7 | 1.43 | .99010 | .14033 | 7.0555 | 81° 56'.0 |
| 1.14 | .90863 | .41759 | 2.1759 | 65° 19'.0 | 1.44 | .99146 | .13042 | 7.6018 | 82° 30'.4 |
| 1.15 | .91276 | .40849 | 2.2345 | 65° 53'.4 | 1.45 | .99271 | .12050 | 8.2381 | 83° 04'.7 |
| 1.16 | .91680 | .39934 | 2.2958 | 66° 27'.8 | 1.46 | .99387 | .11057 | 8.9886 | 83° 39'.1 |
| 1.17 | .92075 | .39015 | 2.3600 | 67° 02'.2 | 1.47 | .99492 | .10063 | 9.8874 | 84° 13'.5 |
| 1.18 | .92461 | .38092 | 2.4273 | 67° 36'.5 | 1.48 | .99588 | .09067 | 10.983 | 84° 47'.9 |
| 1.19 | .92837 | .37166 | 2.4979 | 68° 10'.9 | 1.49 | .99674 | .08071 | 12.350 | 85° 22'.2 |
| 1.20 | .93204 | .36236 | 2.5722 | 68° 45'.3 | 1.50 | .99749 | .07074 | 14.101 | 85° 56'.6 |
| 1.21 | .93562 | .35302 | 2.6503 | 69° 19'.7 | 1.51 | .99815 | .06076 | 16.428 | 86° 31'.0 |
| 1.22 | .93910 | .34365 | 2.7328 | 69° 54'.1 | 1.52 | .99871 | .05077 | 19.670 | 87° 05'.4 |
| 1.23 | .94249 | .33424 | 2.8198 | 70° 28'.4 | 1.53 | .99917 | .04079 | 24.498 | 87° 39'.8 |
| 1.24 | .94578 | .32480 | 2.9119 | 71° 02'.8 | 1.54 | .99953 | .03079 | 32.461 | 88° 14'.1 |
| 1.25 | .94898 | .31532 | 3.0096 | 71° 37'.2 | 1.55 | .99978 | .02079 | 48.078 | 88° 48'.5 |
| 1.26 | .95209 | .30582 | 3.1133 | 72° 11'.6 | 1.56 | .99994 | .01080 | 92.621 | 89° 22'.9 |
| 1.27 | .95510 | .29628 | 3.2236 | 72° 45'.9 | 1.57 | *1.0000 | *.00080 | *1255.8 | 89° 57'.3 |
| 1.28 | .95802 | .28672 | 3.3413 | 73° 20'.3 | 1.58 | .99996 | -.00920 | -108.65 | 90° 31'.6 |
| 1.29 | .96084 | .27712 | 3.4672 | 73° 54'.7 | 1.59 | .99982 | -.01920 | -52.067 | 91° 06'.0 |
| 1.30 | .96356 | .26750 | 3.6021 | 74° 29'.1 | 1.60 | .99957 | -.02920 | -34.233 | 91° 40'.4 |

 π radians = 180°

1 radian = 57° 17' 44".806 = 57.2957795

 $\pi = 3.14159265$

3600" = 60' = 1° = 0.01745329 radian

*1 right angle = 90° = $\pi/2$ radians = 1.5707963 radians

Table IV a — Radians to Degrees

| | RADIANS | TENTHS | HUNDREDTHS | THOUSANDTHS | TEN-THOUSANDTHS |
|---|--------------|-------------|------------|-------------|-----------------|
| 1 | 57°17'44".8 | 5°43'46".5 | 0°34'22".6 | 0° 3'26".3 | 0° 0'20".6 |
| 2 | 114°35'29".6 | 11°27'33".0 | 1° 8'45".3 | 0° 6'52".5 | 0° 0'41".3 |
| 3 | 171°53'14".4 | 17°11'19".4 | 1°43'07".9 | 0°10'18".8 | 0° 1'01".9 |
| 4 | 229°10'59".2 | 22°55'05".9 | 2°17'30".6 | 0°13'45".1 | 0° 1'22".5 |
| 5 | 286°28'44".0 | 28°38'52".4 | 2°51'53".2 | 0°17'11".3 | 0° 1'43".1 |
| 6 | 343°46'28".8 | 34°22'38".9 | 3°26'15".9 | 0°20'37".6 | 0° 2'03".8 |
| 7 | 401° 4'13".6 | 40° 6'25".4 | 4° 0'38".5 | 0°24'03".9 | 0° 2'24".4 |
| 8 | 458°21'58".4 | 45°50'11".8 | 4°35'01".2 | 0°27'30".1 | 0° 2'45".0 |
| 9 | 515°39'43".3 | 51°33'58".3 | 5° 9'23".8 | 0°30'56".4 | 0° 3'05".6 |

AMOUNT OF ONE DOLLAR PRINCIPAL AT COMPOUND INTEREST AFTER n YEARS

| n | 2 % | 2½ % | 3 % | 3½ % | 4 % | 4½ % | 5 % | 6 % | 7 % |
|-----|--------|--------|--------|--------|--------|--------|---------|---------|---------|
| 1 | 1.0200 | 1.0250 | 1.0300 | 1.0350 | 1.0400 | 1.0450 | 1.0500 | 1.0600 | 1.0700 |
| 2 | 1.0404 | 1.0506 | 1.0609 | 1.0712 | 1.0816 | 1.0920 | 1.1025 | 1.1236 | 1.1449 |
| 3 | 1.0612 | 1.0769 | 1.0927 | 1.1087 | 1.1249 | 1.1412 | 1.1576 | 1.1910 | 1.2250 |
| 4 | 1.0824 | 1.1038 | 1.1255 | 1.1475 | 1.1699 | 1.1925 | 1.2155 | 1.2625 | 1.3108 |
| 5 | 1.1041 | 1.1314 | 1.1593 | 1.1877 | 1.2167 | 1.2462 | 1.2763 | 1.3382 | 1.4026 |
| 6 | 1.1262 | 1.1597 | 1.1941 | 1.2293 | 1.2653 | 1.3023 | 1.3401 | 1.4185 | 1.5007 |
| 7 | 1.1487 | 1.1887 | 1.2299 | 1.2723 | 1.3159 | 1.3609 | 1.4071 | 1.5036 | 1.6058 |
| 8 | 1.1717 | 1.2184 | 1.2668 | 1.3168 | 1.3686 | 1.4221 | 1.4775 | 1.5938 | 1.7182 |
| 9 | 1.1951 | 1.2489 | 1.3044 | 1.3629 | 1.4233 | 1.4861 | 1.5513 | 1.6895 | 1.8385 |
| 10 | 1.2190 | 1.2801 | 1.3439 | 1.4106 | 1.4802 | 1.5530 | 1.6289 | 1.7908 | 1.9672 |
| 11 | 1.2434 | 1.3121 | 1.3842 | 1.4600 | 1.5395 | 1.6229 | 1.7103 | 1.8983 | 2.1049 |
| 12 | 1.2682 | 1.3449 | 1.4258 | 1.5111 | 1.6010 | 1.6959 | 1.7959 | 2.0122 | 2.2522 |
| 13 | 1.2936 | 1.3785 | 1.4685 | 1.5640 | 1.6651 | 1.7722 | 1.8856 | 2.1329 | 2.4098 |
| 14 | 1.3195 | 1.4130 | 1.5126 | 1.6187 | 1.7317 | 1.8519 | 1.9799 | 2.2609 | 2.5785 |
| 15 | 1.3459 | 1.4483 | 1.5580 | 1.6753 | 1.8009 | 1.9353 | 2.0789 | 2.3966 | 2.7590 |
| 16 | 1.3728 | 1.4845 | 1.6047 | 1.7340 | 1.8730 | 2.0224 | 2.1829 | 2.5404 | 2.9522 |
| 17 | 1.4002 | 1.5216 | 1.6528 | 1.7947 | 1.9479 | 2.1134 | 2.2920 | 2.6928 | 3.1588 |
| 18 | 1.4282 | 1.5597 | 1.7024 | 1.8575 | 2.0258 | 2.2085 | 2.4066 | 2.8543 | 3.3799 |
| 19 | 1.4568 | 1.5987 | 1.7535 | 1.9225 | 2.1068 | 2.3079 | 2.5270 | 3.0256 | 3.6165 |
| 20 | 1.4859 | 1.6386 | 1.8061 | 1.9898 | 2.1911 | 2.4117 | 2.6533 | 3.2071 | 3.8697 |
| 21 | 1.5157 | 1.6796 | 1.8603 | 2.0594 | 2.2788 | 2.5202 | 2.7860 | 3.3996 | 4.1406 |
| 22 | 1.5460 | 1.7216 | 1.9161 | 2.1315 | 2.3699 | 2.6337 | 2.9253 | 3.6035 | 4.4304 |
| 23 | 1.5769 | 1.7646 | 1.9736 | 2.2061 | 2.4647 | 2.7522 | 3.0715 | 3.8197 | 4.7405 |
| 24 | 1.6084 | 1.8087 | 2.0328 | 2.2833 | 2.5633 | 2.8760 | 3.2251 | 4.0489 | 5.0724 |
| 25 | 1.6406 | 1.8539 | 2.0938 | 2.3632 | 2.6658 | 3.0051 | 3.3864 | 4.2919 | 5.4274 |
| 26 | 1.6734 | 1.9003 | 2.1566 | 2.4460 | 2.7725 | 3.1407 | 3.5557 | 4.5494 | 5.8074 |
| 27 | 1.7069 | 1.9478 | 2.2213 | 2.5316 | 2.8834 | 3.2820 | 3.7335 | 4.8223 | 6.2139 |
| 28 | 1.7410 | 1.9965 | 2.2879 | 2.6202 | 2.9987 | 3.4297 | 3.9201 | 5.1117 | 6.6488 |
| 29 | 1.7758 | 2.0464 | 2.3566 | 2.7119 | 3.1187 | 3.5840 | 4.1161 | 5.4184 | 7.1143 |
| 30 | 1.8114 | 2.0976 | 2.4273 | 2.8068 | 3.2434 | 3.7453 | 4.3219 | 5.7435 | 7.6123 |
| 31 | 1.8476 | 2.1500 | 2.5001 | 2.9050 | 3.3731 | 3.9139 | 4.5380 | 6.0881 | 8.1451 |
| 32 | 1.8845 | 2.2038 | 2.5751 | 3.0067 | 3.5081 | 4.0900 | 4.7649 | 6.4534 | 8.7153 |
| 33 | 1.9222 | 2.2589 | 2.6523 | 3.1119 | 3.6484 | 4.2740 | 5.0032 | 6.8406 | 9.3253 |
| 34 | 1.9607 | 2.3153 | 2.7319 | 3.2209 | 3.7943 | 4.4664 | 5.2533 | 7.2510 | 9.9781 |
| 35 | 1.9999 | 2.3732 | 2.8139 | 3.3336 | 3.9461 | 4.6673 | 5.5160 | 7.6861 | 10.6766 |
| 36 | 2.0399 | 2.4325 | 2.8983 | 3.4503 | 4.1039 | 4.8774 | 5.7918 | 8.1473 | 11.4239 |
| 37 | 2.0807 | 2.4933 | 2.9852 | 3.5710 | 4.2681 | 5.0969 | 6.0814 | 8.6361 | 12.2236 |
| 38 | 2.1223 | 2.5557 | 3.0748 | 3.6960 | 4.4388 | 5.3262 | 6.3855 | 9.1543 | 13.0793 |
| 39 | 2.1647 | 2.6196 | 3.1670 | 3.8254 | 4.6164 | 5.5659 | 6.7048 | 9.7035 | 13.9948 |
| 40 | 2.2080 | 2.6851 | 3.2620 | 3.9593 | 4.8010 | 5.8164 | 7.0400 | 10.2857 | 14.9745 |
| 41 | 2.2522 | 2.7522 | 3.3599 | 4.0978 | 4.9931 | 6.0781 | 7.3920 | 10.9029 | 16.0227 |
| 42 | 2.2972 | 2.8210 | 3.4607 | 4.2413 | 5.1928 | 6.3516 | 7.7616 | 11.5570 | 17.1443 |
| 43 | 2.3432 | 2.8915 | 3.5645 | 4.3897 | 5.4005 | 6.6374 | 8.1497 | 12.2505 | 18.3444 |
| 44 | 2.3901 | 2.9638 | 3.6715 | 4.5433 | 5.6165 | 6.9361 | 8.5572 | 12.9855 | 19.6285 |
| 45 | 2.4379 | 3.0379 | 3.7816 | 4.7024 | 5.8412 | 7.2482 | 8.9850 | 13.7646 | 21.0025 |
| 46 | 2.4866 | 3.1139 | 3.8950 | 4.8669 | 6.0748 | 7.5744 | 9.4343 | 14.5905 | 22.4726 |
| 47 | 2.5363 | 3.1917 | 4.0119 | 5.0373 | 6.3178 | 7.9153 | 9.9060 | 15.4659 | 24.0457 |
| 48 | 2.5871 | 3.2715 | 4.1323 | 5.2136 | 6.5705 | 8.2715 | 10.4013 | 16.3939 | 25.7289 |
| 49 | 2.6388 | 3.3533 | 4.2562 | 5.3961 | 6.8333 | 8.6437 | 10.9213 | 17.3775 | 27.5299 |
| 50 | 2.6916 | 3.4371 | 4.3839 | 5.5849 | 7.1067 | 9.0326 | 11.4674 | 18.4202 | 29.4570 |

AMOUNT OF AN ANNUITY OF ONE DOLLAR PER YEAR AFTER n YEARS

| n | 2 % | 2½ % | 3 % | 3½ % | 4 % | 4½ % | 5 % | 6 % | 7 % |
|-----|---------|---------|----------|----------|----------|----------|----------|----------|----------|
| 1 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 2 | 2.0200 | 2.0350 | 2.0500 | 2.0650 | 2.0800 | 2.0950 | 2.1100 | 2.1250 | 2.1400 |
| 3 | 3.0604 | 3.0756 | 3.0909 | 3.1062 | 3.1216 | 3.1370 | 3.1525 | 3.1686 | 3.1849 |
| 4 | 4.1216 | 4.1525 | 4.1836 | 4.2149 | 4.2465 | 4.2782 | 4.3101 | 4.3746 | 4.4399 |
| 5 | 5.2040 | 5.2563 | 5.3091 | 5.3625 | 5.4163 | 5.4707 | 5.5256 | 5.6371 | 5.7507 |
| 6 | 6.3081 | 6.3877 | 6.4684 | 6.5502 | 6.6330 | 6.7169 | 6.8019 | 6.9753 | 7.1533 |
| 7 | 7.4343 | 7.5474 | 7.6625 | 7.7794 | 7.8983 | 8.0192 | 8.1420 | 8.3938 | 8.6540 |
| 8 | 8.5830 | 8.7361 | 8.8923 | 9.0517 | 9.2142 | 9.3800 | 9.5491 | 9.8975 | 10.2598 |
| 9 | 9.7546 | 9.9545 | 10.1591 | 10.3685 | 10.5828 | 10.8021 | 11.0266 | 11.4913 | 11.9780 |
| 10 | 10.9497 | 11.2034 | 11.4639 | 11.7311 | 12.0061 | 12.2882 | 12.5779 | 13.1808 | 13.8164 |
| 11 | 12.1687 | 12.4835 | 12.8078 | 13.1420 | 13.4864 | 13.8412 | 14.2068 | 14.9716 | 15.7836 |
| 12 | 13.4121 | 13.7956 | 14.1920 | 14.6020 | 15.0258 | 15.4640 | 15.9171 | 16.8699 | 17.8885 |
| 13 | 14.6803 | 15.1404 | 15.6178 | 16.1130 | 16.6268 | 17.1599 | 17.7130 | 18.8821 | 20.1406 |
| 14 | 15.9739 | 16.5190 | 17.0863 | 17.6770 | 18.2919 | 18.9321 | 19.5986 | 21.0151 | 22.5505 |
| 15 | 17.2934 | 17.9319 | 18.5989 | 19.2957 | 20.0236 | 20.7811 | 21.5786 | 23.2760 | 25.1290 |
| 16 | 18.6393 | 19.3802 | 20.1569 | 20.9710 | 21.8215 | 22.7193 | 23.6575 | 25.6725 | 27.8881 |
| 17 | 20.0121 | 20.8647 | 21.7616 | 22.7050 | 23.6975 | 24.7417 | 25.8404 | 28.2129 | 30.8402 |
| 18 | 21.4123 | 22.3863 | 23.4144 | 24.4997 | 25.6454 | 26.8551 | 28.1324 | 30.9057 | 33.9990 |
| 19 | 22.8406 | 23.9460 | 25.1169 | 26.3572 | 27.6712 | 29.0636 | 30.5390 | 33.7600 | 37.3790 |
| 20 | 24.2974 | 25.5447 | 26.8704 | 28.2797 | 29.7781 | 31.3714 | 33.0660 | 36.7856 | 40.9953 |
| 21 | 25.7833 | 27.1833 | 28.6765 | 30.2695 | 31.9692 | 33.7831 | 35.7193 | 39.9927 | 44.8652 |
| 22 | 27.2990 | 28.8629 | 30.5368 | 32.3289 | 34.2480 | 36.3034 | 38.5052 | 43.3923 | 49.0057 |
| 23 | 28.8450 | 30.5844 | 32.4529 | 34.4604 | 36.6179 | 38.9370 | 41.4305 | 46.9958 | 53.4361 |
| 24 | 30.4219 | 32.3490 | 34.4265 | 36.6665 | 39.0826 | 41.6892 | 44.5020 | 50.8156 | 58.1767 |
| 25 | 32.0303 | 34.1578 | 36.4595 | 38.9499 | 41.6459 | 44.5652 | 47.7271 | 54.8645 | 63.2490 |
| 26 | 33.6709 | 36.0117 | 38.5530 | 41.3131 | 44.3117 | 47.5706 | 51.1135 | 59.1564 | 68.6765 |
| 27 | 35.3443 | 37.9120 | 40.7096 | 43.7591 | 47.0842 | 50.7113 | 54.6691 | 63.7058 | 74.4838 |
| 28 | 37.0512 | 39.8598 | 42.9309 | 46.2906 | 49.9676 | 53.9933 | 58.4026 | 68.5281 | 80.6977 |
| 29 | 38.7922 | 41.8563 | 45.2189 | 48.9108 | 52.9663 | 57.4230 | 62.3227 | 73.6398 | 87.3465 |
| 30 | 40.5681 | 43.9027 | 47.5754 | 51.6227 | 56.0849 | 61.0071 | 66.4388 | 79.0582 | 94.4608 |
| 31 | 42.3794 | 46.0003 | 50.0027 | 54.4295 | 59.3283 | 64.7524 | 70.7608 | 84.8017 | 102.0730 |
| 32 | 44.2270 | 48.1503 | 52.5028 | 57.3315 | 62.7015 | 68.6662 | 75.2988 | 90.8898 | 110.2182 |
| 33 | 46.1116 | 50.3540 | 55.0778 | 60.3412 | 66.2095 | 72.7562 | 80.0638 | 97.3432 | 118.9334 |
| 34 | 48.0338 | 52.6129 | 57.7302 | 63.4532 | 69.8579 | 77.0303 | 85.0670 | 104.1838 | 128.2588 |
| 35 | 49.9945 | 54.9289 | 60.4621 | 66.6740 | 73.6522 | 81.4966 | 90.3203 | 111.4348 | 138.2369 |
| 36 | 51.9944 | 57.3014 | 63.2759 | 70.0076 | 77.5983 | 86.1640 | 95.8363 | 119.1209 | 148.9135 |
| 37 | 54.0343 | 59.7339 | 66.1742 | 73.4579 | 81.7022 | 91.0413 | 101.6281 | 127.2681 | 160.3374 |
| 38 | 56.1149 | 62.2273 | 69.1594 | 77.0289 | 85.9703 | 96.1382 | 107.7095 | 135.9042 | 172.5610 |
| 39 | 58.2372 | 64.7830 | 72.2342 | 80.7249 | 90.4091 | 101.4644 | 114.0950 | 145.0585 | 185.6403 |
| 40 | 60.4020 | 67.4026 | 75.4013 | 84.5503 | 95.0255 | 107.0303 | 120.7998 | 154.7620 | 199.6351 |
| 41 | 62.6100 | 70.0876 | 78.6633 | 88.5095 | 99.8265 | 112.8467 | 127.8398 | 165.0477 | 214.6096 |
| 42 | 64.8622 | 72.8398 | 82.0232 | 92.6074 | 104.8196 | 118.9248 | 135.2318 | 175.9505 | 230.6322 |
| 43 | 67.1595 | 75.6608 | 85.4839 | 96.8486 | 110.0124 | 125.2764 | 142.9933 | 187.5076 | 247.7765 |
| 44 | 69.5027 | 78.5523 | 89.0484 | 101.2383 | 115.4129 | 131.9138 | 151.1430 | 199.7580 | 266.1209 |
| 45 | 71.8927 | 81.5161 | 92.7199 | 105.7817 | 121.0294 | 138.8500 | 159.7002 | 212.7435 | 285.7493 |
| 46 | 74.3306 | 84.5540 | 96.5015 | 110.4840 | 126.8706 | 146.0982 | 168.6852 | 226.5081 | 306.7518 |
| 47 | 76.8172 | 87.6679 | 100.3965 | 115.3510 | 132.9454 | 153.6726 | 178.1194 | 241.0986 | 329.2244 |
| 48 | 79.3535 | 90.8596 | 104.4084 | 120.3883 | 139.2632 | 161.5879 | 188.0254 | 256.5645 | 353.2701 |
| 49 | 81.9406 | 94.1311 | 108.5406 | 125.6018 | 145.8337 | 169.8594 | 198.4267 | 272.9584 | 378.9990 |
| 50 | 84.5794 | 97.4843 | 112.7969 | 130.9979 | 152.6671 | 178.5030 | 209.3480 | 290.3359 | 406.5289 |

INDEX

A

| | |
|--|-------------------|
| Aeronautical engineering | 38 |
| Air, the atmosphere | 106 |
| Air conditioning | 513-522 |
| Alcohols, fats, soaps | 118, 120 |
| Algebra, fundamental operation | 271-275 |
| Algebraic equations | 285 |
| Alloys | 141 |
| Alternating current | 430-435 |
| Aluminum and its alloys | 136, 138 |
| Angles, complementary | 326 |
| Angles, triangles | 266, 309-336 |
| Angles and their trigonometric functions | 309, 317-329, 546 |
| Aptitudes and interests | 16 |
| Architecture | 47 |
| Atomic and molecular weights | 86 |
| Atomic theory | 58 |
| Automotive engine | 492, 493 |
| Auxiliary views in drawing | 210-213 |

B

| | |
|---------------------------------------|---------------|
| Batteries and cells | 131, 413 |
| Bolts and machine fasteners | 236-240 |
| Books in technology | 3, 21, 23, 24 |
| Boyle's law | 67, 352 |
| Bridge trusses | 472, 474 |

C

| | |
|---|----------|
| Carbohydrates | 122 |
| Carbon, its oxides and carbides | 82 |
| Careers in technology | 5 |
| Cement | 147, 149 |
| Center of gravity | 464 |
| Chemical engineering | 31 |
| Chemistry, plastics | 28, 125 |
| Chemistry and its language | 53, 61 |
| Chlorine, bromine, etc. | 91 |
| Circle, equation of | 345 |
| City planning | 48 |
| Civil engineering | 35 |
| Common denominator | 255 |

| | |
|-------------------------------------|----------|
| Communication engineering | 43, 452 |
| Complex numbers | 293 |
| Components of forces | 469 |
| Compounds in chemistry | 56 |
| Condenser capacity | 432 |
| Conics | 346, 353 |
| Conservation of mass | 58 |
| Culture and technology | 4 |
| Curve fitting | 355 |

D

| | |
|--|--------------------|
| Decimal numbers | 256 |
| Diesel engine | 463 |
| Dimensioning drawings | 214-221 |
| Drawing | 152 |
| inclined letters | 182 |
| instruments and equipment | 155, 156, 167 |
| shape description | 185 |
| surfaces and edges | 192-196 |
| technical sketches | 231, 233 |
| to scale | 163, 165 |
| vertical letters | 177 |
| with ink | 170, 175 |
| with instruments | 166 |
| with pencil | 157, 173, 175 |
| Drawing auxiliary views | 210-213 |
| Drawing axonometric projections | 222 |
| Drawing equipment | 155, 156, 167 |
| Drawing isometrics | 224 |
| Drawing oblique projections | 222, 226 |
| Drawing perspectives | 222, 227 |
| Drawing pictorials | 222, 236 |
| Drawing projections, the glass box | 186, 188 |
| Drawing sections | 198, 202, 204, 207 |
| Drawings, dimensioning | 214-221 |
| Drawings, idea sketches | 242, 252 |
| Drawings for patents | 247 |
| Drawings used in production | 244-247 |

E

| | |
|-------------------------------------|----------|
| Electric eye | 458 |
| Electric generator design | 422, 424 |
| Electric motors | 425-430 |
| Electrical engineering | 43, 391 |
| Electrical units | 414, 415 |
| Electricity, illumination | 391-409 |
| Electrolysis and ions | 97 |

| | |
|--|-------------------|
| Electromagnets | 419 |
| Electronic amplification and detection | 451-454 |
| Electronics | 11, 392, 439, 458 |
| Electrons and atoms | 60, 410 |
| Electron tubes and their operation | 411-459 |
| Elements and atomic weights | 62 |
| Elements and compounds | 55 |
| Elements and the periodic table | 102 |
| Ellipse, equation of | 350 |
| Engines, aviation and automotive | 38, 492, 493 |
| Equations, quadratic | 292-298 |
| simultaneous | 289 |
| solution of algebraic | 285 |
| Equations of circle | 345 |
| ellipse | 350 |
| hyperbola | 351 |
| parabola | 348 |
| straight lines | 341 |
| Equilibrium | 466 |
| Errors | 280-282 |
| Exponents and radicals | 298-303 |

F

| | |
|------------------------------------|----------|
| Factoring | 255-275 |
| Families of curves | 358 |
| Filters, electrical | 447 |
| Fluorescent lamps | 395, 408 |
| Force, pressure, gravity | 461, 462 |
| Force components | 469 |
| Force problems | 332 |
| Forces of acceleration | 478, 479 |
| Fractions, operations on | 257 |
| Friction coefficient | 475, 476 |

G

| | |
|--|------------|
| Gases and gas laws | 67, 69, 70 |
| Geometrical constructions in drawing | 172 |
| Glass and fused quartz | 126-129 |
| Graphical aids to solving equations | 287 |
| Grid tubes | 448 |

H

| | |
|-----------------------------------|----------|
| Heat power | 491 |
| Horsepower | 415, 496 |
| Humidity control of air | 513-519 |
| Hydrocarbons | 111 |
| Hydrogen | 74 |
| Hyperbola, equation of | 351 |

I

| | |
|---------------------------------------|----------|
| Illumination by electricity | 391-409 |
| Impact and momentum | 486 |
| Incandescent lamp | 392, 408 |
| Industrial engineering | 39 |
| Insulation and home comfort | 521 |
| Interest tables | 573, 574 |
| Ions and electrolysis | 96, 444 |
| Iron and steel | 46, 134 |
| Isometric drawings | 223 |

J

| | |
|--------------------------|-----|
| Jet propulsion | 489 |
|--------------------------|-----|

L

| | |
|---|------------------|
| Law of inverse squares | 284, 401 |
| Laws of gases, Boyle's, Charles' | 67, 69, 261, 352 |
| Laws of thermodynamics | 501, 506 |
| Laws, sine and cosine | 335 |
| Least squares | 355 |
| Leonardo da Vinci's idea sketch of airplane | 252 |
| Lettering | 177 184 |
| Lever, principle | 465 |
| Light and electricity | 391 |
| Lighting, candlepower, footcandle | 399, 400 |
| Lighting, design for | 405, 409 |
| Light reflection and refraction | 402-404 |
| Line fitting points | 356-358 |
| Lines, equations of | 341-344 |
| Lines, straight | 337-344 |
| Linework in drawing | 159 |
| Liquids and solids | 71, 72 |
| Logarithms, tables of | 525 |
| Logarithms and exponents | 303-309 |
| Logarithms on the slide rule | 379 |

M

| | |
|--|----------|
| Machine fasteners | 236-240 |
| Magnetism | 411 |
| Mathematics | 253-36 |
| Mechanical engineering | 3 |
| Mechanics, the effects of forces | 461 |
| Metallic compounds | 141 |
| Metallurgy | 40 |
| Metals, activity series | 76 |
| Metals and alloys | 133, 141 |

Index

579

| | |
|--|----------|
| Metals fail by fatigue | 35-37 |
| Mining and petroleum engineering | 49 |
| Model experiments | 40-42 |
| Moisture in saturated air | 519 |
| Molecular and atomic weights | 86 |
| Molecular structure | 117 |
| Moment of force | 465, 467 |
| Momentum | 485 |

N

| | |
|--------------------------------|----------|
| Nitrogen and ammonia | 107 |
| Nomographs | 359 |
| Numbers, complex | 293 |
| negative, zero | 262, 263 |
| rounding off | 279 |

O

| | |
|---|----------|
| Oblique projection | 222, 226 |
| Ohm's law | 43, 416 |
| Organic chemistry, hydrocarbons | 111 |
| Oxygen and its components | 64 |

P

| | |
|--|---------------|
| Paint | 150 |
| Parabola, equation of | 348 |
| Parlor test for technical traits | 17 |
| Per cent, percentage | 259 |
| Periodic table | 102 |
| Perspective drawings | 222, 227 |
| Petroleum products | 113, 115 |
| Phosphorous group | 110 |
| Photoelastic models | 42 |
| Physics as engineering science | 26 |
| Pictorial drawings | 222, 236 |
| Plastics, chemistry | 28, 125 |
| Power age | 11 |
| Power from machines | 415, 495, 505 |
| Production drawings | 244-247 |
| Production illustrations | 230, 231 |
| Professions and their activities | 12 |
| Projection of views | 188, 190 |
| Proportion | 261 |
| Pythagorean theorem | 267, 268 |

Q

| | |
|-------------------------------|---------|
| Quadratic equations | 292-298 |
|-------------------------------|---------|

R

| | |
|--|---------------|
| Radian measure of angles | 313, 571 |
| Radioactivity | 103 |
| Radio's development | 44 |
| Radio transmission and reception | 455-457 |
| Raman spectrum | 27 |
| Ratio and proportion | 260 |
| Refrigeration | 508, 510, 512 |
| Resistances, series and parallel | 416 |
| Resultant force or velocity | 332, 470, 471 |
| Roots | 275, 298 |
| Roots obtained by slide rule | 372 |
| Rubbers | 116 |

S

| | |
|---|--------------------|
| Screws and machine fasteners | 236-240 |
| Sectioning in drawing | 198, 202, 204, 208 |
| Simultaneous equations | 289 |
| Sketching, technical | 231, 235 |
| Slide rule | 363 |
| circular type | 388 |
| duplex type | 382-385 |
| log-log type | 385-387 |
| Mannheim | 364 |
| Slide rule operations | 367-372 |
| Solutions in chemistry | 88 |
| Square roots | 275 |
| Steam engines and power plants | 500-505 |
| Steam generation | 499, 504, 505 |
| Steam in saturated air | 519 |
| Steels | 143 |
| Sugar and starch | 123 |
| Sulphur and sulphuric acid | 99, 101 |
| Surveying problems and trigonometry | 331 |

T

| | |
|--|----------|
| Tables for mathematical calculations | 523-574 |
| compound interest | 573 |
| logarithms | 525 |
| precise numbers | 545 |
| radian measure | 571 |
| trigonometric functions | 546 |
| Tacoma bridge collapse | 32 |
| Technical drawing | 15 |
| Test for technological traits | 14, 1 |
| Thermodynamic laws | 501, 506 |
| Thermodynamics | 491 |

Index

581

| | |
|---|---------------|
| Threads on bolts and screws | 237-239 |
| Transforming a-c voltage | 435-438 |
| Triangles | 266, 309-336 |
| solution of | 329, 333 |
| Trigonometric functions of angles | 309, 317-329 |
| Trigonometric functions on slide rule | 367, 375-379 |
| Trigonometric tables | 52, 325, 546 |
| Trusses | 472, 474, 475 |

V

| | |
|--------------------------------------|----------|
| Valence | 77, 80 |
| Velocity and acceleration | 332, 477 |
| Views in pictorial drawing | 222 |
| Views in projection | 188, 190 |
| Voltage production | 420 |

W

| | |
|---|----------|
| Water | 81 |
| Work and power of gasoline engine | 495 |
| Work from steam | 500, 504 |
| Work, mechanical | 491, 483 |

